Gromov compactness

Def. $\mathcal{M} := \left\{ u \in C^\infty(\mathbb{R}\times S^1, M) \;\middle|\; \begin{array}{l} u \text{ solves (FE)} \\ E(u) < +\infty \end{array} \right\}$

$$\mathcal{M}(x_-, x_+) := \left\{ u \in \mathcal{M} \;\middle|\; u(s,\cdot) \xrightarrow[s\to\pm\infty]{} x_\pm \right\}$$

$$\underline{\mathcal{M}}(x_-, x_+) := \mathcal{M}(x_-, x_+)/\mathbb{R}$$

Rmk. $u \in \mathcal{M}(x_-, x_+) \implies \mathbb{E}(u) = \mathbb{A}_H(x_-) - \mathbb{A}_H(x_+)$.

We choose (H, J) such that transversality holds.

Goal $CZ(x_-) = CZ(x_+) + 1 \implies \underline{\mathcal{M}}(x_-, x_+)$ compact, hence a finite set.

$$\implies CF_k(H) := \bigoplus_{\substack{x \in \mathcal{P}(H) \\ CZ(x) = k}} \mathbb{Z}_2 \langle x \rangle, \quad \text{and}$$

$$\partial \langle x_- \rangle := \sum_{\substack{x_+ \in \mathcal{P}(H) \\ CZ(x_+) = CZ(x_-) + 1}} \left(\# \underline{\mathcal{M}}(x_-, x_+) \bmod 2 \right) \cdot \langle x_+ \rangle.$$

Theorem $\mathcal{M}$ is compact in C^∞_{loc} - topology.

Lemma C^0_{loc} - convergence $\implies$ C^∞_{loc} - convergence.
More precisely: $u_k \in C^\infty(\mathbb{R}^2, \mathbb{R}^{2n})$, $f_k \in C^\infty(\mathbb{R}^2, \mathbb{R}^{2n})$ s.t.

$$\partial_s u_k + J(u_k)\partial_t u_k = f_k$$

and $u_k \to u$, $f_k \to f$ in C^0_{loc} with u, f smooth s.t.

$$\partial_s u + J(u)\partial_t u = f$$

$\implies$ convergence in C^∞_{loc}.

proof.

$p > 2$. Then Calderon-Zygmund implies

$$\| \nabla (u_k - u) \|_p \leq c \cdot \| \overset{\partial_s + J(u_k)\partial_t}{\bar{\partial}} (u_k - u) \|_p$$

$$= c \cdot \| f_k - f + (J(u_k) - J(u)) \partial_t u \|_p$$

$$\leq c \cdot \left(\| f_k - f \|_p + \| (J(u_k) - J(u)) \partial_t u \|_p \right)$$

$$\xrightarrow[k \to +\infty]{} 0$$

$\Longrightarrow$ $u_k \to u$ in $W^{1,p}_{loc}$ $\Longrightarrow$ $f_k \to f$ in $W^{1,p}_{loc}$ because of the equation $\Longrightarrow$ elliptic bootstrapping implies the claim. ∎

Lemma (weak) For $u \in \mathcal{M}$ $\exists$ $c = c(u) > 0$ s.t. $\| \nabla u(s,t) \| \leq c$, $\forall$ $(s,t) \in \mathbb{R} \times S^1$.

proof.

Assume $R_k := \| \nabla u(s_k, t_k) \| \to +\infty$. Then can pick sequence $(\varepsilon_k) \subseteq \mathbb{R}$ s.t. $\varepsilon_k \to 0$ but $\varepsilon_k R_k \to +\infty$.

Wlog: $\leftsquigarrow$ not entirely obvious but standard

$$\| \nabla u(s,t) \| < 2 R_k, \quad \forall \ (s,t) \in B_{\varepsilon_k}(s_k, t_k).$$

Also

$$\varepsilon_k > \varepsilon_{k-1}, \quad R_{k+1} > 2 \cdot R_k, \quad \forall \ k \in \mathbb{N}.$$

We define $v_k \in C^\infty(\mathbb{R}^2, M)$ by

$$v_k(s,t) := u\left(s_k + s/R_k, \ t_k + t/R_k\right).$$

Then,

$$\|\nabla v_k(0,0)\| = 1 \,, \quad \|\nabla v_k(s,t)\| < 2 \,, \quad \forall (s,t) \in B_{\varepsilon_k R_k}(0)$$

Moreover

$$\partial_s v_k + J(v_k)\partial_t v_k = \frac{1}{R_k} \cdot \nabla H_{t_k + t/R_k} \xrightarrow{k \to +\infty} 0 \,,$$

(v_k) is totally bounded and equicontinuous

$\underset{\text{Arzela-Ascoli}}{\Longrightarrow}$ $v_k \to v$ in C^0_{loc}, and v solves (weakly)

$$\partial_s v + J(v)\partial_t v = 0$$

$\Rightarrow$ v smooth $\overset{\text{previous lemma}}{\Longrightarrow}$ $v_k \to v$ in C^∞_{loc}.

Now fix $R > 0$. Then

$$\mathbb{E}(v) = \int_{B_R(0)} |\partial_s v|^2 \, ds\, dt$$

$$= \int_{B_R(0)} \lim_{k \to +\infty} |\partial_s v_k|^2 \, ds\, dt$$

$$= \lim_{k \to +\infty} \int_{B_R(0)} |\partial_s v_k|^2 \, ds\, dt$$

$$= \lim_{k \to +\infty} \int_{B_{R/R_k}(s_k,t_k)} |\partial_s u|^2 \, ds\, dt \,.$$

Since $\varepsilon_k \cdot R_k \longrightarrow +\infty$, for k large enough $R/R_k < \varepsilon_k/2$

$\Rightarrow$ $B_{R/R_k}(s_k,t_k)$ are disjoint since $(s_{k+1}, t_{k+1}) \notin B_{\varepsilon_k}(s_k,t_k)$

$$\leadsto \sum_{k > k_0} \int_{B_{R/R_k}} |\partial_s u|^2 \, ds\, dt \leq \mathbb{E}(u) < +\infty$$

$\Rightarrow$ $\displaystyle\lim_{k \to +\infty} \int_{B_{R/R_k}} |\partial_s u|^2 \, ds\, dt = 0 \Longrightarrow \mathbb{E}(v) = 0 \Longrightarrow v \equiv$ constant, in contradiction with $\|\nabla v(0,0)\| = 1$

Theorem $\mathcal{M} = \bigcup_{x_-, x_+ \in \mathcal{P}(H)} \mathcal{M}(x_-, x_+)$.

Lemma $u \in \mathcal{M}$, $(s_k) \subseteq \mathbb{R}$ s.t. $s_k \to +\infty$. Then, $\exists\, x \in \mathcal{P}(H)$ s.t. $u(s_k + \cdot, \cdot) \longrightarrow x$

proof.
Since $\|\nabla u\| \leq c$, then $(u(s_k, \cdot))_k$ equicontinuous and totally bdd $\Longrightarrow$ by Arzelà-Ascoli $u(s_k + \cdot, \cdot) \to v$ and as above we conclude $v \in \mathcal{M}$. Furthermore

$$\mathbb{E}(v) = \lim_{R \to +\infty} \int_{-R}^{R} \int_{S^1} |\partial_s v|^2 \, ds\, dt$$

$$= \lim_{R \to +\infty} \underbrace{\lim_{k \to +\infty} \int_{-R+s_k}^{R+s_k} |\partial_s u|^2 \, ds\, dt}_{=0 \text{ because } E(u) < +\infty \text{ (argument as above)}}$$

$$= 0$$

$\Rightarrow v$ trivial cylinder over some $x \in \mathcal{P}(H)$, and $u(s_k, t) \to x(t)$ for $k \to +\infty$.

proof (theorem)
$u \in \mathcal{M}$ has asymptotic limit which is unique.
Assume by contradiction $(a_k), (b_k)$ are s.t. $a_k, b_k \to +\infty$ and $u(a_k + \cdot, \cdot) \to x$, $u(b_k + \cdot, \cdot) \longrightarrow y$.
On ΛM define

$$d(x,y) := \sup_{t \in S^1} d(x(t), y(t)).$$

Since the number of periodic orbits is finite, $\exists\, \varepsilon > 0$ s.t. $\forall\, x \neq y \in \mathcal{P}(H)$, $B_\varepsilon(x) \cap B_\varepsilon(y) = \phi$.
Wlog assume $a_k < b_k < a_{k+1}$, $\forall\, k \in \mathbb{N}$.
For k large enough,

$$u(a_k + \cdot, \cdot) \in B_\varepsilon(x), \quad u(b_k + \cdot, \cdot) \in B_\varepsilon(y).$$

$\Rightarrow \exists\ a_k < c_k < b_k$ s.t. $u(c_k + \cdot, \cdot) \notin B_\varepsilon(z)$, $\forall z \in P(H)$ contradicting the lemma above. ∎

Lemma (strong) $\exists\ c > 0$ s.t. $\|\nabla u\| \leq c$, $\forall u \in \mathcal{M}$.

proof.

Assume $R_k := \|\nabla u_k(s_k, t_k)\| \longrightarrow +\infty$.

Arguing as above, get a limit v with

$$\mathbb{E}(v) = \int |\partial_s v|^2\, ds\, dt$$

$$= \lim_{k \to +\infty} \int |\partial_s v_k|^2\, ds\, dt$$

$$= \lim_{k \to +\infty} \int |\partial_s u_k|^2\, ds\, dt$$

$$\leq \sup_{k \in \mathbb{N}} \mathbb{E}(u_k).$$

But $\mathbb{E}(u_k) = \mathbb{A}_H(x_k^-) - \mathbb{A}_H(x_k^+) \leq \max_{x,y \in P(H)} \mathbb{A}_H(x) - \mathbb{A}_H(y)$ uniformly bounded.

$\Longrightarrow \mathbb{E}(v) < +\infty$. Contradiction comes from the next lemma and finishes the proof. ∎

Lemma $v \in C^\infty(\mathbb{C}, M)$ s.t. $\partial_s v + J(v)\,\partial_t v = 0$ and $\mathbb{E}(v) < +\infty \Rightarrow v \equiv \text{const}$.

proof.

Define $u \in C^\infty(\mathbb{R} \times S^1, M)$ by

$$u(s,t) := v\left(e^{2\pi(s+it)}\right)$$

$\Rightarrow$ then $u \in \mathcal{M}_{(H \equiv 0)}$. For $(s_k) \subseteq \mathbb{R}$, $s_k \to +\infty$,

$u(s_k + \cdot, \cdot) \longrightarrow q \in \mathcal{M}$.

Then can take $\xi_k \in C^\infty(\mathbb{D}, M)$ s.t. $\xi_k(e^{2\pi i t}) = u(s_k, t)$

not claiming that these converge !!

and ξ_k in a small nbhd of q. $(\int_{\mathbb{D}} \xi_k^* \omega \to 0)$.
Then, $v \# \xi_k \in {}^*C^0(S^2, M)$ and

$$\Rightarrow E(v) = \int_{\mathbb{C}} v^* \omega = \underbrace{\int_{S^2} (v \# \xi_k)^* \omega}_{\text{because } \omega \text{ symplectically aspherical}} - \int_{\mathbb{D}} \xi_k^* \omega \to 0. \qquad \blacksquare$$

<u>Theorem</u> $\mathcal{M}$ is compact in C^0_{loc}.

<u>Proof.</u>
Take $(u_k) \subseteq \mathcal{M}$. Since $\|\nabla u_k\| \leq c$, the sequence is equicontinuous $\Longrightarrow$ Arzela-Ascoli implies the claim. $\blacksquare$

From now on $x_- \neq x_+ \in \mathcal{P}(H)$.

<u>Def.</u> $(u^1, .., u^m) \in \mathcal{M}^m$ is a <u>broken flow line</u> if $\exists\, x_0, .., x_m \in \mathcal{P}(H)$ s.t. $u^j \in \mathcal{M}(x_{j-1}, x_j)$.

<u>Def.</u> We say that $(\underline{u}_k) \subseteq \mathcal{M}(x_-, x_+)$ <u>converges to a broken flow line</u>, $\underline{u}_k \to (u^1, .., u^m)$, if $\exists\, u_k \in \underline{u}_k$, $(s_k^j) \subseteq \mathbb{R}$, $j = 1, .., m$, s.t. $s_k^j < s_k^{j+1}$ and

$$u_k(s_k^j + \cdot, \cdot) \to u^j(\cdot, \cdot) \quad \text{in } C^\infty_{loc},$$

and $x_- = x_0$, $x_+ = x_m$.

~~* see next page~~

<u>Prop.</u> $(u_k) \subseteq \mathcal{M}(x_-, x_+)$, $(a_k), (b_k) \subseteq \mathbb{R}$ s.t.

$$u_k(a_k + \cdot, \cdot) \to u, \quad u_k(b + \cdot, \cdot) \to v$$

for $u \in \mathcal{M}(x^-, y)$, $v \in \mathcal{M}(x^-, z)$ $\Longrightarrow$ $y = z$ and $\underline{u} = \underline{v}$.

* same philosophy as in Morse homology

Theorem $(\underline{u}_k) \subseteq \underline{\mathcal{M}}(x_-, x_+)$, then $\exists$ subsequence converging to a broken flow line. Moreover, $m \leq CZ(x_-) - CZ(x_+)$.