Matrix measures on the unit circle, moment spaces, orthogonal polynomials and the Geronimus relations

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Abstract

We study the moment space corresponding to matrix measures on the unit circle. Moment points are characterized by non-negative definiteness of block Toeplitz matrices. This characterization is used to derive an explicit representation of orthogonal polynomials with respect to matrix measures on the unit circle and to present a geometric definition of canonical moments. It is demonstrated that these geometrically defined quantities coincide with the Verblunsky coefficients, which appear in the Szegö recursions for the matrix orthogonal polynomials. Finally, we provide an alternative proof of the Geronimus relations which is based on a simple relation between canonical moments of matrix measures on the interval [-1,1] and the Verblunsky coefficients corresponding to matrix measures on the unit circle.

Keyword and Phrases: Matrix measures on the unit circle, orthogonal polynomials, canonical moments, Verblunsky coefficients, Geronimus relations.

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1 Introduction

In recent years considerable interest has been shown in moment problems, orthogonal polynomials, continued fractions and quadrature formulas corresponding to matrix measures on the real line or on the unit circle. Early work dates back to Krein (1949), while more recent results on matrix measures on the real line can be found in the papers of Rodman (1990), Duran (1995, 1996) and Defez et al. (2000) among many others. Additionally, several authors have discussed matrix measures on the unit circle [see Delsarte et al. (1978), Geronimo (1981), Marcellán and Rodriguez (1989), Sinap and Van Assche (1994, 1996), Yakhlef and Marcellán (2001, 2002), Cantero et al. (2003)].

The purpose of the present paper is to investigate some geometric properties of the moment space corresponding to matrix measures on the unit circle. In Section 2 we present a characterization of the moment space in terms of nonnegative definiteness of block Toeplitz matrices. We also provide a geometric definition of canonical moments of matrix measures on the unit circle, which generalizes the scalar case discussed by Dette and Studden (1997) in a nontrivial way. In Section 3 an explicit determinantal representation of orthogonal matrix polynomials with respect to matrix measures on the unit circle is presented, which generalizes the classical representation in the one-dimensional case [see e.g. Geronimus (1962)]. These results are used to identify the canonical moments as Verblunsky coefficients, which appear in the Szegö relations for the corresponding orthonormal and reversed matrix polynomials [see Delsarte et al. (1978), Sinap and Van Assche (1996) or Damanik et al. (2008)]. In particular our results provide a geometric definition of Verblunsky coefficients corresponding to matrix measures on the unit circle. Roughly speaking, the Verblunsky coefficient of order m can be characterized as the distance of the mth trigonometric moment to a center of a matrix disc relative to the diameter of this disc (see Section 3 for more details). Finally, in Section 4 these results are used to present an alternative proof of the Geronimus relations for monic orthogonal polynomials, which describe the relation between the coefficients in the three-term recursive relation of orthogonal polynomials with respect to a matrix measure on a compact interval and the coefficients in the Szegö recursion of an associated matrix measure on the unit circle.

2 The moment space of matrix measure on the unit circle

A matrix measure μ on the unit circle is defined as a $p \times p$ matrix of complex valued Borel measures $\mu = (\mu_{ij})_{i,j=1,\dots,p}$ on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} |\, |z| = 1\}$ such that for each Borel set $A \subset \partial\mathbb{D}$ the matrix $\mu(A)$ is nonnegative definite, i.e. $\mu(A) \geq 0$. Throughout this paper we use the usual parametrization $z = e^{i\theta}$, $\theta \in [-\pi, \pi)$ and the notation $\mu(\theta)$ for the sake of simplicity.

The kth moment of a matrix measure μ on the unit circle is defined by

$$\Gamma_k = \Gamma_k(\mu) = \int_{-\pi}^{\pi} e^{ik\theta} d\mu(\theta) = \alpha_k + i\beta_k \quad k \in \mathbb{Z} \tag{2.1}$$

where $\alpha_k = \alpha_k(\mu) = \int_{-\pi}^{\pi} \cos(k\theta) d\mu(\theta)$, $\beta_k = \beta_k(\mu) \int_{-\pi}^{\pi} \sin(k\theta) d\mu(\theta)$ $(k = 0, 1, \dots)$ are the trigonometric moments and the dependence on the given measure μ is omitted in the notation, whenever it is clear from the context. Throughout this paper let $m \in \mathbb{N}_0$ $\lambda(\mu) = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in (\mathbb{C}^{p\times p})^{2m+1}$ denote the vector of trigonometric moments of order m and define

$$\mathcal{M}_{2m+1} = \{\lambda(\mu) \mid \mu \text{ is a matrix measure on } \partial\mathbb{D}\} \subset (\mathbb{C}^{p\times p})^{2m+1} \tag{2.2}$$

as the $(2m+1)$th moment space of matrix measures on the unit circle. The set $\mathcal{M}_{2m+1}$ and its interior $\mathrm{Int}(\mathcal{M}_{2m+1})$ can be characterized as follows.

Theorem 2.1 $\lambda = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \mathcal{M}_{2m+1}$ *if and only if*

$$\sum_{i=0}^{m} \sum_{j=0}^{m} \mathrm{trace}(B_i B_j^* \Gamma_{i-j}) \geq 0 \quad \forall\ B_0, \dots, B_m \in \mathbb{C}^{p\times p}, \tag{2.3}$$

where the matrices $\Gamma_{-m}, \Gamma_{-m+1}, \dots, \Gamma_m$ *are defined in (2.1).*
$\lambda = (\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \mathrm{Int}(\mathcal{M}_{2m+1})$ *if and only if there is strict inequality in (2.3) except if* $B_0 = \cdots = B_m = 0$.

Proof: We start with a proof of the first part. Assume that $\lambda \in \mathcal{M}_{2m+1}$ and consider matrices $B_0, \dots, B_m \in \mathbb{C}^{p\times p}$. With the notation

$$B(\theta) = \sum_{k=0}^{m} B_k e^{ik\theta} \quad (\theta \in [-\pi, \pi)) \tag{2.4}$$

it follows that the polynomial $P(\theta) = B(\theta)(B(\theta))^*$ is obviously nonnegative definite, i.e.

$$P(\theta) = B(\theta)(B(\theta))^* \geq 0 \quad \forall \theta \in [-\pi, \pi). \tag{2.5}$$

A straightforward calculation shows that the polynomial P can be represented as

$$P(\theta) = D_0 + \sum_{k=1}^{m} D_k \cos(k\theta) + E_k \sin(k\theta), \tag{2.6}$$

where the hermitian $p \times p$ matrices $D_0, \dots, D_m, E_1, \dots, E_m$ are defined by $D_0 = A_0$, and for $k = 1, \dots, m$

$$D_k = A_k + A_{-k}, \quad E_k = i(A_k - A_{-k})$$

and

$$A_k = \sum_{l=0}^{m-k} B_{k+l} B_l^* \quad \text{and} \quad A_{-k} = A_k^*.$$

Because it is easy to see that the moment space $\mathcal{M}_{2m+1}$ is the convex hull of the set

$$\Big\{ (aa^*, \cos(\theta) aa^*, \sin(\theta) aa^*, \dots, \cos(m\theta) aa^*, \sin(m\theta) aa^*) \;\Big|\; a \in \mathbb{C}^p,\ \theta \in [-\pi, \pi) \Big\},$$

a similar argument as in Corollary 2.2 of Dette and Studden (2002) now shows that (2.5) and (2.6) imply

$$\begin{aligned}
0 &\leq \operatorname{trace}(D_0 \alpha_0) + \sum_{k=1}^m \operatorname{trace}(D_k \alpha_k) + \operatorname{trace}(E_k \beta_k) \\
&= \operatorname{trace}\left(\int_{-\pi}^{\pi} d(D_0 \mu(\theta)) + \sum_{k=1}^m \int_{-\pi}^{\pi} \cos(k\theta) d(D_k \mu(\theta)) + \int_{-\pi}^{\pi} \sin(k\theta) d(E_k \mu(\theta)) \right) \\
&= \operatorname{trace}\left(\int_{-\pi}^{\pi} \sum_{k=-m}^m e^{ik\theta} d(A_k \mu(\theta)) \right) \\
&= \operatorname{trace}\left(\int_{-\pi}^{\pi} \sum_{k=0}^m e^{ik\theta} d\left(\sum_{l=0}^{m-k} B_{k+l} B_l^* \mu(\theta) \right) + \int_{-\pi}^{\pi} \sum_{k=1}^m e^{-ik\theta} d\left(\sum_{l=0}^{m-k} B_l B_{k+l}^* \mu(\theta) \right) \right) \\
&= \operatorname{trace}\left(\sum_{k=0}^m \sum_{l=0}^m \int_{-\pi}^{\pi} e^{i(k-l)\theta} d(B_k B_l^* \mu(\theta)) \right) \\
&= \sum_{k=0}^m \sum_{l=0}^m \operatorname{trace}(B_k B_l^* \Gamma_{k-l}),
\end{aligned}$$

which proves (2.3). On the other hand assume that the inequality (2.3) is satisfied for all matrices $B_0, \dots, B_m \in \mathbb{C}^{p \times p}$ and consider a nonnegative definite matrix polynomial

$$P(\theta) = D_0 + \sum_{k=1}^m D_k \cos(k\theta) + E_k \sin(k\theta) \geq 0 \quad \forall \theta \in [-\pi, \pi). \tag{2.7}$$

with hermitian matrices $D_0, \dots, D_m, E_1, \dots, E_m \in \mathbb{C}^{p \times p}$. It now follows from Malyshev (1982) that there exists a matrix polynomial

$$B(\theta) = \sum_{k=0}^m B_k e^{ik\theta}$$

such that $P(\theta) = B(\theta)(B(\theta))^*$, and the same calculation as in the first part of the proof yields

$$\mathrm{trace}(D_0\alpha_0) + \sum_{k=1}^{m} \mathrm{trace}(D_k\alpha_k) + \mathrm{trace}(E_k\beta_k) = \sum_{i=0}^{m}\sum_{j=0}^{m} \mathrm{trace}(B_iB_j^*\Gamma_{i-j}) \geq 0.$$

By similar arguments as in Lemma 2.3 of Dette and Studden (2002) it follows that this is sufficient for $\lambda \in \mathcal{M}_{2m+1}$.

Finally, the second part of the Theorem is shown similarly observing the fact that $(\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m) \in \mathrm{Int}(\mathcal{M}_{2m+1})$ if and only if

$$\mathrm{trace}(D_0\alpha_0) + \sum_{k=1}^{m} \mathrm{trace}(D_k\alpha_k) + \mathrm{trace}(E_k\beta_k) > 0$$

for any nonnegative definite polynomial $P(\theta)$ of the form (2.6) with $P(\theta) \neq 0 \;\forall \theta \in [-\pi, \pi)$. This characterization can be shown by the same arguments as presented in Dette and Studden (2002) who proved a corresponding statement for the moment space of matrix measures on the interval $[0, 1]$. $\Box$.

Throughout this paper let

$$T_m = T_m(\mu) = \begin{pmatrix} \Gamma_0 & \cdots & \Gamma_m \\ \vdots & \ddots & \vdots \\ \Gamma_{-m} & \cdots & \Gamma_0 \end{pmatrix} \in \mathbb{C}^{p(m+1)\times p(m+1)} \tag{2.8}$$

denote the Block Toeplitz matrix, where the blocks $\Gamma_i = \Gamma_i(\mu)$ $(i = -m, \ldots, m)$ are the moments of a matrix measure μ on the unit circle defined by (2.1) (note that T_m is hermitian). The following characterization of the moment space $\mathcal{M}_{2m+1}$ by nonnegative definiteness of Toeplitz matrices is now easily obtained.

Corollary 2.2 *Assume that $\lambda = (\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m) \in (\mathbb{R}^{p\times p})^{2m+1}$ and that T_m is defined by (2.8) with $\Gamma_k = \alpha_k + i\beta_k$ and $\Gamma_{-k} = \alpha_k - i\beta_k$.*

(a) $\lambda \in \mathcal{M}_{2m+1}$ if and only if $T_m \geq 0$.

(b) $\lambda \in Int(\mathcal{M}_{2m+1})$ if and only if $T_m > 0$.

Proof: We only proof part (a); part (b) is shown by similar arguments. First assume that $\lambda \in \mathcal{M}_{2m+1}$, then we obtain from Theorem 2.1 for all matrices $B_0, \ldots, B_m \in \mathbb{C}^{p\times p}$

$$\sum_{i=0}^{m}\sum_{j=0}^{m} \mathrm{trace}(B_iB_j^*\Gamma_{j-i}) \geq 0.$$

Consequently, if $a_0, \dots, a_m \in \mathbb{C}^p$, $a = (a_0^T, \dots, a_m^T)^T \in \mathbb{C}^{p(m+1)}$ we put $B_i = (a_i, 0, \dots, 0) \in \mathbb{C}^{p\times p}$ $(i = 0, \dots, m)$ and it follows

$$a^* T_m a \;=\; \operatorname{trace}(aa^* T_m) = \sum_{i=0}^m \sum_{j=0}^m \operatorname{trace}(a_i a_j^* \Gamma_{j-i}) = \sum_{i=0}^m \sum_{j=0}^m \operatorname{trace}(B_i B_j^* \Gamma_{j-i}) \geq 0,$$

which shows that the matrix T_m is nonnegative definite. To prove the converse assume that $T_m \geq 0$, i.e.

$$0 \leq a^* T_m a = \sum_{i=0}^m \sum_{j=0}^m \operatorname{trace}(a_i a_j^* \Gamma_{j-i}). \tag{2.9}$$

for all $a = (a_0^T, \dots, a_m^T)^T \in \mathbb{C}^{p(m+1)}$. If $B_0, \dots, B_m \in \mathbb{C}^{p\times p}$, and $a_j^{(i)}$ denotes the ith column of the matrix B_j $(j = 0, \dots, m,\ i = 1, \dots, p)$, then

$$B_j B_k^* = \sum_{i=1}^p a_j^{(i)} \left(a_k^{(i)}\right)^*$$

and we obtain from (2.9)

$$\sum_{i=0}^m \sum_{j=0}^m \operatorname{trace}(B_i B_j^* \Gamma_{j-i}) = \sum_{k=1}^p \sum_{i=0}^m \sum_{j=0}^m \operatorname{trace}\left(a_i^{(k)} \left(a_j^{(k)}\right)^* \Gamma_{j-i}\right) \geq 0.$$

By Theorem 2.1 it follows that $\lambda \in \mathcal{M}_{2m+1}$, which completes the proof of the Corollary. $\square$

With the aid of Theorem 2.1 and Corollary 2.2 we are now able to define geometrically canonical moments for matrix measures on the unit circle. It turns out that these geometrically defined quantities are exactly the Verblunsky coefficients of matrix measures on the unit circle as introduced by Damanik et al. (2008) (see Section 3 where we prove this identity). For this purpose let W denote a $p \times p$ matrix and define

$$A = A(W) = \begin{pmatrix} \Gamma_0 & \Gamma_1 & \cdots & \Gamma_m & W \\ \Gamma_{-1} & \Gamma_0 & \cdots & \Gamma_{m-1} & \Gamma_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Gamma_{-m} & \Gamma_{-m+1} & \cdots & \Gamma_0 & \Gamma_1 \\ W^* & \Gamma_{-m} & \cdots & \Gamma_{-1} & \Gamma_0 \end{pmatrix} \in \mathbb{C}^{p(m+2)\times p(m+2)}. \tag{2.10}$$

Let $\Gamma^{(m)} = (\Gamma_{-m}, \Gamma_{-m+1}, \dots, \Gamma_{m-1}, \Gamma_m) \in (\mathbb{C}^{p\times p})^{2m+1}$ denote a vector of moments of a matrix measure on the unit circle, that is $(\alpha_0, \alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \in \mathcal{M}_{2m+1}$, where $\Gamma_k = \alpha_k + i\beta_k$. Define $\mathcal{P}_{\Gamma^{(m)}}$ as the set of all matrix measures μ on the unit circle with moments of order m given by $\Gamma^{(m)}$, that is $\Gamma_j = \int_{-\pi}^{\pi} e^{ik\theta} d\mu(\theta)$ $(j = -m, \ldots, m)$. By Corollary 2.2 it follows that the matrix W is the $(m+2)$th moment of a matrix measure $\mu \in \mathcal{P}_{\Gamma(m)}$ if and only if $A(W) \geq 0$. We assume without loss of generality that $(\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m) \in \mathrm{Int}(\mathcal{M}_{2m+1})$ which is equivalent to $T_m > 0$ by Corollary 2.2 . From Theorem 1 in Fritzsche and Kirstein (1987) it follows that

$$A(W) \geq 0$$

if and only if there exists a $p \times p$ matrix U with $UU^* \leq I_p$ such that the matrix W can be represented as

$$W = (\Gamma_1 \ldots \Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_{-m} \ldots \Gamma_{-1})^* + L_m^{1/2} U R_m^{1/2}, \tag{2.11}$$

where the matrices L_m and R_m are defined by

$$L_m = \Gamma_0 - (\Gamma_1 \ldots \Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_1 \ldots \Gamma_m)^*, \tag{2.12}$$

$$R_m = \Gamma_0 - (\Gamma_{-m} \ldots \Gamma_{-1})\, T_{m-1}^{-1}\, (\Gamma_{-m} \ldots \Gamma_{-1})^*, \tag{2.13}$$

respectively. Note that the matrices L_m and R_m are Schur complements of the positive definite matrix T_m and as a consequence are also positive definite [see Horn and Johnsohn (1985)]. This means that that the matrix W is the $(m+2)$th moment of the matrix measure $\mu \in \mathcal{P}_{\Gamma(m)}$, if and only if it is an element of the "ball"

$$K_m := \left\{ W \in \mathbb{C}^{p \times p} | L_m^{-1/2} (W - M_m) R_m^{-1/2} = U, UU^* \leq I_p \right\}, \tag{2.14}$$

where the "center" of the ball is given by the matrix

$$M_m = (\Gamma_1 \ldots \Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_{-m} \ldots \Gamma_{-1})^* . \tag{2.15}$$

We are now in a position to define the canonical moments of a matrix measure on the unit circle (or Verblunsky coefficients as shown in Section 3).

Definition 2.3 *Let μ denote a matrix measure on the unit circle with moments $\Gamma_k = \alpha_k + i\beta_k$ $(k \geq 0)$, $\lambda_{2m+1}(\mu) = (\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m) \in (\mathbb{R}^{p \times p})^{m+1}$ $(m \geq 0)$ and define*

$$N(\mu) = \min \left\{ m \in \mathbb{N} \mid \lambda_{2m+1}(\mu) \in \partial \mathcal{M}_{2m+1} \right\}, \tag{2.16}$$

as the minimum number $m \in \mathbb{N}$ such that λ_{2m+1} is a boundary point of the moment space $\mathcal{M}_{2m+1}$ (if $\lambda_{2m+1} \in \mathrm{Int}(\mathcal{M}_{2m+1})$ for all $m \in \mathbb{N}$ we put $N(\mu) = \infty$). For each $m = 0, \ldots, N(\mu) - 1$ the *quantity*

$$A_{m+1} = A_{m+1}(\mu) = L_m^{-1/2}\left(\Gamma_{m+1} - M_m\right) R_m^{-1/2} \tag{2.17}$$
$$= \left[\Gamma_0 - (\Gamma_1, \ldots, \Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_1, \ldots, \Gamma_m)^*\right]^{-1/2}$$
$$\times \left(\Gamma_{m+1} - (\Gamma_1, \ldots, \Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_{-m}, \ldots, \Gamma_{-1})^*\right)$$
$$\times \left[\Gamma_0 - (\Gamma_{-m}, \ldots, \Gamma_{-1})\, T_{m-1}^{-1}\, (\Gamma_{-m}, \ldots, \Gamma_{-1})^*\right]^{-1/2}$$

is called the $(m+1)$*th canonical moment of the matrix measure* μ*.*

Definition 2.3 is a generalization of the definition of canonical moments of scalar measures on the unit circle in Dette and Studden (1997). In general the explicit representation of the canonical moments in terms of the moments $\Gamma_0, \Gamma_1, \ldots$ is very difficult. For example if $m = 0$ we have

$$A_1 = \Gamma_0^{-1/2}\Gamma_1\Gamma_0^{-1/2} \tag{2.18}$$

and in the case $m = 1$ we obtain from Definition 2.3

$$A_2 = \left(\Gamma_0 - \Gamma_1\Gamma_0^{-1}\Gamma_{-1}\right)^{-1/2}\left(\Gamma_2 - \Gamma_1\Gamma_0^{-1}\Gamma_1\right)\left(\Gamma_0 - \Gamma_{-1}\Gamma_0^{-1}\Gamma_1\right)^{-1/2} \tag{2.19}$$

In the following section we will demonstrate that the quantities defined by Definition 2.3 are the well known Verblunsky coefficients, which are usually obtained from the recursive relations of the orthonormal polynomials with respect to matrix measures on the unit circle [see for example Delsarte et al. (1978) where these matrices do not have any special name, Sinap and Van Assche (1996) where they are called reflection coefficients or Damanik et al. (2008)]. For this purpose we use an explicit determinant representation of the matrix orthogonal polynomials, which is of interest by itself and given in the following section.

3 Orthogonal matrix polynomials

A $p \times p$ matrix polynomial is a $p \times p$ matrix with polynomial entries. It is of degree n if all the polynomial entries are of degree less than or equal to n and is usually written in the form

$$P(z) = \sum_{i=0}^{n} A_i z^i. \tag{3.1}$$

with coefficients $A_i \in \mathbb{C}^{p\times p}$ and $z \in \mathbb{C}$. Recall that for matrix polynomials P and Q the right and left inner product are defined by

$$\langle P, Q\rangle_R = \int_{-\pi}^{\pi} P(e^{i\theta})^* d\mu(\theta) Q(e^{i\theta}), \tag{3.2}$$

$$\langle P, Q\rangle_L = \int_{-\pi}^{\pi} P(e^{i\theta}) d\mu(\theta) Q(e^{i\theta})^*, \tag{3.3}$$

respectively [see for example Sinap and Van Assche (1996)]. The matrix polynomials P and Q are called orthogonal with respect to the right inner product $\langle \cdot, \cdot \rangle_R$ if

$$\langle P, Q \rangle_R = 0 \tag{3.4}$$

and orthogonality with respect to the left inner product $\langle \cdot, \cdot \rangle_L$ is defined analogously. The matrix polynomials $P_0(z), P_1(z), P_2(z), \ldots$ are called orthonormal with respect to the right inner product if for each $m \in \mathbb{N}_0$ $P_m(z)$ is of degree m, $P_m(z)$ and $P_{m'}(z)$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_R$ whenever $m \neq m'$ and

$$\langle P_m, P_m \rangle_R = I_p, \tag{3.5}$$

where I_p denotes the $p \times p$ identity matrix. Orthonormal polynomials with respect to the left inner product $\langle \cdot, \cdot \rangle_L$ are defined analogously. Orthonormal polynomials with respect to the inner products $\langle \cdot, \cdot \rangle_R$ and $\langle \cdot, \cdot \rangle_L$ are determined uniquely up to multiplication by unitary matrices. In the following discussion we will derive an explicit representation of these polynomials in terms of the moments of matrix measure μ. A representation very similar to the well known determinant representation in the scalar case [see for example Geronimus (1946)] was given by Miranian (2005, 2009) in the matrix case on the real line and on the circle. Here we develop another explicit representation using determinants.

For this purpose consider a matrix measure μ on the unit circle with moments $\Gamma_{-m}, \ldots, \Gamma_m$ and recall the definition of the corresponding block Toeplitz matrix T_m in (2.8). We define for $m \in \mathbb{N}$ matrix polynomials by

$$\Psi_m^R(z) = \left(T_{ij}^R(z)\right)_{i,j=1,\ldots,p}, \tag{3.6}$$

$$\Psi_m^L(z) = \left(T_{ij}^L(z)\right)_{i,j=1,\ldots,p}, \tag{3.7}$$

where the elements $T_{ij}^R(z)$ and $T_{ij}^L(z)$ in these matrices are given by the determinants

$$T_{ij}^R(z) = \begin{vmatrix} \Gamma_0 & \Gamma_1 & \ldots & \Gamma_m \\ \Gamma_{-1} & \Gamma_0 & \ldots & \Gamma_{m-1} \\ \vdots & \vdots & & \vdots \\ \Gamma_{-m+1} & \Gamma_{-m+2} & \ldots & \Gamma_1 \\ \Gamma_{-m}^{ij}(z) & \Gamma_{-m+1}^{ij}(z) & \ldots & \Gamma_0^{ij}(z) \end{vmatrix}; \quad i, j = 1, \ldots, p \tag{3.8}$$

and

$$T_{ij}^L(z) = \begin{vmatrix} \tilde{\Gamma}_0^{ij}(z) & \Gamma_1 & \ldots & \Gamma_m \\ \tilde{\Gamma}_{-1}^{ij}(z) & \Gamma_0 & \ldots & \Gamma_{m-1} \\ \vdots & \vdots & & \vdots \\ \tilde{\Gamma}_{-m}^{ij}(z) & \Gamma_{-m+1} & \ldots & \Gamma_0 \end{vmatrix}; \quad i, j = 1, \ldots, p, \tag{3.9}$$

respectively, and the matrices Γ^{ij}_{-m+k} (and $\tilde{\Gamma}^{ij}_{-m+k}$) are obtained replacing the jth row (and the ith column) in the matrix Γ_{-m+k} by $e_i^T z^k$ (and $e_j z^{m-k}$). The following result shows that these polynomials are orthogonal with respect to the given matrix measure μ.

Theorem 3.1 *For a given matrix measure μ on the unit circle let $\Psi^R_m(z)$ and $\Psi^L_m(z)$ $(m \in \mathbb{N})$ denote the matrix polynomials defined by (3.6) and (3.7), respectively, then we have*

$$\langle z^k I_p, \Psi^R_m \rangle_R = 0 \ , \quad (k = 0, \ldots, m-1) \ ; \quad \langle z^m I_p, \Psi^R_m \rangle_R = |T_m| I_p \tag{3.10}$$

$$\langle \Psi^L_m, z^k I_p \rangle_L = 0 \ , \quad (k = 0, \ldots, m-1) \ ; \quad \langle \Psi^L_m, z^m I_p \rangle_L = |T_m| I_p .$$

Proof: We will only give a proof for the polynomials $\Psi^R_m(z)$, the remaining part of Theorem 3.1 is shown similarly. The element B^R_{ij} in the position (i, j) of the matrix

$$B^R := \langle z^k I, \Psi^R_m \rangle_R = \int_{-\pi}^{\pi} e^{-ik\theta} d\mu(\theta) \left(T^R_{ij}(e^{i\theta}) \right)_{i,j=1,\ldots,p} \qquad (k = 0, \ldots, m),$$

is given by

$$B^R_{ij} = \sum_{l=1}^{p} \int_{-\pi}^{\pi} e^{-ik\theta} T^R_{lj}(e^{i\theta}) d\mu_{il}(\theta). \tag{3.11}$$

An expansion of the determinant $T^R_{lj}(e^{i\theta})$ with respect to the $(mp + j)$th row yields

$$T^R_{lj}(e^{i\theta}) = \sum_{n=0}^{m} (-1)^{(m+n)p+j+l} e^{in\theta} \left| T_m^{(mp+j),(np+l)} \right| , \tag{3.12}$$

where the matrix $T_m^{(mp+j),(np+l)}$ is obtained from T_m by deleting the $(mp+j)$th row and $(np+l)$th column. If $\gamma_{n,ij} = \int_{-\pi}^{\pi} e^{in\theta} d\mu_{ij}$ denotes the element of the matrix Γ_n in the position (i, j), where $n \in \{-m, \ldots, m\}$, it follows that

$$B^R_{ij} = \sum_{n=0}^{m} \sum_{l=1}^{p} (-1)^{(m+n)p+j+l} \left| T_m^{(mp+j),(np+l)} \right| \gamma_{n-k,il}. \tag{3.13}$$

Now it is easy to see that the right hand side of (3.13) is the determinant of the matrix T_m, where the $(mp + j)$th row has been replaced by the vector

$$(\gamma_{-k,i1}, \ldots, \gamma_{-k,ip}, \gamma_{-k+1,i1}, \ldots, \gamma_{-k+1,ip}, \ldots, \gamma_{m-1-k,i1}, \ldots, \gamma_{m-1-k,ip}, \gamma_{m-k,i1}, \ldots, \gamma_{m-k,ip})$$

Consequently, if $k \in \{0, \ldots, m-1\}$ the $(mp + j)$th and $(kp + i)$th row in this matrix coincide and we have $B^R_{ij} = 0$, which proves the first identity in (3.10).

For a proof of the second identity we note that in the case $k = m$ and $i \neq j$ the same argument yields $B_{ij} = 0$. If $k = m$ and $i = j$ it follows that B_{ij} is exactly the determinant of the matrix T_m, which completes the proof of the first assertion of Theorem 3.1. $\square$

In the following discussion we derive several consequences of the representations (3.6) and (3.7), which will be useful to identify the canonical moments as Verblunsky coefficients. In particular we determine the corresponding leading coefficients and identify the orthonormal polynomials with respect to the measure μ. For this purpose recall that a matrix polynomial of the form (3.1) is called monic, if the coefficient of the leading term is the identity matrix, that is $A_n = I_p$.

Corollary 3.2 *For a given matrix measure μ on the unit circle let $\Psi_m^R(z)$ and $\Psi_m^L(z)$ be defined by (3.6) and (3.7) and consider for $m \leq N(\mu)$ the matrix polynomials*

$$\Phi_m^R(z) = \Psi_m^R(z)|T_m|^{-1}R_m, \tag{3.14}$$

$$\Phi_m^L(z) = |T_m|^{-1}L_m\Psi_m^L(z), \tag{3.15}$$

where the matrices R_m and L_m are defined by (2.13) *and* (2.12), *respectively. The polynomials $\Phi_m^R(z)$ (and $\Phi_m^L(z)$) are monic orthogonal matrix polynomials with respect to the right (and left) inner product $\langle\cdot,\cdot\rangle_R$ (and $\langle\cdot,\cdot\rangle_L$).*
Similarly, define for $m \leq N(\mu)$

$$\phi_m^R(z) = \Psi_m^R(z)|T_m|^{-1}R_m^{1/2}, \tag{3.16}$$

$$\phi_m^L(z) = |T_m|^{-1}L_m^{1/2}\Psi_m^L(z), \tag{3.17}$$

then the matrix polynomial $\phi_m^R(z)$ (and $\phi_m^L(z)$) are orthonormal polynomials with respect to the right (and left) inner product $\langle\cdot,\cdot\rangle_R$ (and $\langle\cdot,\cdot\rangle_L$). The leading coefficients of $\phi_m^R(z)$ and $\phi_m^L(z)$ are given by $R_m^{-1/2}$ and $L_m^{-1/2}$, respectively.

Proof: In the first part we will prove that the leading coefficients of the polynomials $\Psi_m^R(z)$ and $\Psi_m^L(z)$ defined by (3.6) and (3.7) are given by

$$L_m^R = |T_m|R_m^{-1}, \tag{3.18}$$

$$L_m^L = |T_m|L_m^{-1}, \tag{3.19}$$

respectively. With these representations we obtain from Theorem 3.1

$$\langle\Psi_m^R,\Psi_m^R\rangle_R = |T_m|(L_m^R)^* \ ; \quad \langle\Psi_m^L,\Psi_m^L\rangle_L = |T_m|(L_m^L)^*,$$

and the assertion of the Corollary follows by a straightforward calculation.

In order to prove (3.18) and (3.19) we restrict ourselves to the first case; the second case is shown similarly. Observing the definition of the determinants $T_{ij}^R(z)$ in (3.8) we obtain for the entry in the position (i,j) of the leading coefficient of the matrix polynomial $\Psi_m^R(z)$

$$\left(L_m^R\right)_{ij} = (-1)^{2mp+i+j} |T_m^{(mp+j),(mp+i)}|,$$

where we have used an expansion of the determinant with respect to the $(mp+j)$ row and the matrix $T_m^{(mp+j),(mp+i)}$ is obtained from T_m by deleting the $(mp+j)$th row and $(mp+i)$th column. This means that $\left(L_m^R\right)_{ij}$ is the entry in the position $(mp+i, mp+j)$ of the adjoint of the matrix T_m $(i,j = 1,\ldots,p)$, and consequently $L_m^R/|T_m|$ is the $p \times p$ block in the position $(m+1, m+1)$ of the matrix T_m^{-1}, which is given by

$$\left(\Gamma_0 - (\Gamma_{-m}\ldots\Gamma_{-1})T_{m-1}^{-1}(\Gamma_{-m}\ldots\Gamma_{-1})^*\right)^{-1} = R_m^{-1}$$

[see e.g. Horn and Johnsohn (1985)]. This proves the assertion (3.18) and completes the proof of the Corollary. $\Box$.

We are now in a position to identify the canonical moments introduced in Definition 2.3 as Verblunsky coefficients which are defined as coefficients in the Szegö relation of the matrix orthonormal polynomials $\phi_n^L(z)$ and $\phi_n^R(z)$. For this purpose we introduce for a given matrix polynomial P_n of degree n the corresponding reversed polynomial

$$\tilde{P}_n(z) = z^n P_n\left(\frac{1}{\overline{z}}\right)^*,$$

where $\overline{z}$ denotes the complex conjugation of $z \in \mathbb{C}$. Obviously we have for any $p \times p$ matrix A

$$\widetilde{AP}_n(z) = \tilde{P}_n(z)A^*.$$

In the following discussion let $\kappa_m^R = R_m^{-1/2}$ and $\kappa_m^L = L_m^{-1/2}$ $(m = 1,\ldots,N(\mu)-1)$ denote the leading coefficients of the orthonormal matrix polynomials $\phi_m^R(z)$ and $\phi_m^L(z)$ with respect to the right and left inner product induced by the matrix measure μ and define the matrices

$$\rho_m^R = \left(\kappa_{m+1}^R\right)^{-1}\kappa_m^R \quad \text{and} \quad \rho_m^L = \kappa_m^L\left(\kappa_{m+1}^L\right)^{-1} \quad (m = 1,\ldots,N(\mu)-1). \tag{3.20}$$

Then it follows from Damanik et al. (2008) that there exist $p \times p$ matrices H_m such that the orthonormal matrix polynomial with respect to the measure μ on the unit circle satisfy the Szegö recursions

$$z\phi_m^L(z) - \rho_m^L\phi_{m+1}^L(z) = H_{m+1}\tilde{\phi}_m^R(z), \tag{3.21}$$

$$z\phi_m^R(z) - \phi_{m+1}^R(z)\rho_m^R = \tilde{\phi}_m^L(z)H_{m+1}. \tag{3.22}$$

The matrices H_m are uniquely determined and called Verblunsky or reflection coefficients, because they were introduced for the scalar case in two seminal papers by Verblunsky (1935, 1936). The final result of this section shows that the Verblunsky coefficients coincide with the canonical moments introduced in Definition 2.3.

Theorem 3.3 *Let μ denote a matrix measure on the unit circle and assume that $0 \leq m < N(\mu)$. If A_{m+1} is the $(m+1)$th canonical moment of μ defined in Definition 2.3 and H_{m+1} is the $(m+1)$th Verblunsky coefficient defined by the Szegö recursions* (3.21) *and* (3.22)*, then*

$$A_{m+1} = H_{m+1}. \tag{3.23}$$

Proof: Integrating the recursion (3.22) we obtain

$$\langle I_p, z\phi_m^R - \phi_{m+1}^R \rho_m^R \rangle_R \;=\; \langle I_p, \tilde{\phi}_m^L H_{m+1} \rangle_R$$

and

$$\langle I_p, z\Psi_m^R \rangle_R |T_m|^{-1} R_m^{1/2} \;=\; \langle I_p, \tilde{\Psi}_m^L \rangle_R |T_m|^{-1} L_m^{1/2} H_{m+1},$$

where we have used the orthogonality of the matrix polynomials $\Psi_{m+1}^R(z)$ stated in Theorem 3.1 and the representations of the orthonormal polynomials ϕ_m^R and ϕ_m^L in Corollary 3.2. Observing Theorem 3.1 and the identity

$$\langle I_p, \tilde{\Psi}_m^L \rangle_R \;=\; \int_{-\pi}^{\pi} d\mu(\theta) e^{im\theta} \left(\Psi_m^L(e^{i\theta}) \right)^* = \langle z^m I_p, \Psi_m^L \rangle_L = |T_m| I_p \tag{3.24}$$

yields

$$\begin{aligned} H_{m+1} \;&=\; L_m^{-1/2} \langle I_p, \tilde{\Psi}_m^L \rangle_R^{-1} \langle I_p, z\Psi_m^R \rangle_R R_m^{1/2} \\ &=\; L_m^{-1/2} |T_m|^{-1} \langle I_p, z\Psi_m^R \rangle_R R_m^{1/2}. \end{aligned} \tag{3.25}$$

The matrix polynomial $\Psi_m^R(z)$ has the representation

$$\Psi_m^R(z) = L_m^R z^m + \sum_{k=0}^{m-1} K_k^R z^k,$$

where $K_0^R, \ldots, K_{m-1}^R$ denote $p \times p$ matrices and the leading coefficient L_m^R is given by (3.18). Integrating with respect to $d\mu(\theta)$ gives

$$\langle I_p, z\Psi_m^R \rangle_R = \langle I_p, z^{m+1} + \sum_{k=0}^{m-1} K_k^R \left(L_m^R \right)^{-1} z^{k+1} \rangle_R |T_m| R_m^{-1},$$

and it follows from (3.25) that

$$H_{m+1} = L_m^{-1/2} \langle I_p, z^{m+1} + \sum_{k=0}^{m-1} K_k^R \left(L_m^R\right)^{-1} z^{k+1} \rangle_R R_m^{-1/2}. \tag{3.26}$$

Observing the definition of the canonical moments in (2.17) and the definition of the center (2.15) the assertion of the Theorem follows if the identity

$$\langle I_p, z^{m+1} + \sum_{k=0}^{m-1} K_k^R \left(L_m^R\right)^{-1} z^{k+1} \rangle_R \;=\; \Gamma_{m+1} - \left(\Gamma_1 \dots \Gamma_m\right) T_{m-1}^{-1} \left(\Gamma_{-m} \dots \Gamma_{-1}\right)^* . \tag{3.27}$$

can be established. For this purpose we determine the matrices K_k^R $(k = 0, \dots, m-1)$ explicitly using the representation of the orthogonal matrix polynomials $\Psi_m^R(z)$ in (3.6). From this definition it follows that the element in the position (i, j) of the matrix K_k^R is obtained by deleting the $(mp + j)$th row and the $(kp + i)$th column in the determinant $T_{ij}^R(z)$ defined by (3.8), that is

$$\left(K_k^R\right)_{ij} = (-1)^{(m+k)p+i+j} |T_m^{(mp+j),(kp+i)}|.$$

Here again $T_m^{(mp+j),(kp+i)}$ denotes the matrix obtained T_m by deleting the $(mp + j)$th row and $(kp + i)$th column, which coincides with the entry in the position $(kp + i, mp + j)$ of the adjoint of the matrix T_m. Consequently, it follows that

$$\left(K_k^R\right)_{ij} = |T_m| (T_m^{-1})_{kp+i,mp+j},$$

and the "vector"

$$\frac{1}{|T_m|} \begin{pmatrix} K_0^R \\ \vdots \\ K_{m-1}^R \end{pmatrix} \in (\mathbb{C}^{p \times p})^m$$

coincides with the right upper block of size $mp \times p$ of the matrix T_m^{-1}. By standard result in linear algebra this block is given by

$$-T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^* R_m^{-1},$$

which yields

$$\begin{aligned} \langle I_p, \sum_{k=0}^{m-1} K_k^R z^{k+1} \rangle_R &= \sum_{k=0}^{m-1} \Gamma_{k+1} K_k^R \\ &= (\Gamma_1 \dots \Gamma_m) \left((K_0^R)^* \dots (K_{m-1}^R)^*\right)^* \\ &= -|T_m| (\Gamma_1 \dots \Gamma_m) T_{m-1}^{-1} (\Gamma_{-m} \dots \Gamma_{-1})^* R_m^{-1}. \end{aligned}$$

Combining this result with the identity $\left(L_m^R\right)^{-1} = R_m |T_m|^{-1}$ finally gives (3.27), which completes the proof Theorem 3.3. $\square$

4 Geronimus relations for monic polynomials

In this section we present a new proof of the Geronimus relations, which provide a representation of the canonical moments (or Verblunsky coefficients) of a symmetric matrix measure on the unit circle in terms of the coefficients in the recurrence relations of a sequence of orthogonal polynomials with respect to an associated matrix measure on the interval $[-1, 1]$. There exists several alternative proofs of these relations in the literature [see Yakhlef and Marcellán (2001) and Damanik et al. (2008)], but the one presented here explicitly uses the theory of canonical moments of matrix measures as introduced in Dette and Studden (2002). As a by-product we derive several interesting properties of the Verblunsky coefficients.

To be precise let μ_C denote a symmetric (with respect to the point 0) matrix measure on the unit disc (i.e. μ_C is invariant with respect to the transformation $\theta \mapsto -\theta$). We associate to μ_C a corresponding matrix measure, say μ_I, on the the interval $[-1, 1]$, which is defined by the property

$$\int_{-1}^{1} f(x) d\mu_I(x) = \int_{-\pi}^{\pi} f(\cos(\theta)) d\mu_C(\theta) \tag{4.1}$$

for all integrable functions f defined on the interval $[-1, 1]$. Note that the relation $Sz : d\mu_C \mapsto d\mu_I$ is called Szegö mapping in the literature, where the matrix measure μ_I is usually defined on the interval $[-2, 2]$. We will work with the interval $[-1, 1]$ in this section, because this interval is also used in the classical papers of Szegö (1922) and Geronimus (1946) and in the monograph on canonical moments by Dette and Studden (1997).

Note that the inverse of the Szegö mapping (4.1) is characterized by the property

$$\int_{-\pi}^{\pi} g(\theta) d\mu_C(\theta) = \int_{-1}^{1} g(\arccos(x)) d\mu_I(x), \tag{4.2}$$

where g denotes any integrable function on $\partial\mathbb{D}$ with $g(\theta) = g(-\theta)$ for all $\theta \in [-\pi, \pi)$. For a proof of the Geronimus relations we need several preparations. Our first results shows that the canonical moments (or Verblunsky coefficients) of a symmetric matrix measure on the unit circle are hermitian matrices. The result was also proved by Damanik et al. (2008). We provide here an alternative proof, because several steps in the proof are used later.

Lemma 4.1 *For any symmetric matrix measure μ_C on the unit circle the corresponding canonical moments A_m are hermitian .*

Proof: By the symmetry of the matrix measure μ_C we have $\Gamma_k = \int_{-\pi}^{\pi} e^{ik\theta} d\mu_C(\theta) = \int_{-\pi}^{\pi} e^{-ik\theta} d\mu_C(\theta) = \Gamma_{-k}$ which yields $\Gamma_k = \int_{-\pi}^{\pi} \cos(k\theta) d\mu_C(\theta)$. Consequently, the block Toeplitz matrix associated with μ_C is given by

$$T_m = \begin{pmatrix} \Gamma_0 & \dots & \Gamma_m \\ \vdots & \ddots & \vdots \\ \Gamma_m & \dots & \Gamma_0 \end{pmatrix}. \tag{4.3}$$

We denote by $[A]_{(k,l)}$ the $p \times p$ block in the position (k,l) of the $mp \times mp-$ block matrix A. We will show at the end of this proof that

$$\left[T_{m-1}^{-1}\right]_{(k,l)} = \left[T_{m-1}^{-1}\right]_{(m+1-k,m+1-l)}. \tag{4.4}$$

From this identity and the property $\Gamma_k = \Gamma_k^*$ we obtain

$$\begin{aligned} (\Gamma_1,\dots,\Gamma_m)T_{m-1}^{-1}(\Gamma_m,\dots,\Gamma_1)^* &= \sum_{k,l=1}^m \Gamma_k \left[T_{m-1}^{-1}\right]_{(k,l)} \Gamma_{m+1-l} = \sum_{k,l=1}^m \Gamma_{m-k+1} \left[T_{m-1}^{-1}\right]_{(m-k+1,m-l+1)} \Gamma_l \\ &= \sum_{k,l=1}^m \Gamma_{m-k+1} \left[T_{m-1}^{-1}\right]_{(k,l)} \Gamma_l = (\Gamma_m,\dots,\Gamma_1)T_{m-1}^{-1}(\Gamma_1,\dots,\Gamma_m)^*, \end{aligned}$$

and by similar arguments

$$(\Gamma_1,\dots,\Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_1,\dots,\Gamma_m)^* = (\Gamma_m,\dots,\Gamma_1)\, T_{m-1}^{-1}\, (\Gamma_m,\dots,\Gamma_1)^*. \tag{4.5}$$

Observing the definition of the canonical moments A_{m+1} it now follows that

$$\begin{aligned} A_{m+1}^* &= \left[\Gamma_0 - (\Gamma_m,\dots,\Gamma_1)\, T_{m-1}^{-1}\, (\Gamma_m,\dots,\Gamma_1)^*\right]^{-1/2} \left(\Gamma_{m+1} - (\Gamma_1,\dots,\Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_m,\dots,\Gamma_1)^*\right)^* \\ &\quad \times \left[\Gamma_0 - (\Gamma_1,\dots,\Gamma_m)\, T_{m-1}^{-1}\, (\Gamma_1,\dots,\Gamma_m)^*\right]^{-1/2} \\ &= A_{m+1} \end{aligned}$$

which proves the remaining assertion of Lemma 4.1.

Proof of the identity (4.4). The element in the position (i,j) of the matrix $\left[T_{m-1}^{-1}\right]_{(k,l)}$ and $\left[T_{m-1}^{-1}\right]_{(m+1-k,m+1-l)}$ are given by

$$|T_{m-1}|^{-1}(-1)^{(l+k)p+i+j}\left|T_{m-1}^{((l-1)p+j),((k-1)p+i)}\right|$$

and

$$|T_{m-1}|^{-1}(-1)^{(2m-l-k)p+i+j}\left|T_{m-1}^{((m-l)p+j),((m-k)p+i)}\right|,$$

respectively, where $T_{m-1}^{((m-l)p+j),((m-k)p+i)}$ denotes the matrix obtained from T_{m-1} by deleting the $(m-l)p+j$ row and $(m-k)p+i$ column (note that both expressions have the same sign). In the following discussion we denote by $A^{(\cdot),(i)}$ and $A^{(j),(\cdot)}$ the matrix obtained from A by deleting the ith column or the jth row, respectively. Then interchanging first columns and then rows yields

$$
\begin{aligned}
\left|T_{m-1}^{((l-1)p+j),((k-1)p+i)}\right| &= \begin{vmatrix}
\Gamma_0 & \dots & \Gamma_{k-2} & \Gamma_{k-1}^{(\cdot),(i)} & \Gamma_k & \dots & \Gamma_{m-1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\Gamma_{l-2} & \dots & \Gamma_{|l-k|} & \Gamma_{|l-k-1|}^{(\cdot),(i)} & \Gamma_{|l-k-2|} & \dots & \Gamma_{m-l+1} \\
\Gamma_{l-1}^{(j),(\cdot)} & \dots & \Gamma_{|l-k+1|}^{(j),(\cdot)} & \Gamma_{|l-k|}^{(j),(i)} & \Gamma_{|l-k-1|}^{(j),(\cdot)} & \dots & \Gamma_{m-l}^{(j),(\cdot)} \\
\Gamma_l & \dots & \Gamma_{|l-k+2|} & \Gamma_{|l-k+1|}^{(\cdot),(i)} & \Gamma_{|l-k|} & \dots & \Gamma_{m-l-1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\Gamma_{m-1} & \dots & \Gamma_{m-k+1} & \Gamma_{m-k}^{(\cdot),(i)} & \Gamma_{m-k-1} & \dots & \Gamma_0
\end{vmatrix} \\
&= (-1)^{\gamma} \begin{vmatrix}
\Gamma_{m-1} & \dots & \Gamma_k & \Gamma_{k-1}^{(\cdot),(i)} & \Gamma_{k-2} & \dots & \Gamma_0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\Gamma_{m-l+1} & \dots & \Gamma_{|l-k-2|} & \Gamma_{|l-k-1|}^{(\cdot),(i)} & \Gamma_{|l-k|} & \dots & \Gamma_{l-2} \\
\Gamma_{m-l}^{(j),(\cdot)} & \dots & \Gamma_{|l-k-1|}^{(j),(\cdot)} & \Gamma_{|l-k|}^{(j),(i)} & \Gamma_{|l-k+1|}^{(j),(\cdot)} & \dots & \Gamma_{l-1}^{(j),(\cdot)} \\
\Gamma_{m-l-1} & \dots & \Gamma_{|l-k|} & \Gamma_{|l-k+1|}^{(\cdot),(i)} & \Gamma_{|l-k+2|} & \dots & \Gamma_l \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\Gamma_0 & \dots & \Gamma_{m-k-1} & \Gamma_{m-k}^{(\cdot),(i)} & \Gamma_{m-k+1} & \dots & \Gamma_{m-1}
\end{vmatrix} \\
&= (-1)^{2\gamma} \begin{vmatrix}
\Gamma_0 & \dots & \Gamma_{m-k-1} & \Gamma_{m-k}^{(\cdot),(i)} & \Gamma_{m-k+1} & \dots & \Gamma_{m-1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\Gamma_{m-l-1} & \dots & \Gamma_{|l-k|} & \Gamma_{|l-k+1|}^{(\cdot),(i)} & \Gamma_{|l-k+2|} & \dots & \Gamma_l \\
\Gamma_{m-l}^{(j),(\cdot)} & \dots & \Gamma_{|l-k-1|}^{(j),(\cdot)} & \Gamma_{|l-k|}^{(j),(i)} & \Gamma_{|l-k+1|}^{(j),(\cdot)} & \dots & \Gamma_{l-1}^{(j),(\cdot)} \\
\Gamma_{m-l+1} & \dots & \Gamma_{|l-k-2|} & \Gamma_{|l-k-1|}^{(\cdot),(i)} & \Gamma_{|l-k|} & \dots & \Gamma_{l-2} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\Gamma_{m-1} & \dots & \Gamma_k & \Gamma_{k-1}^{(\cdot),(i)} & \Gamma_{k-2} & \dots & \Gamma_0
\end{vmatrix} \\
&= \left|T_{m-1}^{((m-l)p+j),((m-k)p+i)}\right|,
\end{aligned}
$$

for some $\gamma \in \mathbb{N}$, because the number of changed columns coincides with the number of changed rows. This implies (4.4) an completes the proof of Lemma 4.1. $\square$

For the next step we need to define canonical moments of matrix measures on the interval $[-1, 1]$. Because the main arguments here are very similar to the proceeding in Dette and Studden (2002), who considered matrix measures on the interval $[0, 1]$, we only state the main differences without proofs. To be precise, define for a matrix measure μ_I on the interval $[-1, 1]$ the moments $S_k =$

$S_k(\mu_I) = \int_{-1}^{1} x^k d\mu_I(x)$ $(k = 0, 1, \ldots)$ and a vector $c_n(\mu_I) = (S_0(\mu_I), \ldots, S_n(\mu_I)) \in (\mathbb{C}^{p\times p})^{n+1}$. We consider the moment space

$$\mathcal{M}_{n+1}^{(I)} = \{c_n(\mu_I) \mid \mu_I \text{ is a matrix measure on } [-1,1]\} \subset (\mathbb{C}^{p\times p})^{n+1} \tag{4.6}$$

corresponding to the first n moments of matrix measures on the interval $[-1, 1]$. For a matrix measure μ_I on the interval $[-1, 1]$ we define the block Hankel matrices $\overline{H}_j$ and $\underline{H}_j$

$$\begin{aligned}
\underline{H}_{2m} &= \begin{pmatrix} S_0 & \ldots & S_m \\ \vdots & \ddots & \vdots \\ S_m & \ldots & S_{2m} \end{pmatrix}, \\
\overline{H}_{2m} &= \begin{pmatrix} S_0 - S_2 & \ldots & S_{m-1} - S_{m+1} \\ \vdots & \ddots & \vdots \\ S_{m-1} - S_{m+1} & \ldots & S_{2m-2} - S_{2m} \end{pmatrix}, \\
\underline{H}_{2m+1} &= \begin{pmatrix} S_0 + S_1 & \ldots & S_m + S_{m+1} \\ \vdots & \ddots & \vdots \\ S_m + S_{m+1} & \ldots & S_{2m} + S_{2m+1} \end{pmatrix}, \\
\overline{H}_{2m+1} &= \begin{pmatrix} S_0 - S_1 & \ldots & S_m - S_{m+1} \\ \vdots & \ddots & \vdots \\ S_m - S_{m+1} & \ldots & S_{2m} - S_{2m+1} \end{pmatrix}.
\end{aligned}$$

We introduce the notation

$$\begin{aligned}
\underline{h}_{2m} &= (S_m, \ldots, S_{2m-1})^T, \quad \overline{h}_{2m} = (S_{m-1} - S_{m+1}, \ldots, S_{2m-3} - S_{2m-1})^T, \\
\underline{h}_{2m+1} &= (S_m + S_{m+1}, \ldots, S_{2m-1} + S_{2m})^T, \quad \overline{h}_{2m+1} = (S_m - S_{m+1}, \ldots, S_{2m-1} - S_{2m})^T,
\end{aligned}$$

and define $S_1^+ = S_0$, $S_2^+ = S_0$,

$$\begin{aligned}
S_{2m}^+ &= S_{2m-2} - \overline{h}_{2m}^T \overline{H}_{2m-2}^{-1} \overline{h}_{2m} \quad (m \ge 2), \\
S_{2m+1}^+ &= S_{2m} - \overline{h}_{2m+1}^T \overline{H}_{2m-1}^{-1} \overline{h}_{2m+1} \quad (m \ge 1),
\end{aligned} \tag{4.7}$$

and $S_1^- = -S_0$,

$$\begin{aligned}
S_{2m}^- &= \underline{h}_{2m}^T \underline{H}_{2m-2}^{-1} \underline{h}_{2m} \quad (m \ge 1), \\
S_{2m+1}^- &= \underline{h}_{2m+1}^T \underline{H}_{2m-1}^{-1} \underline{h}_{2m+1} - S_{2m} \quad (m \ge 1).
\end{aligned} \tag{4.8}$$

Note that the quantities S_n^+ and S_n^- are determined by $S_0, \ldots, S_{n-1}$. It can be shown by the same argument as in Dette and Studden (2002) that for $(S_0, \ldots, S_{n-1}) \in \mathrm{Int}(\mathcal{M}_n)$ and any matrix measure μ_I on the interval $[-1, 1]$ with moments satisfying $S_j(\mu_I) = S_j$ $(j = 0, \ldots, n-1)$, the moment of order n $S_n(\mu_I) = \int_{-1}^1 x^n d\mu_I(x)$ satisfies

$$S_n^- \leq S_n(\mu_I) \leq S_n^+, \tag{4.9}$$

With these preparations we can define the canonical moments of a matrix measure on the interval $[-1, 1]$ with moments $S_0, \ldots, S_{n-1}$.

Definition 4.2 *Let μ_I denote a matrix measure on the interval $[-1, 1]$ with moments $S_k = S_k(\mu_I) = \int_{-1}^1 x^k d\mu_I(x)$ $(k = 0, 1, \ldots)$ and define*

$$N(\mu_I) = \min \left\{ k \in \mathbb{N} \mid (S_0, \ldots, S_k) \in \partial \mathcal{M}_{k+1}^{(I)} \right\}. \tag{4.10}$$

For any $n = 0, \ldots, N(\mu_I) - 1$ the (hermitian) canonical moments of the matrix measure μ_I are defined by

$$U_{n+1} = \left(S_{n+1}^+ - S_{n+1}^-\right)^{-1/2} \left(S_{n+1} - S_{n+1}^-\right) \left(S_{n+1}^+ - S_{n+1}^-\right)^{-1/2}, \tag{4.11}$$

where the quantities S_{n+1}^+ and S_{n+1}^- are given by (4.7) and (4.8), respectively.

Note that Dette and Studden (2002) use a non hermitian definition of canonical moments of matrix measures on the interval $[0, 1]$, that is

$$\bar{U}_{n+1} = \left(S_{n+1}^+ - S_{n+1}^-\right)^{-1} \left(S_{n+1} - S_{n+1}^-\right). \tag{4.12}$$

This non hermitian definition turns out to be more useful when working with monic orthogonal polynomials but in the present context the hermitian version has advantages. We are now in a position to prove the main result of this section, which relates the canonical moments of a symmetric matrix measure on the unit circle and the canonical moments of the associated matrix measure on the interval $[-1, 1]$ by the Szegö mapping. For this purpose recall the definition of the matrix ball K_m in (2.14) and the defintion for the matrices L_m, R_m and M_m (2.12), (2.13) and (2.15), respectively. If the given measure μ_C on the unit circle is symmetric, then it follows from (4.5)

$$L_m = R_m. \tag{4.13}$$

The following result is the main step for the proof of the Geronimus relations.

Theorem 4.3 *Let μ_C denote a symmetric matrix measure on the unit circle and denote by $\mu_I = Sz(\mu_C)$ the associated matrix measure on the interval $[-1,1]$ defined by the Szegö mapping (4.1). The canonical moments A_n and U_n of the matrix measures μ_C and μ_I satisfy*

$$A_n = 2U_n - I_p \ ; \quad n = 1, \dots, N(\mu_C).$$

Similarly, the non symmetric canonical moments $\overline{U}_n$ defined in (4.12) satisfy

$$2\overline{U}_n - I_p = \overline{A}_n \ ; \quad n = 1, \dots, N(\mu_C), \tag{4.14}$$

where the quantities $\overline{A}_n$ are given by

$$\overline{A}_n = L_{n-1}^{-1/2} A_n L_{n-1}^{1/2}. \tag{4.15}$$

Proof: We only prove the first part of the Theorem. The second part is shown by similar arguments. Assume that $m < N(\mu_C)$ and let $\Gamma_0, \Gamma_1, \dots,$ denote moments of the matrix measure on the unit circle μ_C. For $j = 0, 1, \dots$ we define $T_j(x) = \cos(j \arccos x)$ as the jth (scalar) Chebychev polynomial of the first kind, then it follows from (4.2) and from Rivlin (1990) that

$$\begin{aligned} \Gamma_j &= \int_{-\pi}^{\pi} \cos(j\theta) d\mu_C(\theta) = \int_{-1}^{1} T_j(x) d\mu_I(x) \\ &= \sum_{k=0}^{\lfloor j/2 \rfloor} (-1)^k \frac{j\Gamma(j-k)}{\Gamma(k+1)\Gamma(j-2k+1)} 2^{j-2k-1} S_{j-2k}, \end{aligned} \tag{4.16}$$

where $S_l = \int_{-1}^{1} x^l d\mu_I(x)$ $(l = 0, 1, \dots)$ denote the moments of the associated matrix measure $\mu_I = Sz(\mu_C)$ on the interval. Recall the definition of S_{m+1}^+ and S_{m+1}^- in (4.7) and (4.8), then there exist matrix measures μ_I^+ and μ_I^- on the interval $[-1,1]$ such that $S_j = S_j(\mu_I^{\pm})$ $(j = 0, \dots, m)$ and

$$S_{m+1}^+ = \int_{-1}^{1} x^{m+1} d\mu_I^+(x) \quad \text{and} \quad S_{m+1}^- = \int_{-1}^{1} x^{m+1} d\mu_I^-(x).$$

We define

$$\Gamma_{m+1}^+ = 2^m S_{m+1}^+ + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k} \tag{4.17}$$

$$\Gamma_{m+1}^- = 2^m S_{m+1}^- + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k}. \tag{4.18}$$

With the inverse Szegö mapping we obtain the symmetric measures $\mu_C^+ = (Sz)^{-1}(\mu_I^+)$ and $\mu_C^- = (Sz)^{-1}(\mu_I^-)$ on the unit circle and the representation (4.16) yields that the measures μ_C^- and μ_C^+ satisfy

$$\int_{-\pi}^{\pi} \cos((m+1)\theta) d\mu_C^+(\theta) = \Gamma_{m+1}^+ \quad \text{and} \quad \int_{-\pi}^{\pi} \cos((m+1)\theta) d\mu_C^-(\theta) = \Gamma_{m+1}^-.$$

Consequently, recalling the definition of the set K_m in (2.14) we have $\Gamma^+_{m+1}, \Gamma^-_{m+1} \in K_m$ and from the extremal property of the moments S^+_{m+1} and S^-_{m+1} we obtain that $\Gamma^+_{m+1}, \Gamma^-_{m+1} \in \partial K_m$. By the definition of the set K_m in (2.14) it therefore follows that the canonical moments A^+_{m+1} and A^-_{m+1} corresponding to matrix measures μ^+_C and μ^-_C, respectively, are unitary. Moreover, Lemma 4.1, implies that the matrices A^+_{m+1} and A^-_{m+1} are hermitian , which yields

$$\left(A^+_{m+1}\right)^2 = I_p \quad \text{and} \quad \left(A^-_{m+1}\right)^2 = I_p.$$

Consequently all eigenvalues of the matrices A^+_{m+1} and A^-_{m+1} are given by -1 and 1.

We now define the matrices

$$\tilde{\Gamma}^+_{m+1} = M_m + L_m \quad \text{and} \quad \tilde{\Gamma}^-_{m+1} = M_m - L_m, \tag{4.19}$$

which are obviously elements of the set K_m because by (4.13) we have $L_m = R_m$. Consequently, there exist matrix measures $\tilde{\mu}^+_C$ and $\tilde{\mu}^-_C$ such that $\Gamma_j(\tilde{\mu}^\pm_C) = \Gamma_j$ $(j = 0, \ldots, m)$ and

$$\begin{aligned} \Gamma_{m+1}(\tilde{\mu}^+_C) &= \tilde{\Gamma}^+_{m+1} \\ \Gamma_{m+1}(\tilde{\mu}^-_C) &= \tilde{\Gamma}^-_{m+1} \end{aligned}$$

Without loss of generality we assume that $\tilde{\mu}^+_C$ and $\tilde{\mu}^-_C$ are symmetric with respect to the point 0 [otherwise use $\frac{1}{2}(\tilde{\mu}^+_C(\theta) + \tilde{\mu}^+_C(-\theta))$] and we define $\tilde{\mu}^+_I = Sz(\tilde{\mu}^+_C)$ and $\tilde{\mu}^-_I = Sz(\tilde{\mu}^-_C)$ as the associated measures on the interval $[-1, 1]$ with $(m+1)$th moments $\tilde{S}^+_{m+1}$ and $\tilde{S}^-_{m+1}$, respectively. These matrices satisfy the identities

$$\begin{aligned} \tilde{\Gamma}^+_{m+1} &= 2^m \tilde{S}^+_{m+1} + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k} \\ \tilde{\Gamma}^-_{m+1} &= 2^m \tilde{S}^-_{m+1} + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} (-1)^k \frac{(m+1)\Gamma(m+1-k)}{\Gamma(k+1)\Gamma(m-2k+2)} 2^{m-2k} S_{m+1-2k} \end{aligned}$$

From the inequalities (4.9) it follows that $S^+_{m+1} \geq \tilde{S}^+_{m+1}$ and $\tilde{S}^-_{m+1} \geq S^-_{m+1}$ (note that $\tilde{S}^+_{m+1}$ and $\tilde{S}^-_{m+1}$ are moments of a matrix measure on the interval $[-1, 1]$ with moments $S_0, \ldots, S_m$). On the other hand we have

$$\begin{aligned} 2^m \left(\tilde{S}^+_{m+1} - S^+_{m+1}\right) &= \tilde{\Gamma}^+_{m+1} - \Gamma^+_{m+1} \\ &= M_m + L_m - (M_m + L_m^{1/2} A^+_{m+1} L_m^{1/2}) \\ &= L_m^{1/2} \left(I_p - A^+_{m+1}\right) L_m^{1/2} \\ &\geq 0, \end{aligned}$$

because the eigenvalues of the matrix $I_p - A_{m+1}$ are given by 0 and 2. So we obtain

$$\tilde{S}^+_{m+1} = S^+_{m+1},$$

while a similar argument shows

$$\tilde{S}^-_{m+1} = S^-_{m+1}.$$

Consequently, it follows that

$$\begin{aligned} A^+_{m+1} &= I_p \ ; \qquad A^-_{m+1} = -I_p \ ; \\ \tilde{\Gamma}^+_{m+1} &= \Gamma^+_{m+1} \ ; \qquad \tilde{\Gamma}^-_{m+1} = \Gamma^-_{m+1} \ ; \end{aligned}$$

and we obtain from the definitions of $\tilde{\Gamma}^+_{m+1}$, $\tilde{\Gamma}^-_{m+1}$ in (4.19)

$$M_m = \frac{1}{2}(\Gamma^+_{m+1} + \Gamma^-_{m+1}), \quad L_m = \frac{1}{2}(\Gamma^+_{m+1} - \Gamma^-_{m+1}).$$

The definition of the $(m+1)$th canonical moment A_{m+1} of the matrix measure μ and (4.17)-(4.18) now imply

$$\begin{aligned} A_{m+1} &= L_m^{-1/2}(\Gamma_{m+1} - M_m)L_m^{-1/2} \\ &= \Big(\frac{1}{2}(\Gamma^+_{m+1} - \Gamma^-_{m+1})\Big)^{-1/2}\Big(\Gamma_{m+1} - \frac{1}{2}(\Gamma^+_{m+1} + \Gamma^-_{m+1})\Big)\Big(\frac{1}{2}(\Gamma^+_{m+1} - \Gamma^-_{m+1})\Big)^{-1/2} \\ &= \left(S^+_{m+1} - S^-_{m+1}\right)^{-1/2}\left(2S_{m+1} - (S^+_{m+1} + S^-_{m+1})\right)\left(S^+_{m+1} - S^-_{m+1}\right)^{-1/2} \\ &= 2\left(S^+_{m+1} - S^-_{m+1}\right)^{-1/2}\left(S_{m+1} - S^-_{m+1}\right)\left(S^+_{m+1} - S^-_{m+1}\right)^{-1/2} - I_p \\ &= 2U_{m+1} - I_p, \end{aligned}$$

where the last equality is a consequence of the definition of canonical moments of matrix measures on the interval $[-1, 1]$. This proves the assertion of the theorem. $\Box$

Our final result gives the Geronimus relations for monic orthogonal matrix polynomials, which generalize the results obtained by Geronimus (1946) and Faybusovich and Gekhtman (1999) for the scalar case. To be precise note that Corollary 3.2 together with (4.13) yield for the monic orthogonal polynomials Φ^R_m and Φ^L_m defined in (3.14) and (3.15), respectively

$$\begin{aligned} \rho^L_m \phi^L_{m+1} &= L_m^{-1/2}\Phi^L_{m+1}, \quad \phi^R_{m+1}\rho^R_m = \Phi^R_{m+1}L_m^{-1/2} \\ \tilde{\phi}^R_m &= L_m^{-1/2}\tilde{\Phi}^R_m, \quad \tilde{\phi}^L_m = \tilde{\Phi}^L_m L_m^{-1/2}. \end{aligned}$$

Using these equations we obtain from (3.21), (3.22) and the second part of Theorem 4.3 the Szegö recursion for the monic orthogonal matrix polynomials with respect to a matrix measure on the unit circle, that is

$$
\begin{aligned}
z\Phi_m^L(z) - \Phi_{m+1}^L(z) &= \overline{A}_{m+1}^{*}\tilde{\Phi}_m^R(z), \\
z\Phi_m^R(z) - \Phi_{m+1}^R(z) &= \tilde{\Phi}_m^R(z)\overline{A}_{m+1}
\end{aligned}
$$

Consequently, the matrices $\overline{A}_{m+1}$ defined by (4.15) are the Verblunsky coefficients corresponding to the monic orthogonal polynomials and we obtain the following result.

Theorem 4.4 *Let μ_C denote a symmetric matrix measure on the unit circle and denote by $\mu_I = Sz(\mu_C)$ the associated matrix measure on the interval $[-1, 1]$ defined by the Szegö mapping (4.1). If P_0, P_1,... be the monic polynomials orthogonal with respect to the matrix measure μ_I satisfying the three term recurrence recursion*

$$
(1+t)P_{m+1}(t) = P_{m+2}(t) + P_{m+1}(t)C_{m+1} + P_m(t)B_m, \tag{4.20}
$$

($P_0(t) = I_p$, $P_{-1}(t) = 0_p$), then the matrices B_m and C_{m+1} satisfy

$$
\begin{aligned}
B_m &= \frac{1}{4}(I_p - \overline{A}_{2m})(I_p - \overline{A}_{2m+1}^{2})(I_p + \overline{A}_{2m+2}), \\
C_{m+1} &= \frac{1}{2}(I_p - \overline{A}_{2m+1})(I_p + \overline{A}_{2m+2}) + \frac{1}{2}(I_p - \overline{A}_{2m+2})(I_p + \overline{A}_{2m+3}),
\end{aligned}
$$

where the quantities $\overline{A}_n$ are defined in (4.15).

Proof: It follows analogously to Dette and Studden (2002) that the matrices B_m and C_{m+1} are given by

$$
\begin{aligned}
B_m &= (S_{2m} - S_{2m}^{-})^{-1}(S_{2m+2} - S_{2m+2}^{-}), \\
C_{m+1} &= (S_{2m+2} - S_{2m+2}^{-})^{-1}(S_{2m+3} - S_{2m+3}^{-}) + (S_{2m+1} - S_{2m+1}^{-})^{-1}(S_{2m+2} - S_{2m+2}^{-}).
\end{aligned}
$$

and that the non hermitian canonical moments defined by (4.12) satisfy

$$
2\overline{V}_{n-1}\overline{U}_n = (S_{n-1} - S_{n-1}^{-})^{-1}(S_n - S_n^{-}),
$$

whenever $n \leq N(\mu_I)$, where $\overline{V}_n = I_p - \overline{U}_n$. Consequently, the assertion follows by a direct application of the second part of Theorem 4.3. $\square$

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