Pivotal tests for relevant differences in the second order dynamics of functional time series

Anne van Delft and Holger Dette

Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany

October 15, 2020

Abstract

Motivated by the need to statistically quantify differences between modern (complex) datasets which commonly result as high-resolution measurements of stochastic processes varying over a continuum, we propose novel testing procedures to detect relevant differences between the second order dynamics of two functional time series. In order to take the between-function dynamics into account that characterize this type of functional data, a frequency domain approach is taken. Test statistics are developed to compare differences in the spectral density operators and in the primary modes of variation as encoded in the associated eigenelements. Under mild moment conditions, we show convergence of the underlying statistics to Brownian motions and construct pivotal test statistics. The latter is essential because the nuisance parameters can be unwieldy and their robust estimation infeasible, especially if the two functional time series are dependent. Besides from these novel features, the properties of the tests are robust to any choice of frequency band enabling also to compare energy contents at a single frequency. The finite sample performance of the tests are verified through a simulation study and are illustrated with an application to fMRI data.

keywords: functional data, time series, spectral analysis, relevant tests, self-normalization, martingale theory

AMS subject classification: Primary: 62M15, 60G10; Secondary 62M10.

1 Introduction

Functional time series analysis is concerned with the development of inference methods to model and analyze data measurements from processes that take values over some continuum like a curve, a surface or a sphere and which exhibit a natural dependency between the observations, each considered as a point in the function space $\mathscr{H}$. In the current day and age of technological advances where measurements of a process can be taken over its entire domain of definition at a high precision, it is not surprising that functional time series analysis is of increased applicability in numerous research areas. Examples can be found in molecular biophysics (Tavakoli and Panaretos, 2016), brain imaging (Aston and Kirch, 2012), climatology (Zhang et al., 2011; Zhang and Shao, 2015), environmental data (Hörmann et al., 2018) or yet economics (Antoniadis et al., 2006; Kowal et al., 2019). Naturally, this has led to an upsurge in the available literature on statistical methodology for the analysis of functional time series.

The main purpose of this paper is to develop frequency domain based inference methods which allow to quantify differences in the second order characteristics of two weakly stationary (possibly dependent) functional time series, say $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$. Comparison of the second order characteristics of two functional time series is of interest in various applications and controlled experiments. The motivation in most cases is to know whether two series are similar or that a joint analysis on the pooled data is relevant to consider. Inherent to this type of sequentially collected functional data is the presence of temporal dependence. The second order structure is therefore more involved than for independent functional data, yet the development of appropriate inference methods are of the same eminent importance; the second order dynamics play a key role in providing information on the smoothness properties of the random functions and optimal dimension reduction techniques.

For independent functional data, statistical inference tools for comparing covariance operators have been developed by Panaretos et al. (2010), Fremdt et al. (2013), Guo et al. (2016) and Paparoditis and Sapatinas (2016). Benko et al. (2009) and Pomann et al. (2016) investigated how far the distribution of two random samples of independent functional data coincide by means of their Karhunen-Loève expansion and developed tests to compare the functional principal components, i.e., the eigenvalues and eigenfunctions of the autocovariance operator. In the context of temporally dependent functional data, methods in this direction have also been considered. Motivated by climate downscaling studies, Zhang and Shao (2015) proposed testing for equality of the 0-lag covariance operators of two functional time series and of their associated eigenvalues and eigenfunctions. More recently Pilavakis et al. (2019), proposed a test for the equality of the 0-lag covariance operators of several independent functional time series.

Time domain methods as considered in aforementioned literature suffer however from important shortcomings when one wants to infer on the second order dynamics of temporally dependent functional data. The autocovariance operator only captures *static* features and the long-run covariance operator, being a sum of the sequence of h-lag covariance operators, only captures crude features of the dynamics. In addition, functional principal component analysis (FPCA) does not provide an optimal dimension reduction since it ignores any temporal dynamics present in the collection of functional observations.

To analyze or compare second order dynamics of functional time series, a frequency domain approach might in fact be more appropriate. Not only does it allow to characterize the full second order dynamics, but the *Cramér-Karhunen-Loève decomposition* (Panaretos and Tavakoli, 2013) – which decomposes the process into uncorrelated functional frequency components and separates the functional and stochastic parts– moreover provides the building block for *harmonic* FPCA, yielding an optimal lower dimensional representation of the functional time series (see also Hörmann et al., 2015; van Delft and Eichler, 2020, on this topic). In particular, a starting point for an optimal lower dimensional representation of a zero-mean $\mathscr{H}$-valued stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ that also captures the temporal dynamics is the functional Cramér representation

$$X_t = \int_{-\pi}^{\pi} e^{\mathrm{i}\omega t} dZ_{X,\omega} ,$$

where $\{Z_{X,\omega}\}_{\omega\in[-\pi,\pi]}$ is a functional orthogonal increment process of which the second order properties are completely described by an operator-valued spectral measure $\mathcal{F}_X(\cdot)$ on $[-\pi,\pi]$ (van Delft and Eichler, 2020). As first noted by Panaretos and Tavakoli (2013), an optimal lower dimensional representation can be obtained by expanding each frequency component in its optimal basis and truncating this at an appropriate level. Assuming for simplicity no points of discontinuity in the spectral measure, the *spectral density operator* $\mathscr{F}_X^{(\omega)}$ is the covariance operator of the infinitesimal increment $dZ_{X,\omega}$. Hence, the optimal basis of $dZ_{X,\omega}$ is given by the eigenfunctions $\{\phi_{X,k}^{(\omega)}\}_{k\geq1}$ of the spectral density operator $\mathscr{F}_X^{(\omega)}$, while the corresponding eigenvalues $\{\lambda_{X,k}^{(\omega)}\}_{k\geq 1}$ provide insight on the relative contribution of each frequency component to the total variation in the process as well as on the dimensionality of each component. The eigenfunctions of the spectral density operator of a functional time series thus encode the smoothness properties of the random functions. In order to compare second order characteristics of functional time series, it is therefore of interest to be able to compare the spectral density operators as well as to compare the primary modes of variation as given by the respective eigenprojectors and eigenvalues.

To make this more precise, under mild regularity conditions stated in Section 3, the full second order dynamics of weakly stationary processes $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$ can respectively be characterized by the spectral density operators

$$\mathscr{F}_X^{(\omega)} = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \mathbb{E}(X_h \otimes X_0)e^{-\mathrm{i}\omega h} \quad \text{and}\ \mathscr{F}_Y^{(\omega)} = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \mathbb{E}(Y_h \otimes Y_0)e^{-\mathrm{i}\omega h}, \quad \omega\in[-\omega,\omega].$$

The information carried by these objects was exploited by Tavakoli and Panaretos (2016), who analyze the molecular dynamic trajectories of DNA minicircles by comparing $\mathscr{F}_X^{(\cdot)}$ and $\mathscr{F}_Y^{(\cdot)}$, restricted to a lower-dimensional subspace, at a set of frequencies. In Leucht et al. (2018), an L^2-distance approach between estimates of $\mathscr{F}_X^{(\cdot)}$ and $\mathscr{F}_Y^{(\cdot)}$ of two linear functional time series was considered to infer on the geographical differences in temperature variation over time. Since $\mathscr{F}_X^{(\omega)}$ and $\mathscr{F}_Y^{(\omega)}$ are non-negative definite Hermitian compact operators, these admit a real-valued discrete spectrum for each ω, which are respectively given by

$$\mathscr{F}_X^{(\omega)} = \sum_{k=1}^{\infty} \lambda_{X,k}^{(\omega)}\Pi_{X,k}^{(\omega)} \quad \text{and}\ \mathscr{F}_Y^{(\omega)} = \sum_{k=1}^{\infty} \lambda_{Y,k}^{(\omega)}\Pi_{Y,k}^{(\omega)},$$

where $\{\lambda_{X,k}^{(\omega)}\}_{k\geq 1}$ is the sequence of eigenvalues of $\mathscr{F}_X^{(\omega)}$ arranged in descending order and where $\Pi_{X,k}^{(\omega)} := \phi_{X,k}^{(\omega)} \otimes \phi_{X,k}^{(\omega)}$ with $\big\{\phi_{X,k}^{(\omega)}\big\}_{k\geq 1}$ denoting the corresponding sequence of eigenfunctions. The operator $\Pi_{X,k}^{(\omega)}$ is a self-adjoint rank-one operator and will be referred to as the k-th *eigenprojector* (at frequency ω) since it projects onto the eigenspace of $\mathscr{F}_X^{(\omega)}$ corresponding to the k-th largest eigenvalue $\lambda_{X,k}^{(\omega)}$. The eigenelements of $\mathscr{F}_Y^{(\omega)}$ are defined in a similar manner. Optimal lower dimensional representations with K degrees of freedom can then be shown to be given by

$$X_t^\star = \int_{(-\pi,\pi]} e^{\mathrm{i}\omega t}\Big(\sum_{j=1}^{K}\Pi_{X,j}^{(\omega)}\Big)dZ_{X,(\omega)} \quad \text{and}\ Y_t^\star = \int_{(-\pi,\pi]} e^{\mathrm{i}\omega t}\Big(\sum_{j=1}^{K}\Pi_{X,j}^{(\omega)}\Big)dZ_{Y,(\omega)}$$

where $\{Z_{X,(\omega)}\}_{\omega\in[-\pi,\pi]}$ and $\{Z_{Y,(\omega)}\}_{\omega\in[-\pi,\pi]}$ are $\mathscr{H}$-valued orthogonal increment processes of which the increments have covariance operators $\{\mathscr{F}_X^{(\omega)}\}_{\omega\in[-\pi,\pi]}$ and $\{\mathscr{F}_Y^{(\omega)}\}_{\omega\in[-\pi,\pi]}$, respectively.

In this paper, our goal is to develop pivotal tests statistics to detect differences in the second order structure between two functional time series based on the spectral density operators and their associated characteristics as given by the eigensystems (eigenprojectors and eigenvalues). The novelty of our approach lies in four different aspects.

(i) Firstly, while methods to test for equality of spectral density operators of two functional time series are available (see e.g. Tavakoli and Panaretos, 2016; Leucht et al., 2018), tests to compare the eigenelements of spectral density operators have, to the best of our knowledge, not yet been considered in existing (function-valued) time series literature. Due to their central role in dimension reduction techniques, these tests are extremely relevant but far from trivial to construct.

(ii) Secondly, our approach is in terms of a relevant testing framework, which means that we are only interested in deviations that surpass a certain threshold. For example, in the context of comparing spectral density operators we do *not* consider the problem of testing for exact equality of the spectral density operators $\mathscr{F}_X^{(\cdot)}$ and $\mathscr{F}_Y^{(\cdot)}$, but propose to investigate hypotheses of the form

$$H_0 : \int_a^b \|\mathscr{F}_X^{(\omega)} - \mathscr{F}_Y^{(\omega)}\|^2 d\omega \leq \Delta \quad a \leq b \in [0, \pi]$$

of no relevant deviation between $\mathscr{F}_X^{(\cdot)}$ and $\mathscr{F}_Y^{(\cdot)}$ over a given frequency band. Here $\|\cdot\|$ denotes an appropriate norm and $\Delta > 0$ is a pre-specified threshold. Note that classical hypotheses as considered in Tavakoli and Panaretos (2016) and Leucht et al. (2018) are obtained with the threshold set to zero. Our motivation for considering relevant hypotheses (i.e., $\Delta > 0$) stems from the observation that in many applications it is not very likely that the second order structures of functional time series $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$ are exactly the same. Moreover, often one might not be interested in small changes and the two series might be merged in the statistical analysis if the difference between $\mathscr{F}_X^{(\cdot)}$ and $\mathscr{F}_Y^{(\cdot)}$ is small. A similar comment applies to the eigenfunctions and eigenprojectors of a spectral density operator, for which relevant hypotheses can be defined similarly (see Section 2.2 for more details).

(iii) Thirdly, tests for hypotheses involving quantities derived from the spectral density operators are of a very complicated nature. The asymptotic distributions of corresponding test statistics oftentimes depend on the unknown objects of interest or on the higher order dynamics of the functional time series. For example, Leucht et al. (2018) consider classical testing problems and use the bootstrap to avoid estimation of a functional of the spectral density operator. However, if relevant hypotheses of the form (1) have to be tested then the construction of a bootstrap procedure is highly non-trivial as one has to mimic the distribution of a test statistic under a null hypothesis, which differs only in a quantitative but not in a qualitative way from the alternative. The situation becomes even more difficult in the construction of testing procedures for relevant hypotheses involving the eigenfunctions and eigenprojectors.

In this paper we solve this problem and develop tests that are pivotal and do neither require the estimation of such nuisance parameters nor a bootstrap approach.

(iv) Fourthly, we derive our results under extremely mild moment conditions, which are much weaker than those available in the literature on (functional) time series (see Section 3 for details). The derivation of the distributional properties of our tests is quite involved and relies upon approximating martingale theory and the proofs might be of interest in their own right.

The structure of this article is as follows. First, we introduce the precise form of our hypotheses, relate this to existing literature, and highlight the importance of considering pivotal test statistics. In Section 2, we introduce our testing frameworks. All proposed test statistics can be expressed as a functional of a 'building block' process, which is introduced in Section 3 and its weak convergence is established. These results can then be used to develop new tests and to investigate their statistical properties. In Section 4, we study the finite sample properties of the proposed tests in a simulation study and showcase an application to resting state fMRI data. Finally, in Section 5 we provide the main argument to establish the weak convergence of the 'building block' process, while most of the technical details are deferred to an online appendix.

2 Relevant hypotheses for second order dynamics

2.1 Notation

We start by introducing some required terminology. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|_{\mathcal{H}}$. We denote the Hilbert tensor product between two Hilbert spaces $(\mathcal{H}_j,\langle\cdot,\cdot\rangle_{\mathcal{H}_j})_{j=1,2}$ by $\mathcal{H}^{\otimes_2} := \mathcal{H}_1 \otimes \mathcal{H}_2$, whose elements are linear combinations of the simple tensors $h_1 \otimes h_2$, $h_j \in \mathcal{H}_j, j = 1,2$. This is a Hilbert space formed from the algebraic tensor product together with a bilinear map $\psi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes_{\text{alg}} \mathcal{H}_2$ satisfying $\langle \psi(h_1,h_2),\psi(g_1,g_2)\rangle = \langle h_1,g_1\rangle_{\mathcal{H}_1}\langle h_2,g_2\rangle_{\mathcal{H}_2}$ for $h_1,g_1 \in \mathcal{H}_1$ and $h_2,g_2 \in H_2$ and then taking the completion with respect to the induced norm. We denote the direct sum of two Hilbert spaces by $\mathcal{H}^{\oplus_2} := \mathcal{H}_1 \oplus \mathcal{H}_2$, of which elements are of the form $h = (h_1,h_2)^\top$, where $(\cdot)^\top$ denotes the transpose operation. Observe that this is again a Hilbert space with inner product $\langle g,h\rangle = \sum_{j=1}^{2}\langle g_j,h_j\rangle_{\mathcal{H}_j}$, for any $g,h \in \mathcal{H}^{\oplus_2}$. For more details on these facts we refer to Kadison and Ringrose (1997). Let $\{\chi_j\}_{j\geq1}$ be an orthonormal basis of $\mathcal{H}_1$. For a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ we define, respectively, the operator norm by $|||A|||_\infty = \sup_{\|g\|_{\mathcal{H}_1}\leq1}\|A(g)\|_{\mathcal{H}_2}$, $g \in \mathcal{H}_1$, the Hilbert-Schmidt norm by $|||A|||_2 = \sum_{j\geq1}\left(\|A(\chi_j)\|^2_{\mathcal{H}_2}\right)^{1/2}$, which is induced by the inner product $\langle A_1,A_2\rangle_{S_2} = \sum_{j\geq1}\langle A_1\chi_j,A_2\chi_j\rangle_{\mathcal{H}_2}$, A_1, $A_2 : \mathcal{H}_1 \to \mathcal{H}_2$, and for $A : \mathcal{H}_1 \to \mathcal{H}_1$ the trace class norm by $|||A|||_1 = \sum_j\langle (AA^\dagger)^{1/2}(\chi_j),\chi_j\rangle_{\mathcal{H}_1}$, where $A^\dagger$ denotes the adjoint of A. We write $A \in S_2(\mathcal{H}_1,\mathcal{H}_2)$ if it has finite Hilbert-Schmidt norm and abbreviate $S_2(\mathcal{H}) := S_2(\mathcal{H},\mathcal{H})$. For a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$, with $|||A|||_1 < \infty$ we write $A \in S_1(\mathcal{H})$. For $f,g,v \in \mathcal{H}$, we denote the tensor product $f \otimes g\colon S_2(\mathcal{H}) \to \mathcal{H}$ as the bounded linear operator $(f \otimes g)v = \langle v,g\rangle f$. We additionally define the Kronecker tensor product as $(A\widetilde{\otimes}B)C = ACB^\dagger$ for $A,B,C \in S_2(H)$.

Next, for a $\mathcal{H}$-valued random element X over a probability space $(\Omega,\mathcal{A},\mathbb{P})$, we shall denote $X \in \mathcal{L}^p_{\mathcal{H}}$ if $\|X\|_{\mathbb{H},p} := (\mathbb{E}\|X\|^p_{\mathcal{H}})^{1/p} < \infty$. Observe that $\mathcal{L}^2_{\mathcal{H}}$ is a Hilbert space consisting of $\mathcal{H}$-valued random elements with finite second order moment. We note moreover that for any $X,Y \in \mathcal{L}^2_{\mathcal{H}}$ with zero mean, the cross-covariance operator is given by $\mathrm{Cov}(X,Y) = \mathbb{E}(X \otimes Y)$ and belongs to $S_1(\mathcal{H})$. For a zero mean process $X = (X_1,X_2)^\top \in \mathcal{L}^2_{\mathcal{H}\oplus\mathcal{H}}$, we note that $\mathrm{Cov}(X,X) = \mathbb{E}X^{\otimes_2}$ consists of the components $\mathbb{E}(X_i \otimes X_j)$, $i,j = 1,2$ which are elements of $S_1(\mathcal{H})$. Furthermore, we denote the imaginary unit by i and $\overline{g}$ denotes the complex conjugate (function) of g. We shall also denote $\Re(\cdot)$ and $\Im(\cdot)$ for the real and imaginary part, respectively, of a complex-valued object. We shall write $a_T \sim b_T$ if $\lim_{T\to\infty}\frac{a_T}{b_T} = 1$. Weak convergence in $D[0,1]$ – the space of right-continuous functions with left-hand limits – with respect to the Skorokhod topology will be denoted by $\rightsquigarrow$, while convergence in distribution as $T \to \infty$ will be denoted by $\underset{T\to\infty}{\Rightarrow}$. Finally, we reserve $\mathbb{B}$ to denote standard Brownian motion on the interval $[0,1]$ and remark that $\lfloor\cdot\rfloor$ denotes the floor function.

2.2 Relevant hypotheses

In this paper, we consider Hilbert-valued processes $\{\boldsymbol{X}_t := (X_t,Y_t)^\top : t \in \mathbb{Z}\} \in \mathcal{L}^2_{\mathcal{H}\oplus\mathcal{H}}$ which are assumed to be weakly stationary. This implies in particular that the h-lag covariance operator of $\{\boldsymbol{X}_t\}_{t\in\mathbb{Z}}$, satisfies

$$\mathrm{Cov}(\boldsymbol{X}_{t+h},\boldsymbol{X}_t) = \begin{pmatrix} \mathrm{Cov}(X_{t+h},X_t) & \mathrm{Cov}(X_{t+h},Y_t) \\ \mathrm{Cov}(Y_{t+h},X_t) & \mathrm{Cov}(Y_{t+h},Y_t) \end{pmatrix} = \mathrm{Cov}(\boldsymbol{X}_h,\boldsymbol{X}_0)$$

for all $t,h \in \mathbb{Z}$. In the following, we introduce the three testing frameworks to test for relevant differences in the second order characteristics of the component processes $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$. As a first option, this can be framed as the following hypothesis testing problem on the spectral density operators;

$$H_0 : \int_a^b \left\|\left\| \mathscr{F}_X^{(\omega)} - \mathscr{F}_Y^{(\omega)} \right\|\right\|_2^2 d\omega \leq \Delta \quad \text{versus} \quad H_A : \int_a^b \left\|\left\| \mathscr{F}_X^{(\omega)} - \mathscr{F}_Y^{(\omega)} \right\|\right\|_2^2 d\omega > \Delta, \tag{2.1}$$

where $[a,b] \subseteq [0,\pi]$ and $\Delta > 0$ is a pre-specified constant that represents the maximal value for which the distance $\int_a^b |||\mathscr{F}_X^{(\omega)} - \mathscr{F}_Y^{(\omega)}|||_2^2 d\omega$ is considered as not relevant. Note that by specifying the choice of a and b, one can compare the spectral density operators within a certain narrow frequency band or even at a single frequency, which is of interest in certain applications. For instance, activities of certain areas of the brain, such as the Nucleas Accumbens, are usually located within a small frequency band around frequency zero (see e.g., Fiecas and Ombao, 2016) and the frequency characteristics of resting-state fMRI data tend to have rather frequency-specific biological interpretations (see e.g., Yuen et al., 2019, and references therein).

Besides from (2.1), the main focus in this paper is on two more refined hypotheses testing problems that allow to infer relevant differences in the primary modes of variation. More specifically, to consider the relevant differences at component k for some $k \in \mathbb{N}$, we are in particularly interested in providing a meaningful test for the hypotheses of no relevant difference between the k-th eigenprojectors, that is

$$H_0 : \int_a^b \left\|\left\| \Pi_{X,k}^{(\omega)} - \Pi_{Y,k}^{(\omega)} \right\|\right\|_2^2 d\omega \leq \Delta_{\Pi,k} \quad \text{versus} \quad H_A : \int_a^b \left\|\left\| \Pi_{X,k}^{(\omega)} - \Pi_{Y,k}^{(\omega)} \right\|\right\|_2^2 d\omega > \Delta_{\Pi,k}, \tag{2.2}$$

where $[a,b] \subseteq [0,\pi]$ and where $\Delta_{\Pi,k} > 0$ denotes, similarly to Δ, a pre-specified constant. It is worth mentioning that the eigenfunctions $\{\phi_{X,k}^{(\omega)}\}_{k\geq 1}$ are complex elements of $\mathscr{H}$ (except at $\omega = 0,\pi$). Due to this, a test statistic based upon the difference of the empirical eigenfunctions is not feasible because these are only identifiable up to a rotation on the unit circle. The testing framework in (2.2) is therefore formulated in terms of the eigenprojectors since their empirical counterparts are rotationally invariant. We come back to this in Section 3. Finally, we also consider the hypotheses of no relevant difference between the k-th eigenvalues, that is

$$H_0 : \int_a^b |\lambda_{X,k}^{(\omega)} - \lambda_{Y,k}^{(\omega)}|^2 d\omega \leq \Delta_{\lambda,k} \quad \text{versus} \quad H_A : \int_a^b |\lambda_{X,k}^{(\omega)} - \lambda_{Y,k}^{(\omega)}|^2 d\omega > \Delta_{\lambda,k}, \tag{2.3}$$

where $[a,b] \subseteq [0,\pi]$ and where $\Delta_{\lambda,k} > 0$ is again a pre-specified constant that represents the maximal value for which the difference between the k-th eigenvalues is deemed not relevant.

In this article, we develop pivotal tests for the hypotheses in (2.1), (2.2) and (2.3). To elaborate on its relevance and to motivate that this is a very challenging problem, observe that a natural approach to test hypotheses of the form (2.1) is to construct an empirical distance measure

$$\widehat{M}^2 = \int_a^b \left\|\left\| \hat{\mathscr{F}}_X^{(\omega)} - \hat{\mathscr{F}}_Y^{(\omega)} \right\|\right\|_2^2 d\omega, \tag{2.4}$$

of the distance $M^2 = \int_a^b |||\mathscr{F}_X^{(\omega)} - \mathscr{F}_Y^{(\omega)}|||_2^2 d\omega$, where $\hat{\mathscr{F}}_X^{(\omega)}$ and $\hat{\mathscr{F}}_Y^{(\omega)}$ are suitable estimators of the spectral density operators $\mathscr{F}_X^{(\omega)}$ and $\mathscr{F}_Y^{(\omega)}$, respectively, and to reject the null hypothesis for large values of (2.4). For classical hypotheses, i.e., where $H_0 : M^2 = \int_a^b |||\mathscr{F}_X^{(\omega)} - \mathscr{F}_Y^{(\omega)}|||_2^2 d\omega = 0$, one then requires the (asymptotic) distribution of the statistic at $M^2 = 0$ in order to determine the critical values, which involves the estimation of certain nuisance parameters. The latter was for example considered by Tavakoli and Panaretos (2016), who construct a test for equality of spectral density operators based upon this distance restricted to a finite-dimensional subspace (see also Panaretos et al., 2010, who considered this approach for covariance operators). A drawback is that the method can be sensitive to the specific choice of several regularization parameters, including an appropriate truncation level for the dimension of which the optimal value is frequency-dependent. Another approach was taken in van Delft and Dette (2020), who introduced a fully functional similarity measure for (time-varying) spectral density operators of possibly nonstationary functional time series where the distance measure is estimated based upon integrated functionals of (localized) periodogram operators. While this avoids sensitivity to certain regularization parameters, the expressions of the asymptotic variance can still become quite involved when certain assumptions, such as independence of the two series, are relaxed. Alternatively, one could consider a bootstrap method to obtain the critical values of the test statistic, an approach taken by Leucht et al. (2018). However, even for classical hypotheses such an approach is computationally expensive.

For testing relevant hypotheses of the form (2.1), (2.2) and (2.3) the problems become substantially more intricate. In particular, the determination of critical values for the relevant hypotheses in (2.1) requires the (asymptotic) distribution of the statistic $\widehat{M}^2$ at *any* point $M^2 \geq 0$ of the alternative. As will be demonstrated in this paper, an appropriately normalized version of $\widehat{M}^2 - M^2$ is in general asymptotically normally distributed but, compared to the classical hypothesesis H_0 : $M^2 = 0$, the variance of the limiting distribution now depends in a much more complicated way on the spectral density operators $\mathscr{F}_X^{(\cdot)}$ and $\mathscr{F}_Y^{(\cdot)}$ and is therefore extremely difficult to estimate. Moreover, for the same reason it is unclear whether a bootstrap method for relevant hypotheses can be developed since one basically has to mimic the (asymptotic) distribution of the test statistic for any pair of time series $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$ such that their spectral density operators satisfy the null hypotheses in (2.1).

The above approaches become even more problematic, if not infeasible, if either classical or relevant tests of the form (2.2) and (2.3) for the eigenelements have to be constructed. As will become clear in the subsequent sections, the distributional properties of the corresponding empirical distance measures depend in a highly complicated manner on the dependence structure of the underlying processes (see, for example, Theorem 3.4 below). This to the extent that the estimation of nuisance parameters becomes close to impossible and such an approach highly unstable. To circumvent this problem, we propose tests based on self-normalized or ratio statistics which are constructed via appropriate standardized estimators of the distance measures in (2.1), (2.2) and (2.3), and have a limiting distribution which does not depend on the dependence structure of the underlying processes. The concept of self-normalization has been used in other settings by numerous authors in the context of testing classical hypotheses (see Shao and Zhang, 2010; Shao, 2010; Zhang et al., 2011; Shao, 2015; Zhang and Shao, 2015, among others). Recently, a new concept of self-normalization for testing relevant hypotheses regarding the mean and covariance operator of functional time series has also been developed by Dette et al. (2020). However, the development of frequency domain based tests for relevant hypotheses and hence that allow to infer on the (full) second order dynamics of these processes is far from trivial. As a further matter, we derive our results under very mild moment conditions which improve upon L^p_m-approximable assumptions and do not require summability of functional cumulant-mixing conditions. For the hypothesis in (2.1), our current work therefore not only provides a stable alternative to existing work but also relaxes upon underlying moment assumptions. Because the construction and the distributional properties of the statistics are highly technical, we start the next section by providing the framework and assumptions and the main ingredient to our method. We then develop the test statistics in full detail for all three hypotheses.

3 Methodology

Suppose that we observe a sample of length T_1 from component process $\{X_t\}_{t\in\mathbb{Z}}$ and of length T_2 from component process $\{Y_t\}_{t\in\mathbb{Z}}$. Central in the construction of the pivotal test statistics and the corresponding asymptotic level α tests for the hypotheses (2.1), (2.2) and (2.3) are processes of the form

$$\Big\{\eta\sqrt{b_1T_1+b_2T_2}\Big(\frac{r_1}{\sqrt{b_1T_1}}\int_a^b \Re\Big(\Big\langle \mathcal{Z}^{X,\omega}_{T,\eta},\mathscr{U}^{(\omega)}_{XY}\Big\rangle_{S_2}\Big)d\omega+\frac{r_2}{\sqrt{b_2T_2}}\int_a^b \Re\Big(\Big\langle \mathcal{Z}^{Y,\omega}_{T,\eta},\mathscr{U}^{(\omega)}_{YX}\Big\rangle_{S_2}\Big)d\omega\Big\}_{\eta\in[0,1]}, \tag{3.1}$$

where $r_1, r_2 \in \mathbb{R}$. Here, the operators $\mathscr{U}^{(\omega)}_{XY}, \mathscr{U}^{(\omega)}_{YX} \in S_2(\mathscr{H})$ are such that $\int_{-\pi}^{\pi} |||\mathscr{U}^{(\omega)}_{\cdot,\cdot}|||_2 d\omega < \infty$, and are of varying nature depending on the specific hypotheses under consideration. The operators $\mathcal{Z}^{X,\omega}_{T,\eta}$ and $\mathcal{Z}^{Y,\omega}_{T,\eta}$ are $S_2(\mathscr{H})$-valued partial sum processes defined by

$$\mathcal{Z}^{X,\omega}_{T,\eta} = \frac{\sqrt{b_1}}{\sqrt{T_1}}\sum_{s=1}^{\lfloor \eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_1\rfloor}\tilde{w}^{(\omega)}_{b_1,s,t}(X_s\otimes X_t)-\mathscr{F}^{(\omega)}_X\Big), \tag{3.2}$$

$$\mathcal{Z}^{Y,\omega}_{T,\eta} = \frac{\sqrt{b_2}}{\sqrt{T_2}}\sum_{s=1}^{\lfloor \eta T_2\rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_2\rfloor}\tilde{w}^{(\omega)}_{b_2,s,t}(Y_s\otimes Y_t)-\mathscr{F}^{(\omega)}_Y\Big), \tag{3.3}$$

where

$$\tilde{w}^{(\omega)}_{b_i,s,t} := (2\pi)^{-1}w(b_i(t-s))e^{\mathrm{i}\omega(t-s)} \tag{3.4}$$

for some window function $w(\cdot)$ and where $b_i := b(T_i)$, $i = 1,2$, are bandwidth parameters which are functions of the corresponding sample lengths T_i. Intuitively, the operators (3.2) and (3.3) can be interpreted as scaled and centered sequential estimators of the spectral density operators $\mathscr{F}^{(\omega)}_X$ and $\mathscr{F}^{(\omega)}_Y$. While perhaps not immediately obvious, we shall demonstrate in the following three sections that the distributional properties of empirical versions of the three distance measures in (2.1), (2.2) and (2.3), respectively, can –after centering around the population distance measure– be derived from those of processes of the form (3.1). For example, we will show in the next section that $\hat{M}^2 - M^2$, with $\hat{M}^2$ as in (2.4), can be expressed in terms of such a process that is evaluated at $\eta = 1$.

In order to make this more precise and to derive the distributional properties of the process defined in (3.1), we require the following technical assumptions. Firstly, we specify the dependence structure of $\{X_t\}_{t\in\mathbb{Z}} \in \mathscr{L}^2_{\mathscr{H}}$ and $\{Y_t\}_{t\in\mathbb{Z}} \in \mathscr{L}^2_{\mathscr{H}}$ jointly in terms of the bivariate functional time series $\{\boldsymbol{X}_t\}_{t\in\mathbb{Z}} = \{(X_t, Y_t)^\top\}_{t\in\mathbb{Z}}$. For this, we consider conditions as given in van Delft (2020), who studied limiting distributions of quadratic form statistics of functional time series under mild moment conditions and provided generalizations of the physical dependence measure (Wu, 2005) to Hilbert-valued processes. A functional time series $\{V_t\}_{t\in\mathbb{Z}}$ taking values in a separable Hilbert space $\mathscr{H}$ is said to have a *physical dependence structure* for some $p > 0$ if:

A.1 The series admits a representation of the form $V_t = g(\epsilon_t, \epsilon_{t-1}, \ldots,)$ where $\{\epsilon_t : t \in \mathbb{Z}\}$ is an i.i.d. sequence of elements in some measurable space S and $g : S^\infty \to \mathscr{H}$ is a measurable function.

A.2 The series' dependence structure is of the following nature. Define the measure

$$\nu_{\mathbb{H},p}(V_t) = \|V_t - \mathbb{E}[V_t|\mathscr{G}_{t,\{0\}}]\|_{\mathbb{H},p},$$

where $\mathscr{G}_{t,\{0\}}$ is the filtration up to time t but with the element at time 0 replaced with an independent copy, i.e., $\mathscr{G}_{t,\{0\}} = \sigma(\epsilon_t, \epsilon_{t-1}, \ldots, \epsilon'_0, \epsilon_{-1}, \ldots)$, for some independent copy ϵ'_0 of ϵ_0 and where the conditional expectation is to be understood in the sense of a Bochner integral. The dependence structure of the process satisfies

$$\sum_{j=0}^{\infty} \nu_{\mathbb{H},p}(V_j) < \infty \tag{3.5}$$

for some $p \geq 0$.

As demonstrated in van Delft (2020), the summability condition in (3.5) is a weaker assumption than the L^p_m- approximability condition introduced by Hörmann and Kokoszka (2010) and is generally weaker than summability conditions of the p-th order cumulant tensor for $p > 2$.

Throughout this paper, we assume the following conditions on the function-valued time series.

Assumption 3.1. *The process $\{\boldsymbol{X}_t\}_{t\in\mathbb{Z}} = \{(X_t, Y_t)^\top\}_{t\in\mathbb{Z}}$ is a centered weakly stationary bivariate functional time series in $\mathscr{L}^p_{\mathscr{H}\oplus\mathscr{H}}$ of which the component processes satisfy **A.1**-**A.2** with $p = 4+\varepsilon$, for some small $\varepsilon > 0$.*

Elementary calculations show that Assumption 3.1 implies the process $\{\boldsymbol{X}_t\}_{t\in\mathbb{Z}}$ satisfies **A.1**-**A.2** with $p = 4+\varepsilon$. Observe furthermore that Assumption 3.1 allows for the scenario of independence between the two component processes. It is worth mentioning that the zero-mean assumption simplifies notation but, in practice, the data can be centered without affecting the results of this paper. Processes which satisfy conditions **A.1**-**A.2** for some $p \geq 2$ have a well-defined spectral density operator. In particular, for processes that satisfy Assumption 3.1, the second order structure arises as elements of $(\mathscr{H}^{\oplus_2})^{\otimes_2}$ and the full second order dynamics are therefore described via the vector of spectral density operators given by

$$\mathscr{F}^{(\omega)}_{\boldsymbol{X}} = \left(\mathscr{F}^{(\omega)}_X, \mathscr{F}^{(\omega)}_{XY}, \mathscr{F}^{(\omega)}_{YX}, \mathscr{F}^{(\omega)}_Y\right)^\top \quad \omega \in [-\pi, \pi],$$

where the operators $\mathscr{F}^{(\omega)}_{XY} = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \mathbb{E}(X_h \otimes Y_0) e^{-\mathrm{i}\omega h}$ and $\mathscr{F}^{(\omega)}_{YX} = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \mathbb{E}(Y_h \otimes X_0) e^{-\mathrm{i}\omega h}$ define the cross-spectral density operators. It can be shown that the convergence of the series is with respect to $|||\cdot|||_1$, uniformly in $\omega \in \mathbb{R}$.

As a starting point for our test statistics, consider the following estimators of $\mathscr{F}^{(\omega)}_X$ and $\mathscr{F}^{(\omega)}_Y$

$$\hat{\mathscr{F}}^{\omega}_X = \frac{1}{T_1}\sum_{s,t=1}^{T_1} \tilde{w}^{(\omega)}_{b_1,s,t}\big(X_s \otimes X_t\big); \tag{3.6}$$

$$\hat{\mathscr{F}}^{\omega}_Y = \frac{1}{T_2}\sum_{s,t=1}^{T_2} \tilde{w}^{(\omega)}_{b_2,s,t}\big(Y_s \otimes Y_t\big), \tag{3.7}$$

where the weights $\tilde{w}^{(\omega)}_{b_{T_i},s,t}$ are given by (3.4). For the construction of pivotal test statistics, we require sequential versions of the lag window estimators in (3.6) and (3.7), which are respectively given by

$$\hat{\mathscr{F}}^{(\omega)}_X(\eta) = \frac{1}{\lfloor \eta T_1 \rfloor}\sum_{s=1}^{\lfloor \eta T_1 \rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_1 \rfloor} \tilde{w}^{(\omega)}_{b_1,s,t}(X_s \otimes X_t)\Big), \tag{3.8}$$

and

$$\hat{\mathscr{F}}^{(\omega)}_Y(\eta) = \frac{1}{\lfloor \eta T_2 \rfloor}\sum_{s=1}^{\lfloor \eta T_2 \rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_2 \rfloor} \tilde{w}^{(\omega)}_{b_2,s,t}(Y_s \otimes Y_t)\Big), \tag{3.9}$$

where $\eta \in [0,1]$. We shall denote the eigenvalues and eigenprojectors of (3.8) and (3.9), respectively, by $\{\hat{\lambda}^{(\omega)}_{X,k}(\eta)\}_{k\geq 1}$, $\{\hat{\lambda}^{(\omega)}_{Y,k}(\eta)\}_{k\geq 1}$ and $\{\hat{\Pi}^{(\omega)}_{X,k}(\eta)\}_{k\geq 1}$ and $\{\hat{\Pi}^{(\omega)}_{Y,k}(\eta)\}_{k\geq 1}$. Empirical versions of the distance measures in (2.1), (2.2) and (2.3) can then be expressed in terms of the sequential estimators evaluated at $\eta = 1$, i.e.,

$$\int_a^b \vert\vert\vert \hat{\mathscr{F}}^{(\omega)}_X(1) - \hat{\mathscr{F}}^{(\omega)}_Y(1) \vert\vert\vert_2^2 d\omega, \quad \int_a^b \vert\vert\vert \hat{\Pi}^{(\omega)}_{X,k}(1) - \hat{\Pi}^{(\omega)}_{Y,k}(1) \vert\vert\vert_2^2 d\omega, \quad \int_a^b \big(\hat{\lambda}^{(\omega)}_{X,k}(1) - \hat{\lambda}^{(\omega)}_{Y,k}(1)\big)^2 d\omega.$$

We assume the following mild requirements on the lag window function w.

Assumption 3.2. *Let w be an even, bounded function on $\mathbb{R}$ with $\lim_{x\to 0} w(x) = 1$ that is continuous except at a finite number of points. Suppose that $\lim_{b\to 0} b\sum_{h\in\mathbb{Z}} w^2(bh) = \kappa$ where $\kappa := \int_{-\infty}^{\infty} w^2(x)dx < \infty$ such that $\sup_{0\leq b\leq 1} b\sum_{|h|\geq L/b} w^2(bh) \to 0$ as $L \to \infty$. Furthermore, we assume $\lim_{x\to 0} |w(x) - 1| = O(x)$.*

Under these conditions, the following consistency result on the lag window estimators can be obtained.

Proposition 3.1. *Suppose $\{V_t\}_{t\in\mathbb{Z}}$ is a centered weakly stationary process in $\mathscr{L}^p_{\mathscr{H}}$ that satisfies conditions **A.1**-**A.2** with $p = 2q, q \geq 1$. Furthermore, let Assumption 3.2 be satisfied and assume*

$$\sum_{h\in\mathbb{Z}} h^\ell \nu_{\mathbb{H},2}(V_h) < \infty \tag{3.10}$$

for some $\ell \geq 1$. Let $\hat{\mathscr{F}}^{\omega}_V = \frac{1}{T}\sum_{s,t=1}^T \tilde{w}^{(\omega)}_{b_T,s,t}\big(V_s \otimes V_t\big)$. Then, for $q \geq 2$,

$$\|\hat{\mathscr{F}}^{\omega}_V - \mathbb{E}\hat{\mathscr{F}}^{\omega}_V\|^q_{S_2,q} = O\big((b_T T)^{-q/2}\big) \text{ and } \|\mathbb{E}\hat{\mathscr{F}}^{\omega}_V - \mathscr{F}^{\omega}_V\|^q_{S_2,q} = O\big(b_T^{\ell q} + T^{-q}\big)$$

uniformly in $\omega \in [-\pi,\pi]$, where $\mathscr{F}^{\omega}_V$ is the spectral density operator of process $\{V_t\}$. In particular $\|\hat{\mathscr{F}}^{\omega}_V - \mathscr{F}^{\omega}_V\|^q_{S_2,q} = O\big((b_T T)^{-q/2}\big) + O\big(b_T^{\ell q}\big)$ uniformly in $\omega \in [-\pi,\pi]$.

Note that the value of ℓ in (3.10) only affects the order of the bias, which decreases faster for processes with shorter memory. It is also worth mentioning that the estimator remains consistent for $\ell = 0$. To ensure consistency in q-th mean, Proposition 3.1 gives rise to the following conditions on the rate of the bandwidth.

Assumption 3.3. *Given $\sum_{h\in\mathbb{Z}} h^\ell \nu_{\mathbb{H},2}(V_h) < \infty$ for some $\ell \geq 1$, we require that $b_T \to 0$ such that $b_T T \to \infty$ and such that $b_T^{1+2\ell} T \to 0$ as $T \to \infty$.*

Observe that the last part of the assumption simply means that larger bandwidths are allowed for processes with a 'smoother" spectral distribution.

Under Assumptions 3.1–3.3 the sequential estimators (3.8) and (3.9) provide us with consistent estimators of $\mathscr{F}^{(\omega)}_X$ and $\mathscr{F}^{(\omega)}_Y$. Furthermore, the elements of their respective eigensystems $\{\hat{\lambda}^{(\omega)}_{X,k}(\eta), \hat{\Pi}^{(\omega)}_{X,k}(\eta)\}_{k\geq 1}$, $\{\hat{\lambda}^{(\omega)}_{Y,k}(\eta), \hat{\Pi}^{(\omega)}_{Y,k}(\eta)\}_{k\geq 1}$, can then be shown to provide us with consistent estimators of their population counterparts for each $\eta \in [0,1], \omega \in [-\pi,\pi]$ (see Lemma B.5). Additionally, under these conditions, we obtain a useful bound on the maximum of partial sum of the estimators of the spectral density operators (see Lemma B.1).

The last assumption concerns the 'balance' of the convergence rates.

Assumption 3.4. *Let b_i, $i \in \{1,2\}$ satisfy Assumption 3.3 for some $\ell \geq 1$. If the component processes $\{X_t\}$ and $\{Y_t\}$ of $\{\boldsymbol{X}_t\}$ are independent, we assume there exists a constant $\theta \in (0,1)$ such that*

$$\lim_{T_1,T_2\to\infty} \frac{b_1 T_1}{b_1 T_1 + b_2 T_2} = \theta \in (0,1).$$

If the processes are dependent, we assume $T_1 = T_2$ and $b_1 \sim b_2$.

We can now state the main technical result of this paper which is crucial for the construction of pivotal tests for the hypotheses (2.1), (2.2) and (2.3) of no relevant difference in the spectral density operators, eigenprojectors or eigenvalues, respectively (see Sections 3.1 - 3.3 for details).

Theorem 3.1. *Suppose Assumptions 3.1-3.4 are satisfied. Then*

$$\Big\{\eta\sqrt{b_1T_1+b_2T_2}\Big(\frac{1}{\sqrt{b_1T_1}}\int_a^b \Re\Big(\Big\langle \mathcal{Z}_{T,\eta}^{X,\omega},\mathcal{U}_{XY}^{(\omega)}\Big\rangle_{S_2}\Big)d\omega-\frac{1}{\sqrt{b_2T_2}}\int_a^b \Re\Big(\Big\langle Z_{T,\eta}^{Y,\omega},\mathcal{U}_{YX}^{(\omega)}\Big\rangle_{S_2}\Big)d\omega\Big)\Big\}_{\eta\in[0,1]}$$

$$\rightsquigarrow \big\{\tau_{XY}\eta\mathbb{B}(\eta)\big\}_{\eta\in[0,1]},$$

where τ_{XY} is a constant, $\mathbb{B}$ is a Brownian motion and the processes $\mathcal{Z}_{T,\eta}^{X,\omega}$ and $\mathcal{Z}_{T,\eta}^{Y,\omega}$ are defined in (3.2) *and* (3.3)*, respectively.*

The proof of this statement relies on approximating martingale theory and is postponed to Section 5. The scaling factor τ_{XY} depends in a rather complicated way on the properties of $\{\mathscr{F}_X^{(\omega)}\}_{\omega\in[a,b]}$ and $\{\mathscr{F}_Y^{(\omega)}\}_{\omega\in[a,b]}$ (see Section 5 for details) and is therefore very difficult to estimate. In the next sections, we develop tests for the hypotheses of relevant differences between the spectral density operators $\{\mathscr{F}_X^{(\omega)}\}_{\omega\in[a,b]}$ and $\{\mathscr{F}_Y^{(\omega)}\}_{\omega\in[a,b]}$ and the associated eigenelements, which do not require estimation of τ_{XY} and are in this sense pivotal.

3.1 No relevant difference in the spectral density operators: the hypothesis (2.1)

We start with the construction of a pivotal test for hypothesis (2.1) of no relevant difference between the spectral density operators. Proofs of the statements can be found in Section A.1 of the Appendix. For fixed $\eta\in[0,1]$ and fixed ω, denote the (pointwise) population distances and empirical distances of the spectral density operators by

$$M_{\mathscr{F}}(\omega):=\mathscr{F}_X^{(\omega)}-\mathscr{F}_Y^{(\omega)}\quad\text{and}\quad \widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega):=\eta\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\hat{\mathscr{F}}_Y^{(\omega)}(\eta)\big),$$

and observe that under Assumption 3.1, these are both well-defined elements of $S_1(\mathscr{H})$ for any $\eta\in[0,1]$. The next step is to define a process which quantifies the difference between the empirical and population measures over a given frequency band, i.e.,

$$\hat{\mathcal{Z}}_{\mathscr{F},T_1,T_2}^{[a,b]}(\eta):=\int_a^b \Big|\Big|\Big|\eta\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\hat{\mathscr{F}}_Y^{(\omega)}(\eta)\big)\Big|\Big|\Big|_2^2-\eta^2\Big|\Big|\Big|\mathscr{F}_X^{(\omega)}-\mathscr{F}_Y^{(\omega)}\Big|\Big|\Big|_2^2 d\omega. \tag{3.11}$$

Elementary calculations show that we can write (3.11) as

$$\begin{aligned}\hat{\mathcal{Z}}_{\mathscr{F},T_1,T_2}^{[a,b]}(\eta)&=\int_a^b \Big|\Big|\Big|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)\Big|\Big|\Big|_2^2-\eta^2\Big|\Big|\Big|M_{\mathscr{F}}(\omega)\Big|\Big|\Big|_2^2 d\omega\\ &=\int_a^b\Big\{\Big|\Big|\Big|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)\Big|\Big|\Big|_2^2+\Big\langle \widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega),\eta M_{\mathscr{F}}(\omega)\Big\rangle_{S_2}\\ &\qquad+\overline{\Big\langle \widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega),\eta M_{\mathscr{F}}(\omega)\Big\rangle_{S_2}}\Big\}d\omega.\end{aligned} \tag{3.12}$$

Moreover, notice that

$$\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)=\eta\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}\big)-\eta\big(\hat{\mathscr{F}}_Y^{(\omega)}(\eta)-\mathscr{F}_Y^{(\omega)}\big)\,. \tag{3.13}$$

The following result, which requires to control the maximum of partial sums of (3.8) and (3.9), shows that the first term of (3.12) is of smaller order than the two other terms.

Lemma 3.1. *Suppose Assumptions 3.1-3.4 are satisfied. Then*

$$\sup_{\eta\in[0,1]}\int_a^b \left|\!\left|\!\left|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)\right|\!\right|\!\right|_2^2 d\omega = o_P\Big(\frac{1}{\sqrt{b_1T_1+b_2T_2}}\Big)\,. \tag{3.14}$$

The next statement in turn then shows that we can approximate the process in (3.12) as a linear combination of functionals of processes of the form in (3.2) and (3.3).

Theorem 3.2. *Suppose Assumptions 3.1-3.4 are satisfied. Then*

$$\begin{aligned}\sqrt{b_1T_1+b_2T_2}\,\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(\eta) &= \sqrt{b_1T_1+b_2T_2}\int_a^b \frac{2}{\sqrt{b_1T_1}}\Re\Big\langle \mathcal{Z}^{X,\omega}_{T,\eta},\eta M_{\mathscr{F}}(\omega)\Big\rangle_{S_2}d\omega\\ &\quad-\sqrt{b_1T_1+b_2T_2}\int_a^b \frac{2}{\sqrt{b_2T_2}}\Re\Big\langle \mathcal{Z}^{Y,\omega}_{T,\eta},\eta M_{\mathscr{F}}(\omega)\Big\rangle_{S_2}d\omega+o_P(1).\end{aligned}$$

We can now use Theorem 3.1 with $\mathscr{U}^{(\omega)}_{XY}=\mathscr{U}^{(\omega)}_{YX}=2M_{\mathscr{F}}(\omega)$ and Theorem 3.2 to find the limiting distribution of the process in (3.11), that is

$$\big\{\sqrt{b_1T_1+b_2T_2}\big(\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(\eta)\big)\big\}_{\eta\in[0,1]}\rightsquigarrow\tau_{\mathscr{F}}\{\eta\mathbb{B}(\eta)\}_{\eta\in[0,1]},\quad \text{as } T_1,T_2\to\infty\,, \tag{3.15}$$

where $\tau_{\mathscr{F}}$ is a constant. To make the test independent of $\tau_{\mathscr{F}}$, we consider the following self-normalizing approach, which is similar in nature to Dette et al. (2020). To be precise, define the statistic

$$\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}=\Big(\int_0^1\Big(\int_a^b \left|\!\left|\!\left|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)\right|\!\right|\!\right|_2^2-\eta^2\left|\!\left|\!\left|\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)\right|\!\right|\!\right|_2^2 d\omega\Big)^2\nu(d\eta)\Big)^{1/2}, \tag{3.16}$$

where ν is a probability measure on the interval $(0,1)$. Then it is easy to see that

$$\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}=\Big(\int_0^1\big(\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(\eta)-\eta^2\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(1)\big)^2\nu(d\eta)\Big)^{1/2}$$

and the continuous mapping theorem and the weak convergence (3.15) imply

$$\sqrt{b_1T_1+b_2T_2}\Big(\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(1),\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}\Big)\underset{T_1,T_2\to\infty}{\Rightarrow}\Big(\tau_{\mathscr{F}}\mathbb{B}(1),\Big(\int_0^1\tau^2_{\mathscr{F}}\eta^2\Big(\mathbb{B}(\eta)-\eta\mathbb{B}(1)\Big)^2\nu(d\eta)\Big)^{1/2}\Big). \tag{3.17}$$

Consequently, a further application of the continuous mapping theorem yields

$$\frac{\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(1)}{\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}}\underset{T_1,T_2\to\infty}{\Rightarrow}\mathbb{D}:=\frac{\mathbb{B}(1)}{\big(\int_0^1\eta^2\big(\mathbb{B}(\eta)-\eta\mathbb{B}(1)\big)^2\nu(d\eta)\big)^{1/2}} \tag{3.18}$$

whenever $\tau_{\mathscr{F}}\neq 0$. From this, we can obtain a pivotal test statistic for the hypothesis (2.1) of no relevant difference between the spectral density operators given by

$$\widehat{\mathbb{D}}^{[a,b]}_{T_1,T_2}=\frac{\int_a^b|||\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)|||_2^2d\omega-\Delta}{\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}}, \tag{3.19}$$

and a natural decision rule is then to reject the null hypothesis in (2.1) whenever

$$\hat{\mathbb{D}}^{[a,b]}_{T_1,T_2}>q_{1-\alpha}(\mathbb{D})\quad\Leftrightarrow\quad\int_a^b|||\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)|||_2^2d\omega>\Delta+\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}\cdot q_{1-\alpha}(\mathbb{D}) \tag{3.20}$$

where $q_{1-\alpha}(\mathbb{D})$ denotes the $(1-\alpha)$-th quantile of the distribution of the random variable $\mathbb{D}$ defined in (3.18). Consequently, the test no longer depends on the unknown nuisance parameter but only on the measure ν used in the definition of the self-normalizing factor $\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}$, which can be chosen by the statistician and is therefore known. Observe further that the quantiles $q_{1-\alpha}(\mathbb{D})$ are straightforward to simulate. The next result now shows that the test in (3.20) provides a consistent and asymptotic level α test.

Theorem 3.3. *Suppose Assumptions 3.1-3.4 are satisfied. Then the decision rule* (3.20) *provides an asymptotic level α test for the hypothesis* (2.1) *of no relevant difference between the spectral density operators $\mathscr{F}^{(\cdot)}_X$ and $\mathscr{F}^{(\omega)}_Y$, i.e.,*

$$\lim_{T_1,T_2\to\infty} \mathbb{P}\Big(\hat{\mathbb{D}}^{[a,b]}_{T_1,T_2} > q_{1-\alpha}(\mathbb{D})\Big) = \begin{cases} 0 & \textit{if } \Delta > \int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega; \\ \alpha & \textit{if } \Delta = \int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega \textit{ and } \tau_{\mathscr{F}} \neq 0; \\ 1 & \textit{if } \Delta < \int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega. \end{cases}$$

Proof. Suppose first that $\int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega = 0$. Then (3.11) becomes $\hat{Z}^{[a,b]}_{\mathscr{F},T_1,T_2}(1) = \int_a^b |||\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)|||_2^2 d\omega$. By (3.17), we have $\hat{Z}^{[a,b]}_{\mathscr{F},T_1,T_2}(1) = o_P(1)$ and $\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2} = o_P(1)$ as $T_1, T_2 \to \infty$. Consequently, we obtain $\lim_{T_1,T_2\to\infty} \mathbb{P}\Big(\hat{\mathbb{D}}^{[a,b]}_{T_1,T_2} > q_{1-\alpha}(\mathbb{D})\Big) = 0$. Next, suppose $\int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega > 0$. In this case, we can write

$$\begin{aligned} &\mathbb{P}\Big(\hat{\mathbb{D}}^{[a,b]}_{T_1,T_2} > q_{1-\alpha}(\mathbb{D})\Big) && (3.21) \\ &= \mathbb{P}\Big(\int_a^b |||\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)|||_2^2 d\omega - \int_a^b |||M_{\mathscr{F}}(\omega)|||_2^2 d\omega > \Delta - \int_a^b |||M_{\mathscr{F}}(\omega)|||_2^2 d\omega + q_{1-\alpha}(\mathbb{D})\, \hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}\Big). \end{aligned}$$

From (3.17) it follows that

$$\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(1) = \int_a^b |||\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)|||_2^2 d\omega - \int_a^b |||M_{\mathscr{F}}(\omega)|||_2^2 d\omega = o_P(1)\ , \quad \hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2} = o_P(1),$$

and consequently the assertion in the cases $\Delta > \int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega$ and $\Delta < \int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega$ follows easily. Finally, if $\tau_{\mathscr{F}} \neq 0$ we have from (3.18) that

$$\frac{\int_a^b |||\widehat{M}_{\hat{\mathscr{F}}}(1,\omega)|||_2^2 d\omega - \int_a^b |||M_{\mathscr{F}}(\omega)|||_2^2 d\omega}{\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}} = \frac{\hat{\mathcal{Z}}^{[a,b]}_{\mathscr{F},T_1,T_2}(1)}{\hat{V}^{[a,b]}_{\mathscr{F},T_1,T_2}} \underset{T_1,T_2\to\infty}{\Rightarrow} \mathbb{D},$$

and we obtain the remaining case $\Delta = \int_a^b |||M^{(\omega)}_{\mathscr{F}}|||_2^2 d\omega$ from (3.21). □

3.2 No relevant difference in the eigenprojectors: hypothesis (2.2)

In this section, we construct a pivotal test for hypothesis (2.2) of no relevant difference between the eigenprojectors $\Pi^{(\omega)}_{X,k}$ and $\Pi^{(\omega)}_{Y,k}$ of the functional time series $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$. The development is of a more intricate nature than for the spectral density operators, which we will elaborate upon. Proofs of the statements provided in this section are relegated to Appendix A. To ease notation, denote the (pointwise) population distances and empirical distances of the k-th eigenprojectors at frequency ω by

$$M_{\Pi,k}(\omega) := \Pi^{(\omega)}_{X,k} - \Pi^{(\omega)}_{Y,k} \quad \text{and} \quad \widehat{M}_{\Pi,k}(\eta,\omega) := \eta\big(\hat{\Pi}^{(\omega)}_{X,k}(\eta,\omega) - \hat{\Pi}^{(\omega)}_{Y,k}(\eta)\big), \quad \eta \in [0,1].$$

As already briefly mentioned in Section 2, we construct a test based on the eigenprojectors rather than on the eigenfunctions because the latter are only defined up to some multiplicative factor c

on the unit circle. To understand the problem, suppose for simplicity that we would like to test for relevant differences in the k-th eigenspace of $\mathscr{F}_X^{(\omega)}$ and $\mathscr{F}_Y^{(\omega)}$. From the estimators in (3.8) and (3.9) we can obtain $c_1\hat{\phi}_{X,k}^{(\omega)}$ and $c_2\hat{\phi}_{Y,k}^{(\omega)}$ for some unknown $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2 = |c_2|^2 = 1$. The empirical eigenfunctions might therefore not be comparable due the unknown rotation in different directions. Moreover, a consequence of this rotation is that a bound on the differences in norm between the population and empirical eigenfunctions does not follow from those of the corresponding operators. This is in contrast with eigenfunctions that strictly belong to the real-valued subspace of $\mathscr{H}$, of which only the sign is unknown for the empirical counterparts. We can however construct a test using the eigenprojectors because these are rotationally invariant since $\hat{\Pi}_{X,k}^{(\omega)} = c_1\overline{c_1}\hat{\phi}_{X,k}^{(\omega)} \otimes \hat{\phi}_{X,k}^{(\omega)}$. As a consequence, $\hat{\Pi}_{X,k}^{(\omega)}$ and $\hat{\Pi}_{Y,k}^{(\omega)}$ are directly comparable and a bound on the differences in norm between the population and empirical eigenprojectors can be derived. The derivation of the theoretical properties of a test based upon eigenprojectors is however more involved than one based upon eigenfunctions (see also Aue et al., 2019, who developed self-normalized tests for relevant changes in the eigenfunctions of the covariance operator).

We start by defining the process

$$\hat{\mathscr{Z}}_{\Pi,T_1,T_2}^{[a,b],(k)}(\eta) = \int_a^b \left|\left|\left|\widehat{M}_{\Pi,k}(\eta,\omega)\right|\right|\right|_2^2 - \eta^2 \left|\left|\left|M_{\Pi,k}(\omega)\right|\right|\right|_2^2 d\omega \tag{3.22}$$

and observe in this case that

$$\widehat{M}_{\Pi,k}(\eta,\omega) - \eta M_{\Pi,k}(\omega) = \eta\Big(\hat{\Pi}_{X,k}^{(\omega)}(\eta) - \Pi_{X,k}^{(\omega)}\Big) - \eta\Big(\hat{\Pi}_{Y,k}^{(\omega)}(\eta) - \Pi_{Y,k}^{(\omega)}\Big).$$

Unlike the terms in (3.13), the properties of the two terms of the right-hand side are not obvious to disentangle. Using perturbation theory on the Kronecker tensors products, we can obtain the following expansion for the sequential eigenprojectors.

Proposition 3.2. *Let $\mathscr{F}^{\omega}$ be the spectral density operator of a weakly stationary functional time series with eigendecomposition $\sum_{k=1}^{\infty} \lambda_i^{(\omega)} \Pi_i^{\omega}$ and suppose the first k eigenvalues of $\mathscr{F}^{\omega}$ are all distinct and positive. Furthermore, let $\{\hat{\lambda}_i^{(\omega)}(\eta)\}_{i\geq 1}$, $\{\hat{\Pi}_i^{(\omega)}(\eta)\}_{i\geq 1}$ be the sequence of eigenvalues and eigenprojectors, respectively, of the sequential estimators $\hat{\mathscr{F}}^{\omega}(\eta)$, $\eta \in [0,1]$, of $\mathscr{F}^{\omega}$. Then*

$$\begin{aligned}\hat{\Pi}_k^{(\omega)} - \Pi_k^{(\omega)} &= \Big\langle \hat{\Pi}_k^{\omega}(\eta) - \Pi_k^{\omega}, \Pi_k^{\omega} \Big\rangle_{S_2} \Pi_k^{(\omega)} \\ &+ \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty} \frac{1}{\lambda_j^{(\omega)}\lambda_{j'}^{(\omega)} - (\lambda_k^{(\omega)})^2} \Big[\Big\langle \big(\mathscr{F}^{\omega} \widetilde{\otimes} \mathscr{F}^{\omega} - \hat{\mathscr{F}}^{\omega}(\eta) \widetilde{\otimes} \hat{\mathscr{F}}^{\omega}(\eta)\big)\Pi_k^{(\omega)}, \Pi_{jj'}^{(\omega)} \Big\rangle_{S_2} \\ &\qquad + \Big\langle E_{k,b_T}^{(\omega)}(\eta), \Pi_{jj'}^{(\omega)} \Big\rangle_{S_2} \Big] \Pi_{jj'}^{(\omega)}\end{aligned}$$

where $\{\cdot\}^{\complement}$ denotes the complement set, $\Pi_{jj'}^{(\omega)} = \phi_j^{(\omega)} \otimes \phi_{j'}^{(\omega)}$, and where

$$E_{k,b_T}^{(\omega)}(\eta) = \Big[(\mathscr{F}^{\omega} \widetilde{\otimes} \mathscr{F}^{\omega} - \hat{\mathscr{F}}^{\omega}(\eta) \widetilde{\otimes} \hat{\mathscr{F}}^{\omega}(\eta)) \Big] \Big[\hat{\Pi}_k^{(\omega)}(\eta) - \Pi_k^{(\omega)} \Big] + \Big[(\hat{\lambda}_k^{(\omega)}(\eta))^2 - (\lambda_k^{(\omega)})^2 \Big] \Big[\hat{\Pi}_k^{(\omega)}(\eta) - \Pi_k^{(\omega)} \Big].$$

In order to make sure the above expansion is well-defined for the eigenprojectors $\Pi_{X,k}^{(\omega)}$ and $\Pi_{Y,k}^{(\omega)}$ we require the following assumption on the eigenvalues.

Assumption 3.5. *The first $k_0 > 1$ eigenvalues of $\mathscr{F}_X^{\omega}$ and $\mathscr{F}_Y^{(\omega)}$ satisfy $\lambda_{X,1}^{(\omega)} > \ldots > \lambda_{X,k_0}^{(\omega)} > 0$ and $\lambda_{Y,1}^{(\omega)} > \ldots > \lambda_{Y,k_0}^{(\omega)} > 0$, respectively.*

This assumption guarantees separability of the eigenvalues and ensures that we can test for relevant differences in the first k_0-1 eigenprojectors. Even though the above expansion expression is quite involved, the next statement shows that the properties are controlled by a functional of a stochastic process, $\widetilde{M}_{\Pi,k}(\eta,\omega)$ that we will be able to link again to a process of the form (3.1).

Lemma 3.2. *Suppose Assumptions 3.1-3.4 are satisfied and let $\mathcal{Z}^{X,\omega}_{T,\eta}$ and $\mathcal{Z}^{Y,\omega}_{T,\eta}$ be given by* (3.2) *and* (3.3)*, respectively. Suppose furthermore that Assumption 3.5 holds true. Then,*

$$\text{(i)} \sup_{\eta\in[0,1]}\int_a^b \left\|\!\left\|\widehat{M}_{\Pi,k}(\eta,\omega)-\eta\Big(\Pi^{(\omega)}_{X,k}-\Pi^{(\omega)}_{Y,k}\Big)-\widetilde{M}_{\Pi,k}(\eta,\omega)\right\|\!\right\|_2 d\omega = o_P\Big(\frac{1}{\sqrt{b_1T_1+b_2T_2}}\Big);$$

$$\text{(ii)} \sup_{\eta\in[0,1]}\int_a^b \left\|\!\left\|\widetilde{M}_{\Pi,k}(\eta,\omega)\right\|\!\right\|_2^2 d\omega = o_P\Big(\frac{1}{\sqrt{b_1T_1+b_2T_2}}\Big),$$

where

$$\begin{aligned}\widetilde{M}_{\Pi,k}(\eta,\omega) = {}& \frac{1}{\sqrt{b_1T_1}}\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{1}{\lambda^{(\omega)}_{X,j}\lambda^{(\omega)}_{X,j'}-(\lambda^{(\omega)}_{X,k})^2}\Big\langle \mathscr{F}^{(\omega)}_X\widetilde{\otimes}\mathcal{Z}^{X,\omega}_{T,\eta}+\mathcal{Z}^{X,\omega}_{T,\eta}\widetilde{\otimes}\mathscr{F}^{(\omega)}_X,\Pi^{(\omega)}_{X,jj'}\widetilde{\otimes}\Pi^{(\omega)}_{X,k}\Big\rangle_{S_2}\Pi^{(\omega)}_{X,jj'}\\ &-\frac{1}{\sqrt{b_2T_2}}\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{1}{\lambda^{(\omega)}_{Y,j}\lambda^{(\omega)}_{Y,j'}-(\lambda^{(\omega)}_{Y,k})^2}\Big\langle \mathscr{F}^{(\omega)}_Y\widetilde{\otimes}\mathcal{Z}^{Y,\omega}_{T,\eta}+\mathcal{Z}^{Y,\omega}_{T,\eta}\widetilde{\otimes}\mathscr{F}^{(\omega)}_Y,\Pi^{(\omega)}_{Y,jj'}\widetilde{\otimes}\Pi^{(\omega)}_{Y,k}\Big\rangle_{S_2}\Pi^{(\omega)}_{X,jj'}.\end{aligned} \tag{3.23}$$

The proof of this result is involved and left to Section A.1 of the Appendix. The expression (3.23) follows from the definition of $\mathcal{Z}^{X,\omega}_{T,\eta}$ and $\mathcal{Z}^{Y,\omega}_{T,\eta}$ together with an application of Lemma B.2 in the Appendix. This subsequently allows us to establish that the process $\hat{\mathcal{Z}}^{[a,b],(k)}_{\Pi,T_1,T_2}$ in (3.22) admits a stochastic expansion of the form as given in Theorem 3.1, from which its distributional properties can be obtained.

Theorem 3.4. *Suppose Assumptions 3.1-3.5 are satisfied. Then*

$$\hat{\mathcal{Z}}^{[a,b],(k)}_{\Pi,T_1,T_2}(\eta)=\eta\int_a^b\Big(\frac{1}{\sqrt{b_2T_2}}\Re\Big\langle\mathcal{Z}^{Y,\omega}_{T,\eta},\widetilde{\Pi}^{(\omega)}_{Y,X,k}\Big\rangle_{S_2}+\frac{1}{\sqrt{b_1T_1}}\Re\Big\langle\mathcal{Z}^{X,\omega}_{T,\eta},\widetilde{\Pi}^{(\omega)}_{X,Y,k}\Big\rangle_{S_2}\Big)d\omega+o_P\Big(\frac{1}{\sqrt{b_1T_1+b_2T_2}}\Big),$$

where

$$\widetilde{\Pi}^{(\omega)}_{X,Y,k}:=4\sum_{j\neq k}\frac{\lambda^{(\omega)}_{X,k}\Re\Big\langle\Pi^{(\omega)}_{X,kj},\Pi^{(\omega)}_{Y,k}\Big\rangle_{S_2}}{\lambda^{(\omega)}_{X,k}\lambda^{(\omega)}_{X,j}-(\lambda^{(\omega)}_{X,k})^2}\Pi^{(\omega)}_{X,jk},\quad \textit{and}\quad \widetilde{\Pi}^{(\omega)}_{Y,X,k}:=4\sum_{j\neq k}\frac{\lambda^{(\omega)}_{Y,k}\Re\Big\langle\Pi^{(\omega)}_{Y,kj},\Pi^{(\omega)}_{X,k}\Big\rangle_{S_2}}{\lambda^{(\omega)}_{Y,k}\lambda^{(\omega)}_{Y,j}-(\lambda^{(\omega)}_{Y,k})^2}\Pi^{(\omega)}_{Y,jk}.$$

By Theorem 3.1 with $\mathscr{U}^{(\omega)}_{XY}=-\widetilde{\Pi}^{(\omega)}_{X,Y,k}$, $\mathscr{U}^{(\omega)}_{YX}=-\widetilde{\Pi}^{(\omega)}_{Y,X,k}$ and Theorem 3.4 we obtain the weak convergence

$$\big\{\sqrt{b_1T_1+b_2T_2}\big(\hat{\mathcal{Z}}^{[a,b],(k)}_{\Pi,T_1,T_2}(\eta)\big)\big\}_{\eta\in[0,1]}\rightsquigarrow\tau_\Pi\{\eta\mathbb{B}(\eta)\}_{\eta\in[0,1]},\quad \text{as } T_1,T_2\to\infty,$$

for some constant τ_Π and an application of the continuous mapping theorem shows that

$$\frac{\hat{\mathcal{Z}}^{[a,b],(k)}_{\Pi,T_1,T_2}(1)}{\hat{V}^{[a,b],(k)}_{\Pi,T_1,T_2}}\underset{T_1,T_2\to\infty}{\Rightarrow}\mathbb{D},$$

where the random variable $\mathbb{D}$ is defined in (3.18) and the normalizing factor $\hat{V}^{[a,b],(k)}_{\Pi,T_1,T_2}$ is given by

$$\hat{V}^{[a,b],(k)}_{\Pi,T_1,T_2} = \Big(\int_0^1 \Big(\int_a^b \Big\|\Big\|\widehat{M}_{\Pi,k}(\eta,\omega)\Big\|\Big\|_2^2 - \eta^2 \Big\|\Big\|\widehat{M}_{\Pi,k}(1,\omega)\Big\|\Big\|_2^2 d\omega\Big)^2 \nu(d\eta)\Big)^{1/2}.$$

Combining these findings with the arguments given in the proof of Theorem 3.3 yields a consistent and asymptotic level α test for the hypothesis (2.2) of no relevant difference in the k-th eigenprojector.

Theorem 3.5. *Suppose Assumptions 3.1-3.5 satisfied. Then the test which rejects the null hypothesis in* (2.2) *of no relevant difference in the k-th eigenprojector whenever*

$$\widehat{\mathbb{D}}^{[a,b],k}_{\Pi,T_1,T_2} = \frac{\int_a^b \|\|\widehat{M}_{\Pi,k}(1,\omega)\|\|_2^2 d\omega - \Delta_{\Pi,k}}{\hat{V}^{[a,b],k}_{\Pi,T_1,T_2}} > q_{1-\alpha}(\mathbb{D})$$

is consistent and has asymptotic level α, i.e.,

$$\lim_{T_1,T_2\to\infty} \mathbb{P}\Big(\widehat{\mathbb{D}}^{[a,b],k}_{\Pi,T_1,T_2} > q_{1-\alpha}(\mathbb{D})\Big) = \begin{cases} 0 & \text{if } \Delta_{\Pi,k} > \int_a^b \|\|M^{(\omega)}_{\Pi,k}\|\|_2^2 d\omega; \\ \alpha & \text{if } \Delta_{\Pi,k} = \int_a^b \|\|M^{(\omega)}_{\Pi,k}\|\|_2^2 d\omega \text{ and } \tau_\Pi \neq 0; \\ 1 & \text{if } \Delta_{\Pi,k} < \int_a^b \|\|M^{(\omega)}_{\Pi,k}\|\|_2^2 d\omega. \end{cases}$$

3.3 No relevant difference in the eigenvalues: hypothesis (2.3)

Finally, we briefly discuss the test for the hypothesis in (2.3) of no relevant difference in the k-th eigenvalue. Denote the (pointwise) population distances and empirical distances of the k-th largest eigenvalues at frequency ω by

$$M_{\lambda,k}(\omega) := \lambda^{(\omega)}_{X,k} - \lambda^{(\omega)}_{Y,k} \quad \text{and} \quad \widehat{M}_{\lambda,k}(\eta,\omega) := \eta\big(\hat{\lambda}^{(\omega)}_{X,k}(\eta) - \hat{\lambda}^{(\omega)}_{Y,k}(\eta)\big) \quad \eta[0,1],$$

and define

$$\hat{\mathcal{Z}}^{[a,b],(k)}_{\lambda,T_1,T_2}(\eta) = \sqrt{b_1 T_1 + b_2 T_2} \int_a^b |\widehat{M}_{\lambda,k}(\eta,\omega)|^2 - \eta^2 |M_{\lambda,k}(\omega)|^2 d\omega. \tag{3.24}$$

We make use of the following proposition, which is proved in Section A.2 of the Appendix.

Proposition 3.3. *Let $\mathscr{F}^\omega$ have eigendecomposition $\sum_{k=1}^\infty \lambda^{(\omega)}_k \Pi^\omega_k$ and let $\big\{\hat{\lambda}^{(\omega)}_k(\eta)\big\}_{k\geq 1}$ $\big\{\hat{\Pi}^{(\omega)}_k(\eta)\big\}_{k\geq 1}$ be the sequence of eigenvalues and eigenprojectors , respectively, of $\hat{\mathscr{F}}^\omega(\eta)$, $\eta \in [0,1]$. Then,*

$$\hat{\lambda}^{(\omega)}_k(\eta) - \lambda^{(\omega)}_k = \Big\langle \hat{\mathscr{F}}^\omega(\eta) - \mathscr{F}^\omega, \Pi^{(\omega)}_k \Big\rangle_{S_2} + \Big\langle E^{(\omega)}_{\lambda,k,b_T}(\eta), \Pi^{(\omega)}_k \Big\rangle_{S_2},$$

where

$$E^{(\omega)}_{\lambda,k,b_T}(\eta) = \big(\hat{\mathscr{F}}^\omega(\eta) - \mathscr{F}^\omega\big)\big(\hat{\Pi}^{(\omega)}_k(\eta) - \Pi^{(\omega)}_k \eta\big) - \big(\hat{\lambda}^{(\omega)}_k(\eta) - \lambda^{(\omega)}_k\big)\big(\hat{\Pi}^{(\omega)}_k(\eta) - \Pi^{(\omega)}_k \eta\big).$$

The following theorem is the counterpart of Theorem 3.2 and Theorem 3.4, and shows that we can express (3.24) into a process of the form (3.1).

Theorem 3.6. *Suppose Assumptions 3.1-3.4 are satisfied. Then*

$$\begin{aligned}\hat{\mathcal{Z}}^{[a,b],(k)}_{\lambda,T_1,T_2}(\eta) = \int_a^b \Big(&\frac{2\eta M_{\lambda,k}(\omega)}{\sqrt{b_1 T_1}} \Re\Big\langle \mathcal{Z}^{X,\omega}_{T,\eta}, \Pi^{(\omega)}_{X,k}\Big\rangle_{S_2} - \frac{2\eta M_{\lambda,k}(\omega)}{\sqrt{b_2 T_2}} \Re\Big\langle \mathcal{Z}^{Y,\omega}_{T,\eta}, \Pi^{(\omega)}_{Y,k}\Big\rangle_{S_2}\Big) d\omega \\ &+ o_P\Big(\frac{1}{\sqrt{b_1 T_1 + b_2 T_2}}\Big),\end{aligned}$$

In this case, an application of Theorem 3.1 with $\mathscr{U}_{XY}^{(\omega)} = 2M_{\lambda,k}(\omega)\Pi_{X,k}^{(\omega)}$ and $\mathscr{U}_{YX}^{(\omega)} = 2M_{\lambda,k}(\omega)\Pi_{Y,k}^{(\omega)}$ yields therefore

$$\left\{\sqrt{b_1T_1+b_2T_2}\left(\hat{\mathscr{Z}}_{\lambda,T_1,T_2}^{[a,b],(k)}(\eta)\right)\right\}_{\eta\in[0,1]} \rightsquigarrow \tau_\lambda\{\eta\mathbb{B}(\eta)\}_{\eta\in[0,1]}, \quad \text{as } T_1,T_2\to\infty,$$

for some constant τ_λ (see Section 5 for details). Now using the same arguments as in Section 3.1 and 3.2 we obtain that the test, which rejects the null hypothesis of no relevant difference in the k-th eigenvector in (2.3), whenever

$$\widehat{\mathbb{D}}_{\lambda,T_1,T_2}^{[a,b],k} = \frac{\int_a^b |||\widehat{M}_{\lambda,k}(1,\omega)|||_2^2 d\omega - \Delta_{\lambda,k}}{\hat{V}_{\lambda,T_1,T_2}^{[a,b],k}} > q_{1-\alpha}(\mathbb{D}) \, ,$$

is consistent and has asymptotic levek α. The proof is omitted for the sake of brevity.

Theorem 3.7. *Suppose Assumptions 3.1-3.4 are satisfied. Then,*

$$\lim_{T_1,T_2\to\infty} \mathbb{P}\left(\widehat{\mathbb{D}}_{\lambda,T_1,T_2}^{[a,b],k} > q_{1-\alpha}(\mathbb{D})\right) = \begin{cases} 0 & \text{if } \Delta_{\lambda,k} > \int_a^b |||M_{\lambda,k}^{(\omega)}|||_2^2 d\omega; \\ \alpha & \text{if } \Delta_{\lambda,k} = \int_a^b |||M_{\lambda,k}^{(\omega)}|||_2^2 d\omega \text{ and } \tau_\lambda \neq 0; \\ 1 & \text{if } \Delta_{\lambda,k} < \int_a^b |||M_{\lambda,k}^{(\omega)}|||_2^2 d\omega. \end{cases}$$

4 Finite sample properties

In this section, we report the results of a simulation study conducted to asses the finite sample properties of the tests proposed in Section 3.1-3.3. In all scenarios, the empirical rejection probabilities are calculated over 1000 repetitions and the functional processes are generated on a grid of 1000 equispaced points in the interval $[0,1]$ and then converted into functional data objects using a Fourier basis. In order to define the self-normalization sequence we used the measure $\nu = \frac{1}{n-1}\sum_{i=1}^{n-1}\delta_{i/n}$, where δ_η denotes the Dirac measure at $\eta \in [0,1]$. Simulations reported below are conducted with $n = 20$. Other values were also considered but we found comparable results for all other choices of n for which the positive mass is sufficiently bounded away from the boundaries. In order to provide the relevant tests, it is important to be able to trace back the distances $\Delta, \Delta_{\Pi,k}$ and $\Delta_{\lambda,k}$, which is not always an obvious task as explicit expressions of the eigenelements of the spectral density operator can be notoriously hard to find. In all simulations, the estimator of the spectral density operator is obtained using a Daniell window with bandwidth $b_T = T^{-1/3}$. The result are given for $a = 0$, $b = \pi$ but tests were also done pointwise, with similar results.

In the first setting, we generate a sequence $\{X_t\}_{t=1}^T$ of independent Brownian bridges with variance upscaled by a factor 2π. It can be shown that the eigenvalues and eigenfunctions of the spectral density operator $2\pi\mathscr{F}_X^{(\omega)}$ are respectively given by $\lambda_{X,k}^{(\omega)} = 1/(\pi k)^2$and $\phi_{X,k}^{\omega}(\tau) = \sqrt{2}\sin(\pi k\tau), \tau \in [0,1], k = 1,2,3,\ldots$, for all $\omega \in \mathbb{R}$. The number of basis functions is chosen to be $d = 21$, which captures more than 95 percent of variation.

- **Scenario 1: shift in the eigenfunctions**. We generate processes $\{Y_t\}_{t=1}^T$ from independent Brownian bridges, with the variance again upscaled with a factor 2π. However, here the first eigenfunction is shifted to $\sqrt{2}\sin(\pi k(\tau+\iota))$ with ι varying between 0 and 0.15. This leads to a change in various eigenfunctions of the spectral density operator and hence to a change in the operator itself. The empirical rejection probabilities corresponding to a true shift $\iota = 0.05$ are depicted in Figure 4.1 for the relevant hypotheses on the spectral density operators (Theorem 3.3) and on the first and second eigenprojectors (Theorem 3.5). This particular shift corresponds to respective relevant hypotheses with $\Delta \approx 0.00047$ (left), $\Delta_{\Pi,1} \approx 0.0474$ (middle) and $\Delta_{\Pi,2} \approx 0.040$ (right). The behavior visible in the three plots clearly corroborates with the theoretical findings stated in Theorem 3.3 and Theorem 3.5, respectively. For shifts belonging to the interior of the null hypothesis, i.e., $\iota < 0.05$, we observe that the empirical rejection probabilities are below the nominal level and are getting closer to zero as the shift gets close to zero. For those values that belong to the interior of the alternative, i.e, $\iota > 0.05$, we observe empirical rejection probability strictly larger than the nominal level and which increase to 1 as the size of the shift increases. At the boundary of the null hypothesis, i.e., where $\iota = 0.05$, the test is close to the nominal level of $\alpha = 0.05$. One moreover finds that this behavior improves as the sample size T increases.

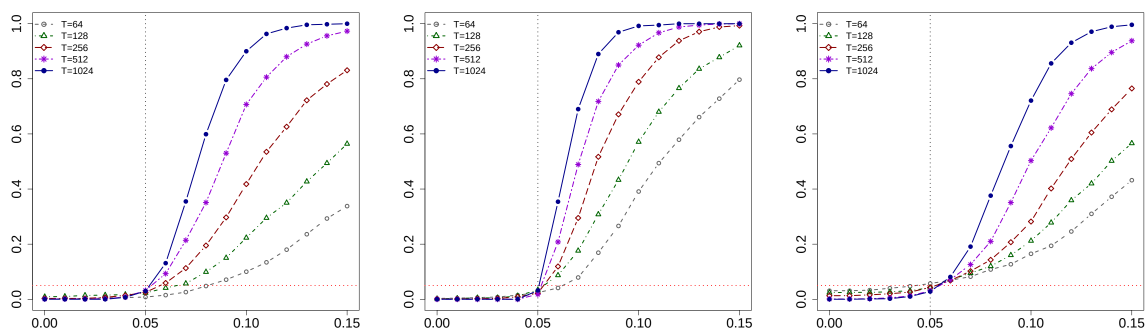

Figure 4.1: *Empirical rejection probabilities under scenario 1 for the relevant hypotheses in Section 3.1 (left panel) and Section 3.2 for the first and second eigenprojectors (middle and right panel) as a function of* ι *at the nominal level 0.05 (horizontal red dotted line). The shift of* $\iota = 0.05$ *is marked in the three panels by the vertical dotted line and corresponds to the respective relevant hypotheses* $\Delta \approx 0.00047$ *(left) and* $\Delta_{\Pi,1} \approx 0.047$ *(middle) and* $\Delta_{\Pi,2} \approx 0.040$ *(right).*

- **Scenario 2: amplitude variation**. We generate processes $\{Y_{\iota,t}\}_{t=1}^{T}$ from independent Brownian bridges multiplied by a factor $\sqrt{2\pi}$ but where the standard deviation is multiplied by a factor $1.2^{\iota}, \iota = 0,1,\dots 8$. Empirical rejection probabilities for $\iota = 3$ are depicted in Figure 4.2 for the relevant hypotheses on the spectral density operators (Theorem 3.3) and the first and second eigenvalues (Theorem 3.7). We observe that all tests behave as prescribed by the theory, where the precision is quite accurate, even for the smallest sample size.

In the second setting, we consider processes of the form

$$X_t(\tau) = \sum_{j=1}^{2} \chi_{t,1,j}\sqrt{2}\sin(2\pi\tau j) + \chi_{t,2,j}\sqrt{2}\cos(2\pi(\tau j + \iota_j)) \quad \tau \in [0,1], \tag{4.1}$$

where the coefficients $\boldsymbol{\chi}_t := (\chi_{t,1,1}, \chi_{t,2,1}, \chi_{t,1,2}, \chi_{t,2,2})$ are generated from a vector autoregressive process, i.e., $\boldsymbol{\chi}_t = c\boldsymbol{\chi}_{t-1} + \sqrt{1-c^2}\epsilon_t$ with $\epsilon_t \in \mathbb{R}^4$. In the following simulations, we fix $\epsilon_t \in \mathbb{R}^4 \sim \mathcal{N}(\mathbf{0}, \mathrm{diag}(4,8,0.5,1.5))$ and vary the strength of dependence and shift the eigenfunction belonging to the largest eigenvalue.

- **Scenario 3: shift in the eigenfunctions**. We generate $\{X_t\}_{t=1}^{T}$ from model (4.1) with $c = .3$ and $\iota_j = 0, j = 1,2$ and do the same for processes $\{Y_t\}_{t=1}^{T}$. However, for $\{Y_t\}_{t=1}^{T}$, ι_1 is varied between 0 and 0.25. Figure 4.3 depicts the empirical rejection probabilities for the test of

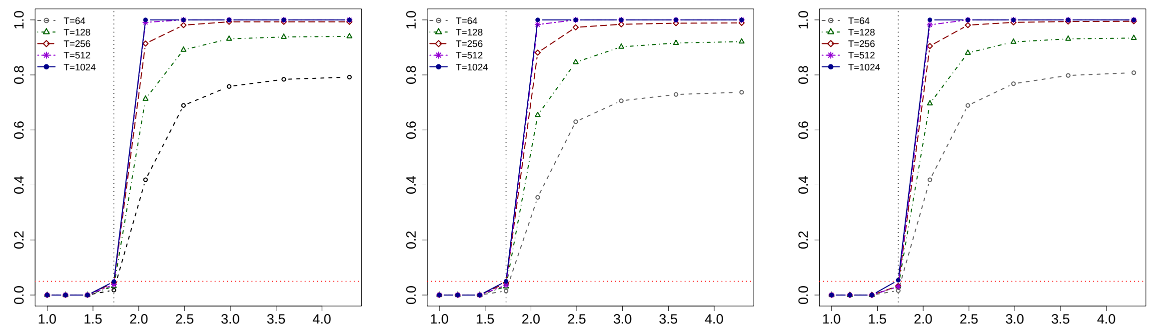

Figure 4.2: *Empirical rejection probabilities under scenario 2 for the relevant hypotheses in Section 3.1 (left panel) and Section 3.3 for* $k = 1, 2$ *(middle and right panel) as a function of* ι *at the nominal level 0.05 (horizontal red dotted line). The vertical dotted line depicts an increase in amplitude with a factor* 1.78^2 *(*$\iota = 3$*). The thresholds for the relevant hypotheses corresponding to this amplitude increase depicted in the three panels are respectively given by* $\Delta \approx 0.044$ *(left),* $\Delta_{\lambda,1} \approx 0.040$ *(middle) and* $\Delta_{\lambda,2} \approx 0.0025$ *(right).*

relevant deviations between the spectral density operators (Theorem 3.3) and between the first eigenprojectors (Theorem 3.5) for a true shift of $\iota_1 = 0.075$. This corresponds to $\Delta \approx 0.81$ (a), $\Delta_{\Pi,1} \approx \Delta_{\Pi,2} \approx 0.89$ (b)–(c) and $\Delta_{\Pi,3} = \Delta_{\Pi,4} = 0$ (d)–(e). Observe that (a) shows that the rejection probabilities closely align with the prescribed theory, which is also the case for the first and second eigenprojectors (b)–(c), albeit the second appears slightly oversized. To understand what we observe in (c)–(e), it is important to note that the second eigenfunctions become orthogonal at 0.150, the third at 0.150 and the fourth eigenfunctions become orthogonal at 0.175, which is where the true distances for the eigenprojectors are equal to 2 and hence the rejection probabilities should jump to 1. This explains the jump from the prescribed level of (approximately) 0.05 to 1 at $\iota_1 = 0.150$ and $\iota_1 = 0.175$, for the third and fourth eigenprojectors, respectively.

- **Scenario 4: change in the strength of dependence**. In the final setting, we generate $\{X_t\}_{t=1}^T$ from model (4.1) with $c = 0$ and $\iota_j = 0, j = 1, 2$ and do the same for processes $\{Y_t\}_{t=1}^T$ but here we vary c from 0 to 0.6. In Figure 4.4, we depict the empirical rejection probablities for the proposed tests of no relevant difference between the spectral density operators (Theorem 3.3) and between the largest eigenvalues (Theorem 3.7) for a true change in dependence of $c = 0.28$. Also in this case, the graphs demonstrate good nominal levels and power for sample sizes $T \geq 128$. Note that this is promising since we vary the strength of dependence –which affects the amplitude of the peaks in the frequency distribution for the second process– while we keep the bandwidth parameter and width of the frequency band fixed. Hence, we expect for the eigenvalues test a larger error for larger values of c in certain areas of the integration region (in this model around frequency 0) due to some oversmoothing. This could explain why the test for the first two eigenvalues are slightly undersized at the boundary of the null hypothesis.

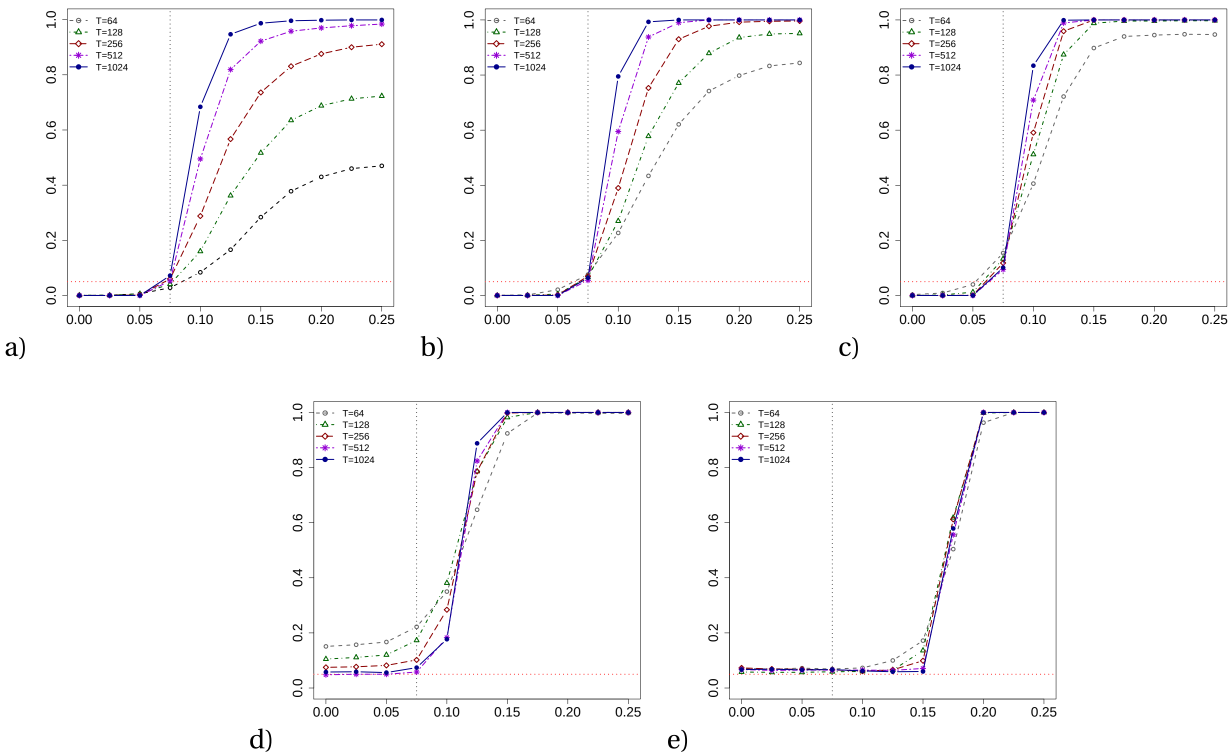

Figure 4.3: *Empirical rejection probabilities under scenario 3 for the relevant hypotheses in Section 3.1 (panel (a)) and Section 3.2 and for the first four eigenprojectors (panels (b) - (e)) as a function of* ι *at the nominal level 0.05 (horizontal red dotted line). The vertical line illustrates the true shift* $\iota_1 = 0.075$. *The thresholds induced by this shift are* $\Delta \approx 0.81$ *(a),* $\Delta_{\Pi,1} \approx \Delta_{\Pi,2} 0.89$ *(b-c) and* $\Delta_{\Pi,3} = \Delta_{\Pi,4} = 0$ *(d-e).*

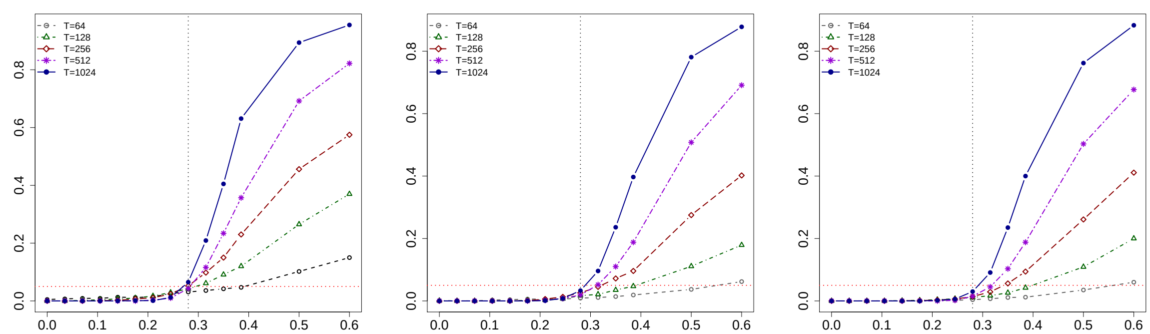

Figure 4.4: *Empirical rejection probabilities under scenario 4 of the tests in Theorem 3.3 (left panel) and in Theorem 3.7 for* $k = 1,2$ *(middle and right panel) as a function of* c *at the nominal level 0.05 (horizontal red dotted line). The vertical line indicates the true strength of dependence given by* $c = 0.28$. *The corresponding thresholds are respectively given by* $\Delta \approx 0.36$ *(left),* $\Delta_{\lambda,1} \approx 0.21$ *(middle) and* $\Delta_{\lambda,2} \approx 0.07$ *(right).*

4.1 Application to resting state fMRI

Next, we demonstrate the methodology developed in this paper by an application to resting state functional Magneting Resonance Imaging (fMRI) data. fMRI data allows to capture brain activity and consists of a sequence of three-dimensional images of the brain recorded every few seconds. Since the brain operates as a single unit of which we record the activity at a large number of spatial locations, it is natural to model the brain as a function and hence the time record as a functional time series, thereby taking into account the present temporal dependencies. The data we use are publicly available as part of the *1000 connectome project* (Biswal et al., 2010). In order to avoid differences in scanner types and locations, we consider testing for relevant differences for 6 subjects of which the data was measured at a single site (Beijing, China). For each subject, the resting state scans are comprised of 225 temporal scans, measured 2 seconds apart, where each temporal scan consists of three dimensional images of size $64\times64\times33$ voxels. The fMRI data set for one of the subjects is depicted in Figure 4.5. In order to correct for technical effects such as scanner drift, a polynomial trend of order 3 was removed from each voxel time series which are voxel-wise normalized (see also Worsley et al., 2002).

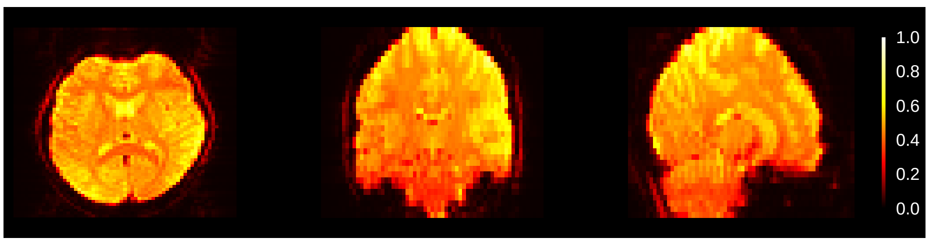

Figure 4.5: Slices of the fMRI data set for one of the subjects.

The high-dimensionality of fMRI data and hence of the corresponding second order dependence structure requires an efficient method to allow for the functional eigenanalysis. To make this computationally efficient and to avoid spurious identifiability issues that come with alternative discretized matrix approaches (see Aston and Kirch, 2012, for a discussion), we shall assume the functional component has a separable structure. The assumption of separable functions in the context of brain imaging is commonly applied as a method to deal with the high dimensionality of this type of data (see e.g., Worsley et al., 2002; Ruttimann et al., 1998; Aston and Kirch, 2012; Stoehr et al., 2019) To make this more precise in our set up, let $\prod_{i=1}^3 \mathscr{S}_i$ be a product of compact sets. Then we model each fMRI data set as a functional time series $\{X_t(\tau_1,\tau_2,\tau_3) : \tau_i \in \mathscr{S}_i, i = 1,2,3\}_{t\in\mathbb{Z}}$ with well-defined spectral density operator in $S_2(\prod_{i=1}^3 \mathscr{S}_i)$ that satisfies

$$\mathscr{F}_X^{(\omega)} = \mathscr{F}_{X,1}^{(\omega)} \widetilde{\bigotimes} \mathscr{F}_{X,2}^{(\omega)} \widetilde{\bigotimes} \mathscr{F}_{X,3}^{(\omega)}, \tag{4.2}$$

where $\mathscr{F}_{X,i}^{(\omega)}$ denotes the i-th *directional spectral density operator* which has a kernel function in $L^2_{\mathbb{C}}(\mathscr{S}_i \times \mathscr{S}_i)$ that replicates across the other two directions. For example, the first directional component has kernel given by

$$\mathscr{F}_{X,1}^{(\omega)}(\tau_1,\sigma_1) = \frac{1}{2\pi} \sum_{h\in\mathbb{Z}} \int_{\mathscr{S}_2} \int_{\mathscr{S}_3} \mathrm{Cov}\big(X_h(\tau_1,\tau_2,\tau_3), X_0(\sigma_1,\tau_2,\tau_3)\big) d\tau_2 d\tau_3 e^{-\mathrm{i}h\omega} \quad \tau_1,\sigma_1 \in \mathscr{S}_1,$$

The kernels of $\mathscr{F}_{X,2}^{(\omega)}$ and $\mathscr{F}_{X,3}^{(\omega)}$ are similarly defined. Let the eigenelements of $\mathscr{F}_{X,i}^{(\omega)}$ be denoted as $\{\lambda_{i,j}^{(\omega)}, \phi_{i,j}^{(\omega)}\}_{j\geq1}$ for $i = 1,2,3$. Then the eigendecomposition of the operator in(4.3) at frequency ω is

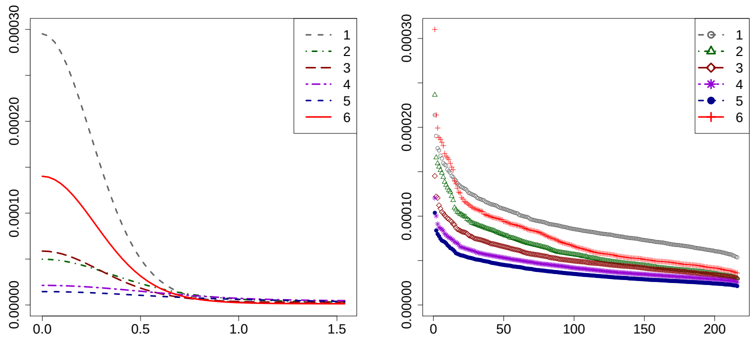

Figure 4.6: *Squared Hilbert-Schmidt norm of the estimator of* (4.3) *as a function of frequency for the 6 subjects (left) and the corresponding largest 216 eigenvalues averaged over* $[0, \pi/2)$ *(right).*

given by

$$\mathscr{F}_X^{(\omega)} := \sum_{j,k,l=1}^{\infty} \lambda_{jkl}^{(\omega)} \Pi_{jkl}^{(\omega)} \bigotimes \Pi_{jkl}^{(\omega)}. \tag{4.3}$$

where the eigenvalues $\lambda_{jkl}^{(\omega)} := \lambda_{1,j}^{(\omega)} \lambda_{2,k}^{(\omega)} \lambda_{3,l}^{(\omega)}$ are ordered in a descending manner, i.e., $\{\lambda_{jkl}^{(\omega)}\}_{jkl} \searrow 0$ and where $\Pi_{jkl}^{(\omega)} = \phi_{1,j}^{(\omega)} \otimes \phi_{2,k}^{(\omega)} \otimes \phi_{3,l}^{(\omega)}$ is the eigenfunction belonging the jkl-th largest eigenvalue at frequency ω (see also subsection A.2). The sequential eigenelements can now be efficiently estimated via consistent estimators of the sequential eigenelements of the directional operators as given in (3.8). For example, the estimator of the first directional sequential spectral density kernels is defined as

$$\hat{\mathscr{F}}_{X,1}^{(\omega)}(\eta)(\tau_1, \sigma_1) = \frac{1}{\lfloor \eta T \rfloor} \sum_{s=1}^{\lfloor \eta T \rfloor} \Bigg(\sum_{t=1}^{\lfloor \eta T \rfloor} \frac{\tilde{w}_{b,s,t}^{(\omega)}}{|\mathscr{S}_2||\mathscr{S}_3|} \sum_{\substack{\tau_2 \in \mathscr{S}_2 \\ \tau_3 \in \mathscr{S}_3}} (X_s(\tau_1, \tau_2, \tau_3)(X_t(\sigma_1, \tau_2, \tau_3) \Bigg), \quad \tau_1, \sigma_1 \in \mathscr{S}_1, \tag{4.4}$$

where $|\mathscr{S}_i|$ is the set of the discrete observations of the function in the i-th direction.

The raw data was converted into functional observations by using cubic b-spline functions on $[0,1]^3$. It is worth mentioning that in the computation of the eigenfunctions we took into account that the B-spline basis functions do not form an orthogonal basis. Estimators of the sequential sequence of directional spectral density kernels were obtained using the same parameters as in the previous section and evaluated at a 100^2 equispaced grid of $[0,1]^2$. We then investigated whether we could find evidence of relevant differences in the second order structure of the different subjects by applying the three tests developed in Section 3.1-3.3 pairwise. Since the sampling rate of the data is 0.5 Hertz per second, we restrict our analysis to the interval $[0, \pi/2)$.

In Figure (4.6), the squared Hilbert-Schmidt norm of the estimator of (4.3) is plotted as a function of frequency for each of the six subjects. We observe most signal in the low-frequency band (<0.1 Hertz). The first 216 estimated eigenvalues averaged over frequences are given in the right graph. For fixed frequency, these were obtained by taking the first $T^{1/3} = (225)^{1/3}$ eigenvalues of each of the directional operators, taking the kronecker product of these eigenvalues and arranging the resulting values in descending order. For all subjects, a clear gap is present after approximately 2 to 3 largest averaged eigenvalues, which then taper off slowly. Tables 1–3 provides the p-values of the test statistics $\widehat{\mathbb{D}}_{T_1,T_2}^{[a,b]}$, $\widehat{\mathbb{D}}_{\Pi,T_1,T_2}^{[a,b],k}$ and $\widehat{\mathbb{D}}_{\lambda,T_1,T_2}^{[a,b],k}$, respectively, for the specified relevant hypotheses. To clarify that the tests were conducted over the frequency band $[a,b]$, the hypotheses values are

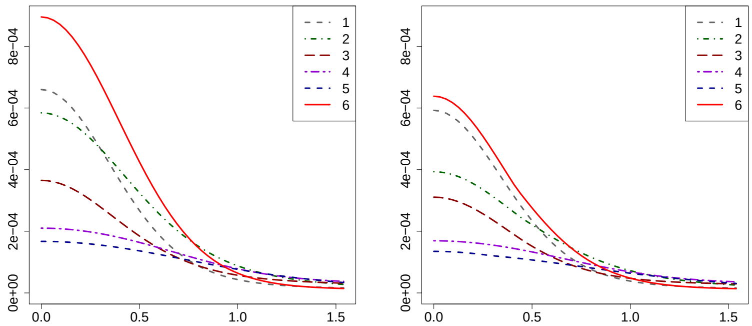

Figure 4.7: *The first (left), second (middle) and third (right) estimated eigenvalues of* (4.3) *as a function of frequency for the six subjects.*

equipped with a superscript $[a,b]$, e.g., we write $\Delta^{[a,b]}$. For the testing frameworks of no relevant differences between the spectral density operator and between the eigenvalues, the thresholds $\Delta^{[a,b]}$ and $\Delta_{\lambda,i}^{[a,b]}$ are specified based upon the overall signal present in the data.

For the hypothesis of a relevant difference of at most $\Delta_{[0,\pi/2)} = 1e-06$ between the spectral density operators, we observe only between subjects 3 and 5 approximately a rejection at the 6% level, but not for $\Delta_{[0,\pi/2)} = 2e-06$. When we restrict to the frequency band where most signal is present (<0.1 Hertz), we find p-values less than 0.05 for subjects 3 and 5 and we find some marginal evidence of relevant differences for between subjects 2 and 4 with a p-value of 0.11 for the hypothesis $\Delta_{[0,\pi/5]} = 1e-06$. For the first two eigenvalues, which are plotted in Figure 4.7, we do not find evidence of significant differences (Table 3), also not when we restrict to $[0,\pi/5]$ or when we change the value of $\Delta_{\lambda,i}^{[0,\pi/2)}$. This appears to be caused by a relatively large variance.

More interestingly, Table 2 provides clear evidence of relevant differences between the eigenprojectors. The null of no relevant differences between the first eigenprojectors with $\Delta_{\Pi,1}^{[0,\pi/2)} = 0.2$ can be rejected for quite a few combinations at the 10% level and in particular for all those combinations with subject 1, for which the p-values are less than 0.07. The p-values corresponding to the hypotheses $\Delta_{\Pi,1}^{[0,\pi/2)} = 0.3$ provides for some pairs less clear evidence of relevant differences in the first eigenprojector. In case of the second eigenprojectors, we do however reject the null of no relevant differences for both the hypotheses $\Delta_{\Pi,2}^{[0,\pi/2)} = 0.2$ and $\Delta_{\Pi,2}^{[0,\pi/2)} = 0.3$ in most cases at a 1% or 5% level. Except for the tests between subject 2 and 6, we in fact reject all cases at 10% level. This behavior did not change when we restricted to $[0,\pi/5]$.

This preliminary analysis would indicate that differences in brain activity between subjects might be driven by differences in shapes in their primary modes of variation. However, given the complexity of the data a more detailed analysis and a longer observation length might be of interest. This is however beyond the scope of this paper and is left for future research.

5 Proof of Theorem 3.1

In this section, we prove Theorem 3.1 and provide exact expressions of the constant τ_{XY} in terms of the spectral density operators and the factorizations of $\mathscr{U}_{XY}^{(\omega)}$ and $\mathscr{U}_{YX}^{(\omega)}$. We remark once more that the exact expression of $\mathscr{U}_{XY}^{(\omega)}$ depends on the specific hypothesis under consideration and that both are allowed to depend on both component processes X and Y. In the following, consider

$$\mathscr{V}_{\boldsymbol{X}}^{(\omega)} := (\mathscr{V}_X^{(\omega)}, \mathscr{V}_Y^{(\omega)})^\top \tag{5.1}$$

	$H_0 : \Delta^{[0,\pi/2)} = 1e-06$					$H_0 : \Delta^{[0,\pi/2)} = 2e-06$				
	2	3	4	5	6	2	3	4	5	6
1	0.336	0.328	0.296	0.269	0.422	0.341	0.337	0.299	0.272	0.434
2		0.375	0.193	0.175	0.227		0.829	0.496	0.287	0.265
3			0.175	0.053	0.176			0.364	0.141	0.218
4				0.957	0.19				0.998	0.208
5					0.185					0.204

	$H_0 : \Delta^{[0,\frac{\pi}{5}]} = 1e-06$					$H_0 : \Delta^{[0,\frac{\pi}{5}]} = 2e-06$				
	2	3	4	5	6	2	3	4	5	6
1	0.332	0.326	0.296	0.265	0.416	0.337	0.328	0.296	0.27	0.422
2		0.242	0.107	0.127	0.214		0.403	0.182	0.167	0.234
3			0.129	0.031	0.15			0.196	0.046	0.177
4				0.615	0.187				0.946	0.2
5					0.182					0.189

Table 1: *p-values corresponding to the test in Theorem 3.3 for the specified null hypotheses and frequency bands.*

	$H_0 : \Delta_{\Pi,1}^{[0,\pi/2)} = 0.2$					$H_0 : \Delta_{\Pi,2}^{[0,\pi/2)} = 0.2$				
	2	3	4	5	6	2	3	4	5	6
1	0.001	0.001	0.061	0.014	0.064	0.004	0.031	0.057	0.034	0.024
2		0.073	0.314	0.289	0.465		0.006	0.001	0.007	0.195
3			0.043	0.122	0.131			0.004	0.007	0.029
4				0.583	0.375				0.019	0.004
5					0.278					0.056

	$H_0 : \Delta_{\Pi,1}^{[0,\pi/2)} = 0.3$					$H_0 : \Delta_{\Pi,2}^{[0,\pi/2)} = 0.3$				
	2	3	4	5	6	2	3	4	5	6
1	0.002	0.001	0.132	0.036	0.17	0.007	0.038	0.076	0.043	0.038
2		0.205	0.482	0.414	0.627		0.014	0.001	0.016	0.24
3			0.117	0.282	0.242			0.007	0.016	0.039
4				0.887	0.508				0.039	0.007
5					0.42					0.08

Table 2: *p-values corresponding to the eigenprojector test (Theorem 3.5) applied to the first two eigenprojectors for the stated hypotheses.*

	$H_0 : \Delta_{\lambda,1}^{[0,\pi/2]} = (2e-06)^2$					$H_0 : \Delta_{\lambda,2}^{[0,\pi/2]} = (2e-06)^2$				
	2	3	4	5	6	2	3	4	5	6
1	0.473	0.342	0.311	0.264	0.341	0.446	0.32	0.302	0.255	0.487
2		0.289	0.239	0.234	0.214		0.295	0.204	0.206	0.204
3			0.304	0.184	0.241			0.319	0.2	0.207
4				0.291	0.215				0.255	0.187
5					0.216					0.196

Table 3: *p-values corresponding to the eigenvalue test (Theorem 3.7) applied to the first two eigenvalues for the stated hypotheses.*

where $\mathcal{V}_X^{(\omega)}, \mathcal{V}_Y^{(\omega)}$ are arbitrary elements of $S_2(\mathscr{H})$ and where the subscript only refers to the index of the component. Note that $\mathcal{V}_{\boldsymbol{X}}^{(\omega)}$ is an element of the Hilbert space $\mathcal{H} := S_2(\mathscr{H}) \oplus S_2(\mathscr{H})$. In order to prove Theorem 3.1, we prove the following statement in detail.

Theorem 5.1. *Suppose Assumptions 3.1-3.4 are satisfied. Let* $\mathcal{V}_{\boldsymbol{X}}^{(\omega)} \in \mathcal{H}$ *be defined by* (5.1), $\omega \in \mathbb{R}$ *and define*

$$\mathcal{Z}_{T,\eta}^{\boldsymbol{X},\omega} := \left(\mathcal{Z}_{T_1,\eta}^{X,\omega}/\sqrt{b_1 T_1}, \mathcal{Z}_{T_2,\eta}^{Y,\omega}/\sqrt{b_2 T_2}\right)^\top \in \mathcal{H}$$

where $\mathcal{Z}_{T_1,\eta}^{X,\omega}$ *and* $\mathcal{Z}_{T_2,\eta}^{Y,\omega}$ *are given by* (3.2) *and* (3.3), *respectively.*

i). If the component processes of $\{\boldsymbol{X}_t\}$ *are dependent, then*

$$\left\{\eta\sqrt{b_1 T_1 + b_2 T_2}\left(\int_a^b \Re\left(\left\langle \mathcal{Z}_{T,\eta}^{\boldsymbol{X},\omega}, \mathcal{V}_{\boldsymbol{X}}^{(\omega)}\right\rangle_{\mathcal{H}}\right) d\omega\right\}_{\eta\in[0,1]} \rightsquigarrow \left\{\tau_{XY}\eta\mathbb{B}(\eta)\right\}_{\eta\in[0,1]},$$

where $\mathbb{B}$ *is a Brownian motion,* $\tau_{XY}^2 = \frac{1}{4}\int_a^b \Re\left[\Gamma_{\boldsymbol{X}}^{\omega}\left(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\right) + \Sigma_{\boldsymbol{X}}^{\omega}\left(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\right)\right] d\omega$ *with*

$$\begin{aligned}\Gamma_{\boldsymbol{X}}^{\omega}\left(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\right) = 4\pi^2 \sum_{l,j\in\{X,Y\}} \Bigg(&\left\langle \left(\mathscr{F}_{l,j}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{j,l}^{\dagger(\omega)}\right)(\mathcal{V}_j^{(\omega)}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2} + 1_{\omega\in\{0,\pi\}}\left\langle \left(\mathscr{F}_{l,j}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{j,l}^{\dagger(\omega)}\right)(\mathcal{V}_j^{(\omega)}), \overline{\mathcal{V}_l^{\dagger(\omega)}}\right\rangle_{S_2} \\ &+ \left\langle \left(\mathscr{F}_{j,l}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{l,j}^{\dagger(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \mathcal{V}_j^{\dagger(\omega)}\right\rangle_{S_2} + 1_{\omega\in\{0,\pi\}}\left\langle \left(\mathscr{F}_{j,l}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{l,j}^{\dagger(\omega)}\right)(\mathcal{V}_j^{(\omega)}), \overline{(\mathcal{V}_j^{\dagger(\omega)})}\right\rangle_{S_2}\Bigg)\end{aligned}$$

and

$$\begin{aligned}\Sigma_{\boldsymbol{X}}^{\omega}\left(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\right) = 4\pi^2 \sum_{l,j\in\{X,Y\}} \Bigg(&1_{\omega\in\{0,\pi\}}\left(\left\langle \left(\mathscr{F}_{l,j}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{j,l}^{\dagger(\omega)}\right)(\overline{\mathcal{V}_j^{(\omega)}}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2} + \left\langle \left(\mathscr{F}_{l,j}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{j,l}^{\dagger(\omega)}\right)(\mathcal{V}_j^{\dagger(\omega)}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2}\right. \\ &+ 1_{\omega\in\{0,\pi\}}\left\langle \left(\mathscr{F}_{j,l,m}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{l,j}^{\dagger(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \overline{\mathcal{V}_j^{\dagger(\omega)}}\right\rangle_{S_2} + \left\langle \left(\mathscr{F}_{j,l}^{(\omega)} \widetilde{\otimes} \mathscr{F}_{l,j}^{\dagger(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \mathcal{V}_j^{(\omega)}\right\rangle_{S_2}\Bigg)\end{aligned}$$

ii). If the component processes of $\{\boldsymbol{X}_t\}$ *are independent, then*

$$\left\{\eta\sqrt{b_1 T_1 + b_2 T_2}\left(\int_a^b \Re\left(\left\langle \mathcal{Z}_{T,\eta}^{\boldsymbol{X},\omega}, \mathcal{V}_{\boldsymbol{X}}^{(\omega)}\right\rangle_{\mathcal{H}}\right) d\omega\right\}_{\eta\in[0,1]} \rightsquigarrow \left\{\eta\left(\frac{\upsilon_X}{\sqrt{\theta}}\mathbb{B}_X(\eta) + \frac{\upsilon_Y}{\sqrt{1-\theta}}\mathbb{B}_Y(\eta)\right)\right\}_{\eta\in[0,1]}$$

where $\mathbb{B}_X$ *and* $\mathbb{B}_Y$ *are independent Brownian motions and* $\upsilon_l^2 = \frac{1}{4}\int_a^b \Re\left[\Gamma_l^{\omega}\left(\mathcal{V}_l^{(\omega)}\right) + \Sigma_l^{\omega}\left(\mathcal{V}_l^{(\omega)}\right)\right] d\omega$, $l \in \{X, Y\}$, *with*

$$\begin{aligned}\Gamma_l^{(\omega)}(\mathcal{V}_l^{(\omega)}) = 4\pi^2\Bigg(&\left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{(\omega)}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2} + 1_{\omega\in\{0,\pi\}}\left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{(\omega)}), \overline{\mathcal{V}_l^{\dagger(\omega)}}\right\rangle_{S_2} \\ &+ \left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \mathcal{V}_l^{\dagger(\omega)}\right\rangle_{S_2} + 1_{\omega\in\{0,\pi\}}\left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{(\omega)}), \overline{(\mathcal{V}_l^{\dagger(\omega)})}\right\rangle_{S_2}\Bigg)\end{aligned}$$

and

$$\begin{aligned}\Sigma_l^{(\omega)}(\mathcal{V}_l^{(\omega)}) = 4\pi^2\Bigg(&1_{\omega\in\{0,\pi\}}\left(\left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\overline{\mathcal{V}_l^{(\omega)}}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2} + \left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2}\right. \\ &+ 1_{\omega\in\{0,\pi\}}\left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \overline{\mathcal{V}_l^{\dagger(\omega)}}\right\rangle_{S_2} + \left\langle \left(\mathscr{F}_l^{(\omega)} \widetilde{\otimes} \mathscr{F}_l^{(\omega)}\right)(\mathcal{V}_l^{\dagger(\omega)}), \mathcal{V}_l^{(\omega)}\right\rangle_{S_2}\Bigg).\end{aligned}$$

Note that the expression for the covariance and pseudo-covariance in both parts of the statement can be further simplified if we further assume that the $\mathcal{V}_l^{(\omega)}, l, j \in \{X, Y\}$ are self-adjoint, which is the case in all our statements.

Proof of Theorem 5.1. The proof is involved and relies on several auxiliary results, which can be found in Appendix B. We will only proof part *i*). The proof under independence follows similarly by verifying the steps for each component process separately and using the independence to conclude it for the linear combination. Following Assumption 3.4, we can ease notation in the dependent scenario and write $T := T_1 = T_2$ throughout this section. Since we only assume very mild moment conditions, it is not obvious how to obtain the distributional properties directly. The principal idea is therefore to construct an approximating process of which the distributional properties can be established and then show that the process limiting distribution is the same as for the approximating process. Before we can introduce this process, we require some necessary terminology. Let

$$\boldsymbol{X}_t^{(m)} = \mathbb{E}[\boldsymbol{X}_t | \sigma(\epsilon_t, \epsilon_{t-1}, \ldots, \epsilon_{t-m})].$$

and define the $\mathcal{H}^{\oplus_2}$-valued stochastic process

$$D_{m,k}^{(\omega)} := (D_{X,m,k}^{(\omega)}, D_{Y,m,k}^{(\omega)})^\top := \frac{1}{\sqrt{2\pi}} \sum_{t=0}^{\infty} \left(\mathbb{E}[\boldsymbol{X}_{t+k}^{(m)} | \mathcal{G}_k] - \mathbb{E}[\boldsymbol{X}_{t+k}^{(m)} | \mathcal{G}_{k-1}]\right) e^{-\mathrm{i}t\omega}, \tag{5.2}$$

where $\mathcal{G}_k = \sigma(\epsilon_k, \ldots \epsilon_{k-1}, \ldots)$. Under Conditions **A.1**-**A.2**, this process is an m-dependent stationary martingale difference sequence w.r.t. the filtration $\{\mathcal{G}_k\}$ in $\mathcal{L}^p_{\mathcal{H}^{\oplus_2}}$ for each $\omega \in [-\pi, \pi]$. Additionally, consider the process

$$\mathcal{M}_{T,m}^{(\omega)} := \sum_{t=2}^{T} \sum_{s=1}^{t-1} \tilde{w}_{b_T,t,s}^{(\omega)} \mathcal{D}_{XY,s,t}^{(\omega)}, \tag{5.3}$$

where $\mathcal{D}_{XY,s,t}^{(\omega)} = \left(D_{X,m,t}^{(\omega)} \otimes D_{X,m,s}^{(\omega)}, D_{Y,m,s}^{(\omega)} \otimes D_{Y,m,t}^{(\omega)}\right)^\top$ and denote $\mathcal{W}_{b_T}^2 = \sum_{t,s=1}^T |\tilde{w}_{b_T,t,s}^{(\omega)}|^2$. Under Assumption 3.1, the process $\mathcal{W}_{b_T}^{-1} \mathcal{M}_{T,m}^{(\omega)}$ is a martingale process in $\mathcal{L}_{\mathcal{H}}^{p/2}$ with respect to the filtration $\{\mathcal{G}_T\}$ for $p \geq 2$. The above claims on the properties of (5.2) and (5.3) can be verified similar to Proposition 3.2 and 3.3 of van Delft (2020) noting that for any $f = (f_1, f_2)^\top \in \mathcal{H}$, we have

$$\|f\|_{\mathcal{H}} = |||f_1|||_{S_2(\mathcal{H})} + |||f_2|||_{S_2(\mathcal{H})} \tag{5.4}$$

To construct the required approximating process, consider the arrays

$$N_{m,T,t}^{(\omega)} = \sum_{s=1}^{t-1} \left\langle \tilde{w}_{b_T,t,s}^{(\omega)} \mathcal{D}_{XY,s,t}^{(\omega)}, \mathcal{V}_{\boldsymbol{X}}^{(\omega)} \right\rangle_{\mathcal{H}} + \sum_{s=1}^{t-1} \left\langle \left(\tilde{w}_{b_T,t,s}^{(\omega)} \mathcal{D}_{XY,s,t}^{(\omega)}\right)^\dagger, \mathcal{V}_{\boldsymbol{X}}^{(\omega)} \right\rangle_{\mathcal{H}}, \quad 2 \leq t \leq T, \tag{5.5}$$

and set $N_{m,T,1}^{(\omega)} = 0$ for all $\omega \in [-\pi, \pi]$. The following statement then provides the distributional properties of the (scaled) partial sum of the real part of (5.5) integrated over the frequency band $[a, b]$. The proof is tedious and postponed Section A.3.

Theorem 5.2. *Let $N_{m,T,t}^{(\omega)}$ be defined as in* (5.5) *and let $\mathcal{V}_{\boldsymbol{X}}^{(\omega)}$ be defined as in* (5.1). *Suppose that Assumptions 3.1-3.3 hold. Then, for fixed m,*

$$\frac{1}{\mathcal{W}_{b_T}} \Big\{ \sum_{t=1}^{\lfloor \eta T \rfloor} \Re\Big(\int_a^b N_{m,T,t}^{(\omega)} d\omega\Big)\Big\}_{\eta \in [0,1]} \rightsquigarrow \tau_{m,XY} \cdot \Big\{\eta \mathbb{B}(\eta)\Big\}_{\eta \in [0,1]}, \quad (T \to \infty)$$

where $\tau_{m,XY}^2 = \frac{1}{4} \int_a^b \Re\left[\Gamma_{\boldsymbol{X},m}^{\omega}\big(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\big) + \Sigma_{\boldsymbol{X},m}^{\omega}\big(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\big)\right] d\omega$ *with*

$$\Gamma_{\boldsymbol{X},m}^{\omega}\big(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\big) = 4\pi^2 \sum_{l,j \in \{X,Y\}} \left\langle \big(\mathcal{F}_{l,j,m}^{(\omega)} \widetilde{\otimes} \mathcal{F}_{j,l,m}^{\dagger(\omega)}\big)(\mathcal{V}_j^{(\omega)}), \mathcal{V}_l^{(\omega)} \right\rangle_{S_2} + 1_{\omega \in \{0,\pi\}} \left\langle \big(\mathcal{F}_{l,j,m}^{(\omega)} \widetilde{\otimes} \mathcal{F}_{j,l,m}^{\dagger(\omega)}\big)(\mathcal{V}_j^{(\omega)}), \overline{\mathcal{V}_l^{\dagger(\omega)}} \right\rangle_{S_2}$$

$$+\Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m}\widetilde{\otimes}\mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{\dagger(\omega)}_{l}),\mathcal{V}^{\dagger(\omega)}_{j}\Big\rangle_{S_2}+1_{\omega\in\{0,\pi\}}\Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m}\widetilde{\otimes}\mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{(\omega)}_{j}),\overline{(\mathcal{V}^{\dagger(\omega)}_{j})}\Big\rangle_{S_2}$$

and

$$\Sigma^{\omega}_{\boldsymbol{X},m}\big(\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\big)=4\pi^2\sum_{l,j\in\{X,Y\}}\Big(\Big\langle \big(\mathscr{F}^{(\omega)}_{l,j,m}\widetilde{\otimes}\mathscr{F}^{\dagger(\omega)}_{j,l,m}\big)(\mathcal{V}^{\dagger(\omega)}_{j}),\mathcal{V}^{(\omega)}_{l}\Big\rangle_{S_2}+\Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m}\widetilde{\otimes}\mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{\dagger(\omega)}_{l}),\mathcal{V}^{(\omega)}_{j}\Big\rangle_{S_2}$$
$$1_{\omega\in\{0,\pi\}}\Big\langle \big(\mathscr{F}^{(\omega)}_{l,j,m}\widetilde{\otimes}\mathscr{F}^{\dagger(\omega)}_{j,l,m}\big)(\overline{\mathcal{V}^{(\omega)}_{j}}),\mathcal{V}^{(\omega)}_{l}\Big\rangle_{S_2}+1_{\omega\in\{0,\pi\}}\Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m}\widetilde{\otimes}\mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{\dagger(\omega)}_{l}),\overline{\mathcal{V}^{\dagger(\omega)}_{j}}\Big\rangle_{S_2}\Big).$$

and where $\mathscr{F}^{(\omega)}_{XY,m}=\mathbb{E}(D^{(\omega)}_{X,m,0}\otimes D^{(\omega)}_{Y,m,0})$.

We shall use this result in order to derive the distributional properties of the process given in Theorem 5.1. We define the processes

$$\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor}:=\eta\int_a^b\widetilde{\mathcal{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}d\omega\quad\text{and}\quad\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m}=\frac{\eta}{\mathscr{W}_{b_T}}\sum_{t=1}^{\lfloor\eta T\rfloor}\int_a^b N^{(\omega)}_{m,T,t}d\omega$$

where

$$\widetilde{\mathcal{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}:=\frac{1}{\mathscr{W}_{b_T}}\Big\langle\sum_{s=1}^{\lfloor\eta T\rfloor}\Big(\sum_{t=1}^{\lfloor\eta T\rfloor}\tilde{w}^{(\omega)}_{b_T,s,t}(X_s\otimes X_t,Y_s\otimes Y_t)^\top-\mathbb{E}\big(\hat{\mathscr{F}}^{\omega}_{X}(\eta),\hat{\mathscr{F}}^{\omega}_{Y}(\eta)\big)^\top\Big),\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}.$$

Let d_S denote the Skorokhod metric on $D[0,1])$ and d_U the uniform metric and recall that $(D[0,1],d_S)$ is a metric space. Let F be a closed set of $D[0,1]$ and denote $F_{d_S,\epsilon}=\{x:d_S(x,y)\le\epsilon,y\in F\}$ Since the Skorokhod metric is weaker than the uniform metric, we have

$$\begin{aligned}&\mathbb{P}\Big(\{\Re(\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor})\}_{\eta\in[0,1]}\in F\Big)\\&\le\mathbb{P}\Big(d_S\big(\{\Re(\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor})\}_{\eta\in[0,1]},\{\Re(\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m})\}_{\eta\in[0,1]}\big)\ge\epsilon\Big)+\mathbb{P}\Big(\{\Re(\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m})\}_{\eta\in[0,1]}\in F_{d_S,\epsilon}\Big)\\&\le\mathbb{P}\Big(d_U\big(\{\Re(\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor})\}_{\eta\in[0,1]},\{\Re(\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m})\}_{\eta\in[0,1]}\big)\ge\epsilon\Big)+\mathbb{P}\Big(\{\Re(\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m})\}_{\eta\in[0,1]}\in F_{d_S,\epsilon}\Big)\end{aligned}$$

We will first prove that

$$\lim_{m\to\infty}\lim_{T\to\infty}\mathbb{P}\Big(d_U\big(\{\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor}\}_{\eta\in[0,1]},\{\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m}\}_{\eta\in[0,1]}\big)\ge\epsilon\Big)=0.\tag{5.6}$$

By Markov's inequality,

$$\mathbb{P}\Big(d_U\big(\{\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor}\}_{\eta\in[0,1]},\{\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m}\}_{\eta\in[0,1]}\big)\ge\epsilon\Big)\le\epsilon^{-\gamma}\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor}-\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m}\Big|\Big)^{\gamma},$$

where we take $\gamma>2$. We find

$$\begin{aligned}\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\widetilde{\mathcal{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor}-\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m}\Big|\Big)^{\gamma}&\le\mathbb{E}\Big(\sup_{\eta\in[0,1]}\eta\int_a^b\Big|\Big(\widetilde{\mathcal{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}-\frac{1}{\mathscr{W}_{b_T}}\sum_{t=1}^{\lfloor\eta T\rfloor}N^{(\omega)}_{m,T,t}\Big)\Big|d\omega\Big)^{\gamma}\\&\le\mathbb{E}\Big(\int_a^b\sup_{\eta\in[0,1]}\eta\Big|\Big(\widetilde{\mathcal{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}-\frac{1}{\mathscr{W}_{b_T}}\sum_{t=1}^{\lfloor\eta T\rfloor}N^{(\omega)}_{m,T,t}\Big)\Big|d\omega\Big)^{\gamma}\\&\le(b-a)^{\gamma}\int_a^b\mathbb{E}\Big(\sup_{\eta\in[0,1]}\eta\Big|\Big(\widetilde{\mathcal{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}-\frac{1}{\mathscr{W}_{b_T}}\sum_{t=1}^{\lfloor\eta T\rfloor}N^{(\omega)}_{m,T,t}\Big)\Big|\Big)^{\gamma}d\omega,\end{aligned}\tag{5.7}$$

where the last inequality follows from an application of Jensen's inequality to the integral since $\gamma > 1$, and from Tonelli's theorem, which allows to interchange the expectation and integral. Continuity of the Hilbert-Schmid inner product with respect to the product topology on $\mathscr{H} \otimes \mathscr{H}$ and the Cauchy-Schwarz inequality imply for the integrand of (5.7)

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\eta\in[0,1]} \eta\Big|\Big(\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor} - \frac{1}{\mathscr{W}_{b_T}}\sum_{t=1}^{\lfloor\eta T\rfloor} N^{(\omega)}_{m,T,t}\Big)\Big|\Big)^{\gamma}\\
&= \mathbb{E}\Big(\sup_{\eta} \eta\Big|\Big\langle \frac{1}{\mathscr{W}_{b_T}}\Big(\sum_{s=1}^{\lfloor\eta T\rfloor}\big(\sum_{t=1}^{\lfloor\eta T\rfloor} \tilde{w}^{(\omega)}_{b_T,s,t}(X_s\otimes X_t, Y_s\otimes Y_t)^{\top} - \mathbb{E}\big(\hat{\mathscr{F}}^{\omega}_X(\eta), \hat{\mathscr{F}}^{\omega}_Y(\eta)\big)^{\top}\big) - \mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m} - \mathscr{M}^{\dagger(\omega)}_{\lfloor\eta T\rfloor,m}\Big), \mathscr{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}\Big|\Big)^{\gamma}\\
&\le \big(\sup_{\omega} \|\!|\!| \mathscr{V}^{(\omega)}_{\boldsymbol{X}} |\!|\!|_2^{\gamma}\big)\frac{1}{\mathscr{W}^{\gamma}_{b_T}}\mathbb{E}\Big(\sup_{\eta}\Big|\!\Big|\!\Big|\sum_{s=1}^{\lfloor\eta T\rfloor}\Big(\sum_{t=1}^{\lfloor\eta T\rfloor} \tilde{w}^{(\omega)}_{b_T,s,t}(X_s\otimes X_t, Y_s\otimes Y_t)^{\top} - \mathbb{E}\big(\hat{\mathscr{F}}^{\omega}_X(\eta), \hat{\mathscr{F}}^{\omega}_Y(\eta)\big)^{\top}\Big) - \mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m} - \mathscr{M}^{\dagger(\omega)}_{\lfloor\eta T\rfloor,m}\Big|\!\Big|\!\Big|_{\mathcal{H}}\Big)^{\gamma},
\end{aligned}
$$

where $\mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m}$ is as defined in (5.3). The statement in (5.6) then follows from Lemma B.3 in the Appendix together with (5.4) and (5.7). Next, write the real part of a complex random variable as a linear combination with its conjugate and apply the triangle inequality to find

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\Re(\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}) - \Re(\mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m})\Big|\Big)^{\gamma} &\le \mathbb{E}\Big(\frac{1}{2}\sup_{\eta\in[0,1]}\Big|\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor} - \mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m}\Big| + \frac{1}{2}\sup_{\eta\in[0,1]}\Big|\overline{\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}} - \overline{\mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m}}\Big|\Big)^{\gamma}\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor} - \mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m}\Big|\Big)^{\gamma} + \frac{1}{2}\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\overline{\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}} - \overline{\mathscr{M}^{(\omega)}_{\lfloor\eta T\rfloor,m}}\Big|\Big)^{\gamma}.
\end{aligned}
$$

Hence, it also follows from Lemma B.3 together with (5.7) that

$$
\begin{aligned}
&\lim_{m\to\infty}\lim_{T\to\infty}\mathbb{P}\Big(d_U\big(\{\Re(\widetilde{\mathscr{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor})\}_{\eta\in[0,1]}, \{\Re(\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m})\}_{\eta\in[0,1]}\big) \ge \epsilon\Big)\\
&\le \lim_{m\to\infty}\lim_{T\to\infty}\epsilon^{-\gamma}(b-a)^{\gamma}\frac{1}{2}\int_a^b\Big\{\mathbb{E}\Big(\sup_{\eta\in[0,1]}\eta\Big|\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor} - \widetilde{\mathscr{M}}^{\boldsymbol{X},(\omega)}_{\lfloor\eta T\rfloor,m}\Big|\Big)^{\gamma} + \mathbb{E}\Big(\sup_{\eta\in[0,1]}\eta\Big|\overline{\widetilde{\mathscr{Z}}^{\boldsymbol{X},\omega}_{\lfloor\eta T\rfloor}} - \overline{\widetilde{\mathscr{M}}^{\boldsymbol{X},(\omega)}_{\lfloor\eta T\rfloor,m}}\Big|\Big)^{\gamma}\Big\}d\omega = 0.
\end{aligned}
$$

Consequently, an application of Theorem 5.2 yields

$$
\begin{aligned}
\lim_{T\to\infty}\mathbb{P}\Big(\big\{\Re(\widetilde{\mathscr{Z}}^{X}_{\lfloor\eta T\rfloor})\big\}_{\eta\in[0,1]} \in F\Big) &\le \lim_{m\to\infty}\lim_{T\to\infty}\mathbb{P}\Big(\big\{\Re(\widetilde{\mathscr{M}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor,m})\big\}_{\eta\in[0,1]} \in F_{d_S,\epsilon}\Big)\\
&\le \lim_{m\to\infty}\mathbb{P}\Big(\Big\{\tau_{m,XY}\cdot\mathbb{B}(\eta)\Big\}_{\eta\in[0,1]} \in F_{d_S,\epsilon}\Big)\\
&= \mathbb{P}\Big(\Big\{\tau_{XY}\cdot\mathbb{B}(\eta)\Big\}_{\eta\in[0,1]} \in F_{d_S,\epsilon}\Big),
\end{aligned}
$$

where the last equality follows by taking the limit with respect to m of $\Gamma^{\omega}_{\boldsymbol{X},m}$ and $\Sigma^{\omega}_{\boldsymbol{X},m}$ to obtain the limiting covariance structure (see Proposition 3.2 of van Delft , 2020). Taking $\epsilon \downarrow 0$ we obtain,

$$
\Big\{\Re(\widetilde{\mathscr{Z}}^{\boldsymbol{X}}_{\lfloor\eta T\rfloor})\Big\}_{\eta\in[0,1]} \rightsquigarrow \Big\{\eta\tau_{XY}\mathbb{B}(\eta)\Big\}_{\eta\in[0,1]}
$$

The result now follows from Lemma B.4 in the Appendix and from noting that

$$
\Big|\frac{\sqrt{b_T}}{\sqrt{\kappa T}} - \frac{1}{\mathscr{W}_{b_T}}\Big| = \Big|\frac{\sqrt{b_T}}{\sqrt{\kappa T}}\frac{\mathscr{W}_{b_T}}{\mathscr{W}_{b_T}} - \frac{1}{\mathscr{W}_{b_T}}\Big| = \frac{1}{\mathscr{W}_{b_T}}\Big|\frac{\sqrt{b_T}}{\sqrt{\kappa T}}\Big(\frac{\sqrt{T}}{\sqrt{b_T}}(\kappa + o(1))\Big) - 1\Big| = o(\frac{1}{\mathscr{W}_{b_T}}) = o(1),
$$

where we used that under Assumption 3.2 $\mathscr{W}^2_{b_T} = T\sum_{|h|<T} w(b_T h)^2 = \frac{T}{b_T}(\kappa^2 + o(1))$. □

Acknowledgements This work has been supported by the Collaborative Research Center "Statistical modeling of nonlinear dynamic processes" (SFB 823, Teilprojekt A1, C1) of the German Research Foundation (DFG).

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A Proofs of main statements

A.1 Proofs of statements from Section 3

Proof of Proposition 3.1. This follows from an adjustment of the proof of Theorem 4.1(ii) of van Delft (2020) for the value of ℓ. Details are omitted. □

Proof of Lemma 3.1. We have

$$\mathbb{E}\Big(\sup_{\eta\in[0,1]}\int_a^b \Big|\Big|\Big|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)\Big|\Big|\Big|_2^2 d\omega\Big) \leq \mathbb{E}\Big(\int_a^b \sup_{\eta\in[0,1]}\Big|\Big|\Big|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)\Big|\Big|\Big|_2^2 d\omega\Big)$$
$$\leq \int_a^b \mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\Big|\Big|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)\Big|\Big|\Big|_2^2\Big) d\omega$$

where the last line follows from Tonelli's theorem. Observe that

$$\Big|\Big|\Big|\widehat{M}_{\hat{\mathscr{F}}}(\eta,\omega)-\eta M_{\mathscr{F}}(\omega)\Big|\Big|\Big|_2^2 \leq 2|||\eta\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}\big)|||_2^2+2|||\eta\big(\hat{\mathscr{F}}_Y^{(\omega)}(\eta)-\mathscr{F}_Y^{(\omega)}\big)|||_2^2,$$

and that $\frac{\eta}{\lfloor\eta T\rfloor}=\frac{\eta T}{\eta T}\frac{\eta}{\lfloor\eta T\rfloor}=\frac{1}{T}\big(\frac{\eta T}{\lfloor\eta T\rfloor}-1\big)+\frac{1}{T}$. Jensen's inequality and the Minkowski's inequality imply therefore for the first term

$$\begin{aligned}
\mathbb{E}\sup_{\eta\in[0,1]}\Big|\Big|\Big|\eta\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}\big)\Big|\Big|\Big|_2^2 &\leq \mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\Big|\Big|\eta\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}\big)\Big|\Big|\Big|_2\Big)^2\\
&\leq \Big(\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\Big|\Big|\frac{\eta}{\lfloor\eta T_1\rfloor}\sum_{s=1}^{\lfloor\eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor\eta T_1\rfloor}\tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t)-\mathscr{F}_X^{(\omega)}\Big)\Big|\Big|\Big|_2\Big)^{(2\gamma/2)}\Big)^{2/\gamma},\\
&\leq \Big(\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\Big|\Big|\frac{1}{T_1}\sum_{s=1}^{\lfloor\eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor\eta T_1\rfloor}\tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t)-\mathscr{F}_X^{(\omega)}\Big)\Big|\Big|\Big|_2\Big)^{\gamma}\Big)^{2/\gamma}\\
&+\sup_{\eta\in[0,1]}\Big|\frac{\eta T}{\lfloor\eta T\rfloor}-1\Big|\Big(\mathbb{E}\Big(\sup_{\eta\in[0,1]}\Big|\Big|\Big|\frac{1}{T_1}\sum_{s=1}^{\lfloor\eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor\eta T_1\rfloor}\tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t)-\mathscr{F}_X^{(\omega)}\Big)\Big|\Big|\Big|_2\Big)^{\gamma}\Big)^{2/\gamma}\\
&=O\Big(1+T_1^{-1}\Big)\Big(O(b_1^{-1-\epsilon/2}T_1^{-2-\epsilon}T_1^{1+\epsilon/2})+O(b_1^{\ell(2+\epsilon)})\Big)^{2/(2+\epsilon)}\\
&=O(b_1^{-1}T_1^{-1})+O(b_1^{2\ell}).
\end{aligned}$$

where we used (B.5) and Lemma B.1. In complete analogy, we obtain

$$\mathbb{E}\sup_{\eta\in[0,1]}\Big|\Big|\Big|\eta\big(\hat{\mathscr{F}}_Y^{(\omega)}(\eta)-\mathscr{F}_Y^{(\omega)}\big)\Big|\Big|\Big|_2^2=O(b_2^{-1}T_2^{-1})+O(b_2^{2\ell}).$$

The statement now follows from Assumption 3.3 and Assumption 3.4. □

Proof of Theorem 3.2. The follows from Lemma 3.1 and Lemma B.2 and from using that $c\in\mathbb{C}$, $c+\overline{c}=2\Re(c)$. □

Proof of Lemma 3.2. Denote the perturbation $\hat{\Delta}_\eta\mathscr{F}^\omega:=\hat{\mathscr{F}}^\omega(\eta)-\mathscr{F}^\omega$. We first consider (i). Observe that by Minkowski's inequality it suffices to show

$$\sup_{\eta\in[0,1]}\int_a^b\eta\Big|\Big|\Big|\hat{\Pi}_{X,k}^{(\omega)}(\eta)-\Pi_{X,k}^{(\omega)}-\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{\Big\langle\mathscr{F}_X^{(\omega)}\widetilde{\otimes}\hat{\Delta}_\eta\mathscr{F}_X^{(\omega)}+\hat{\Delta}_\eta\mathscr{F}_X^{(\omega)}\widetilde{\otimes}\mathscr{F}_X^{(\omega)},\Pi_{X,jj'}^{(\omega)}\widetilde{\otimes}\Pi_{X,k}^{(\omega)}\Big\rangle_{S_2}\Pi_{X,jj'}^{(\omega)}}{\lambda_{X,j}^{(\omega)}\lambda_{X,j'}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\Big|\Big|\Big|_2 d\omega$$
$$=O_p\Big(\frac{\log^{2/\gamma}(T_1)}{b_1T_1}\Big) \tag{A.1}$$

and

$$\sup_{\eta\in[0,1]}\int_a^b \eta\Big\|\Big\|\hat{\Pi}^{(\omega)}_{Y,k}(\eta)-\Pi^{(\omega)}_{Y,k}-\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{\Big\langle \mathscr{F}^{(\omega)}_{Y}\widetilde{\otimes}\hat{\Delta}_{\eta}\mathscr{F}^{(\omega)}_{Y}+\hat{\Delta}_{\eta}\mathscr{F}^{(\omega)}_{Y}\widetilde{\otimes}\mathscr{F}^{(\omega)}_{Y},\Pi^{(\omega)}_{Y,jj'}\widetilde{\otimes}\Pi^{(\omega)}_{Y,k}\Big\rangle_{S_2}\Pi^{(\omega)}_{Y,jj'}}{\lambda^{(\omega)}_{Y,j}\lambda^{(\omega)}_{Y,j'}-(\lambda^{(\omega)}_{Y,k})^2}\Big\|\Big\|_2 d\omega$$

$$=O_p\Big(\frac{\log^{2/\gamma}(T_2)}{b_2T_2}\Big).$$

As the proof for both processes is the same, we shall focus on (A.1) and drop the subscript X in the following. From Proposition 3.2, we have

$$\begin{aligned}\hat{\Pi}^{(\omega)}_k-\Pi^{(\omega)}_k &= \Big\langle \hat{\Pi}^{\omega}_k(\eta)-\Pi^{\omega}_k,\Pi^{\omega}_k\Big\rangle\Pi^{(\omega)}_k\\ &+\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{1}{\lambda^{(\omega)}_j\lambda^{(\omega)}_{j'}-(\lambda^{(\omega)}_k)^2}\Big[\Big\langle\big(\mathscr{F}^{\omega}\widetilde{\otimes}\mathscr{F}^{\omega}-\hat{\mathscr{F}}^{\omega}(\eta)\widetilde{\otimes}\hat{\mathscr{F}}^{\omega}(\eta)\big)\Pi^{(\omega)}_k,\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}+\Big\langle E^{(\omega)}_{k,b_T}(\eta),\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\Big]\Pi^{(\omega)}_{jj'},\end{aligned}$$

where

$$E^{(\omega)}_{k,b_T}(\eta)=\Big[(\mathscr{F}^{\omega}\widetilde{\otimes}\mathscr{F}^{\omega}-\hat{\mathscr{F}}^{\omega}(\eta)\widetilde{\otimes}\hat{\mathscr{F}}^{\omega}(\eta))\Big]\Big[\hat{\Pi}^{(\omega)}_k-\Pi^{(\omega)}_k\Big]+\Big[\big(\lambda^{(\omega)}_k(\eta)\big)^2-\big(\lambda^{(\omega)}_k\big)^2\Big]\Big[\hat{\Pi}^{(\omega)}_k-\Pi^{(\omega)}_k\Big]. \tag{A.2}$$

Elementary calculations yield

$$\hat{\mathscr{F}}^{\omega}(\eta)\widetilde{\otimes}\hat{\mathscr{F}}^{\omega}(\eta)-\mathscr{F}^{\omega}\widetilde{\otimes}\mathscr{F}^{\omega}=\mathscr{F}^{\omega}\widetilde{\otimes}\big(\hat{\Delta}_{\eta}\mathscr{F}^{\omega}\big)+\big(\hat{\Delta}_{\eta}\mathscr{F}^{\omega}\big)\widetilde{\otimes}\mathscr{F}^{\omega}+\big(\hat{\Delta}_{\eta}\mathscr{F}^{\omega}\big)\widetilde{\otimes}\big(\hat{\Delta}_{\eta}\mathscr{F}^{\omega}\big).$$

Therefore, we will show that

$$\sup_{\eta\in[0,1]}\eta\int_a^b\Big\|\Big\|\Big\langle\hat{\Pi}^{\omega}_k(\eta)-\Pi^{\omega}_k,\Pi^{\omega}_k\Big\rangle_{S_2}\Pi^{(\omega)}_k+\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{\Big[\Big\langle\hat{\Delta}_{\eta}\mathscr{F}^{\omega}\widetilde{\otimes}\hat{\Delta}_{\eta}\mathscr{F}^{\omega})\Pi^{(\omega)}_k,\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}+\Big\langle E^{(\omega)}_{k,b_T}(\eta),\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\Big]\Pi^{(\omega)}_{jj'}}{\lambda^{(\omega)}_j\lambda^{(\omega)}_{j'}-(\lambda^{(\omega)}_k)^2}\Big\|\Big\|_2 d\omega$$

$$=O_p\Big(\frac{\log^{2/\gamma}(T_1)}{b_{T_1}T_1}\Big).$$

By Minkowski's inequality,

$$\begin{aligned}&\sup_{\eta\in[0,1]}\eta\int_a^b\Big\|\Big\|\Big\langle\hat{\Pi}^{\omega}_k(\eta)-\Pi^{\omega}_k,\Pi^{\omega}_k\Big\rangle_{S_2}\Pi^{(\omega)}_k+\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{\Big[\Big\langle\hat{\Delta}_{\eta}\mathscr{F}^{\omega}\widetilde{\otimes}\hat{\Delta}_{\eta}\mathscr{F}^{\omega})\Pi^{(\omega)}_k,\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}+\Big\langle E^{(\omega)}_{k,b_T}(\eta),\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\Big]\Pi^{(\omega)}_{jj'}}{\lambda^{(\omega)}_j\lambda^{(\omega)}_{j'}-(\lambda^{(\omega)}_k)^2}\Big\|\Big\|_2 d\omega\\ &\le\sup_{\eta\in[0,1]}\eta\int_a^b\Big\{\Big\|\Big\|\Big\langle\hat{\Pi}^{\omega}_k(\eta)-\Pi^{\omega}_k,\Pi^{\omega}_k\Big\rangle_{S_2}\Pi^{(\omega)}_k\Big\|\Big\|_2+\Big\|\Big\|\frac{\Big\langle\big(\hat{\mathscr{F}}^{\omega}(\eta)-\mathscr{F}^{\omega}\big)\widetilde{\otimes}\big(\hat{\mathscr{F}}^{\omega}(\eta)-\mathscr{F}^{\omega}\big)\Pi^{(\omega)}_k,\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\Pi^{(\omega)}_{jj'}}{\lambda^{(\omega)}_j\lambda^{(\omega)}_{j'}-(\lambda^{(\omega)}_k)^2}\Big\|\Big\|_2\\ &\qquad+\Big\|\Big\|\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{\Big\langle E^{(\omega)}_{k,b_T}(\eta),\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}}{\lambda^{(\omega)}_j\lambda^{(\omega)}_{j'}-(\lambda^{(\omega)}_k)^2}\Pi^{(\omega)}_{jj'}\Big\|\Big\|_2\Big\}d\omega.\end{aligned}\tag{A.3}$$

We treat these terms separately. Firstly, observe that for any $A,B\in S_2(\mathscr{H})$, we can write

$$|||A|||_2^2-|||B|||_2^2=|||A-B|||_2^2+\langle A,B\rangle_{S_2}+\overline{\langle A,B\rangle}_{S_2}.$$

Moreover, since $|||\hat{\Pi}^{\omega}_k(\eta)|||_2=|||\hat{\Pi}^{\omega}_k(\eta)|||_2=1$ and

$$\Big\langle\hat{\Pi}^{\omega}_k(\eta)-\Pi^{\omega}_k,\Pi^{\omega}_k\Big\rangle_{S_2}=|\langle\hat{\phi}^{(\omega)}_k,\phi^{(\omega)}_k\rangle|^2-1=|\overline{\langle\hat{\phi}^{(\omega)}_k,\phi^{(\omega)}_k\rangle}|^2-1=\overline{\Big\langle\hat{\Pi}^{\omega}_k(\eta)-\Pi^{\omega}_k,\Pi^{\omega}_k\Big\rangle}_{S_2}$$

Rearranging terms yields $\left\langle \hat{\Pi}_k^\omega(\eta) - \Pi_k^\omega, \Pi_k^\omega \right\rangle_{S_2} = -\frac{1}{2}\|\hat{\Pi}_k^\omega(\eta) - \Pi_k^\omega\|_2^2$. We obtain using Lemma B.5

$$\begin{aligned}
\mathbb{E}\sup_{\eta\in[0,1]} \eta \int_a^b \left\| \left\langle \hat{\Pi}_k^\omega(\eta) - \Pi_k^\omega, \Pi_k^\omega \right\rangle_{S_2} \Pi_k^{(\omega)} \right\|_2 d\omega &\le \mathbb{E}\frac{1}{2}\int_a^b \sup_{\eta\in[0,1]} \eta \|\hat{\Pi}_k^\omega(\eta) - \Pi_k^\omega\|_2^2 d\omega \qquad (A.4)\\
&\le C \int_a^b \mathbb{E}\sup_{\eta\in[0,1]} \sum_{i=2}^4 \eta \|\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)}\|_2^i d\omega\\
&\le C \sum_{i=2}^4 \int_a^b \mathbb{E}\Big(\sup_{\eta\in[0,1]} \|\eta^{1/i}\big(\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)}\big)\|_2\Big)^i d\omega
\end{aligned}$$

for some constant $C > 0$, where we applied Tonelli's theorem in the second inequality. Consequently, from Theorem B.1 we obtain

$$\mathbb{E}\sup_{\eta\in[0,1]} \eta \int_a^b \left\| \left\langle \hat{\Pi}_k^\omega(\eta) - \Pi_k^\omega, \Pi_k^\omega \right\rangle_{S_2} \Pi_k^{(\omega)} \right\|_2 d\omega = O\Big(\frac{\log^{2/\gamma}(T)}{b_T T}\Big) + o_p\big(\frac{1}{b_T T}\big) = o\Big(\frac{1}{\sqrt{b_{T_1}T_1 + b_{T_2}T_2}}\Big).$$

To treat the other two terms of (A.3), we first observe that

$$\sqrt{\frac{1}{\big(\lambda_j^{(\omega)}\lambda_{j'}^{(\omega)} - (\lambda_k^{(\omega)})^2\big)^2}} = \frac{1}{|\lambda_j^{(\omega)}\lambda_{j'}^{(\omega)} - (\lambda_k^{(\omega)})^2|} \le C$$

for some bounded constant $C > 0$. Let $G_{X,k} = \inf_{l\neq k}\big|\lambda_{X,k}^{(\omega)} - \lambda_{X,l}^{(\omega)}\big|$. And observe that

$$\lambda_j^{(\omega)}\lambda_{j'}^{(\omega)} - (\lambda_k^{(\omega)})^2 = \Big|\lambda_j^{(\omega)}\big(\lambda_{j'}^{(\omega)} - \lambda_k^{(\omega)}\big) + \lambda_k^{(\omega)}\big(\lambda_j^{(\omega)} - \lambda_k^{(\omega)}\big)\Big|$$

We distinguish three cases. If $\lambda_k < \lambda_j$ and $\lambda_k < \lambda'_j$ are positive this is bounded from below by $2\inf_j \lambda_j^{(\omega)} G_{X,k}$. By the reverse triangle inequality

$$\Big|\lambda_j^{(\omega)}\big(\lambda_{j'}^{(\omega)} - \lambda_k^{(\omega)}\big) + \lambda_k^{(\omega)}\big(\lambda_j^{(\omega)} - \lambda_k^{(\omega)}\big)\Big| \ge \Big|\lambda_j^{(\omega)}\big|\big(\lambda_{j'}^{(\omega)} - \lambda_k^{(\omega)}\big)\big| - \lambda_k^{(\omega)}\big|\big(\lambda_j^{(\omega)} - \lambda_k^{(\omega)}\big)\big|\Big|$$

if $\lambda_k > \lambda_j$ and $\lambda_k > \lambda'_j$ then by the reverse triangle inequality

$$\Big|\lambda_j^{(\omega)}\big|\big(\lambda_{j'}^{(\omega)} - \lambda_k^{(\omega)}\big)\big| - \lambda_k^{(\omega)}\big|\big(\lambda_j^{(\omega)} - \lambda_k^{(\omega)}\big)\big|\Big| \ge \lambda_j\Big|\lambda_k^{(\omega)} - \lambda_{j'}^{(\omega)} + \lambda_j^{(\omega)} - \lambda_k^{(\omega)}\Big| \ge \lambda_j\Big|\lambda_j^{(\omega)} - \lambda_{j'}^{(\omega)}\Big| \ge \inf_j \lambda_j^{(\omega)} G_{X,k}$$

if $\lambda_k < \lambda_j$ and $\lambda_k > \lambda'_j$ then under Assumption 3.5,

$$\begin{aligned}
\Big|\lambda_j^{(\omega)}\big|\big(\lambda_{j'}^{(\omega)} - \lambda_k^{(\omega)}\big)\big| - \lambda_k^{(\omega)}\big|\big(\lambda_j^{(\omega)} - \lambda_k^{(\omega)}\big)\big|\Big| &\ge \lambda_k\Big|\big|\big(\lambda_{j'}^{(\omega)} - \lambda_k^{(\omega)}\big)\big| - \big|\big(\lambda_j^{(\omega)} - \lambda_k^{(\omega)}\big)\big|\Big|\\
&\ge \lambda_k\Big|2\lambda_k - \lambda_j^{(\omega)} - \lambda_{j'}^{(\omega)}\Big| > 0.
\end{aligned}$$

The same holds true for the case $\lambda_k > \lambda_j$ and $\lambda_k < \lambda'_j$. By the Cauchy-Schwarz inequality and Holder's inequality for operators, we obtain

$$\begin{aligned}
&\sup_{\eta\in[0,1]} \eta \int_a^b \left\| \frac{\left\langle \big(\hat{\mathscr{F}}^\omega(\eta) - \mathscr{F}^\omega\big)\widetilde{\otimes}\big(\hat{\mathscr{F}}^\omega(\eta) - \mathscr{F}^\omega\big)\Pi_k^{(\omega)}, \Pi_{jj'}^{(\omega)} \right\rangle_{S_2} \Pi_{jj'}^{(\omega)}}{\lambda_j^{(\omega)}\lambda_{j'}^{(\omega)} - (\lambda_k^{(\omega)})^2} \right\|_2 d\omega\\
&\le C \sup_{\eta\in[0,1]} \eta \int_a^b \left\| \hat{\mathscr{F}}^\omega(\eta) - \mathscr{F}^\omega \right\|_2^2 d\omega = O_p\big(\frac{\log^{2/\gamma}(T_1)}{b_{T_1}T_1}\big) = o_p\big(\frac{1}{\sqrt{b_{T_1}T_1 + b_{T_2}T_2}}\big),
\end{aligned}$$

where we used that the eigenprojectors are rank-one operators (and hence elements of $S_1(\mathscr{H})$) and where the order follows from Lemma B.1. For the last term of (A.3), Parseval's identity and orthogonality of the eigenprojectors yield

$$\sup_{\eta\in[0,1]} \eta \int_a^b \left\| \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^\complement}}^\infty \frac{\left\langle E_{k,b_T}^{(\omega)}(\eta), \Pi_{jj'}^{(\omega)} \right\rangle_{S_2}}{\lambda_j^{(\omega)}\lambda_{j'}^{(\omega)} - (\lambda_k^{(\omega)})^2} \Pi_{jj'}^{(\omega)} \right\|_2 d\omega$$

$$\leq C \sup_{\eta\in[0,1]} \eta \int_a^b \Big(\sum_{r,s=1}^{\infty} \Big| \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty} \Big\langle E^{(\omega)}_{k,b_T}(\eta), \Pi^{(\omega)}_{jj'} \Big\rangle_{S_2} \Big\langle \Pi^{(\omega)}_{jj'}, \Pi^{(\omega)}_{r,s} \Big\rangle_{S_2} \Big|^2 \Big)^{1/2} d\omega$$

$$\leq C \sup_{\eta\in[0,1]} \eta \int_a^b \Big|\Big|\Big| E^{(\omega)}_{k,b_T}(\eta) \Big|\Big|\Big|_2 d\omega.$$

We find using Lemma B.5 and Holder's inequality

$$\Big|\Big|\Big| E^{(\omega)}_{k,b_T}(\eta) \Big|\Big|\Big|_2 \leq \Big\{ 2|||\mathscr{F}^{(\omega)}_X|||_2 |||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2 + |||\hat{\mathscr{F}}^{\omega}_X(\eta) - \mathscr{F}^{\omega}|||_2^2 + C_1 |||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2 + |||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2^2 \Big\}$$
$$\times C\Big(|||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2 + |||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2^2 \Big) = \tilde{C} \sum_{i=2}^{4} |||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2^i$$

for some constants $C, \tilde{C}$. Similar to (A.4), we thus obtain from Theorem B.1

$$\tilde{C} \mathbb{E} \sup_{\eta\in[0,1]} \eta \int_a^b \Big|\Big|\Big| E^{(\omega)}_{k,b_T}(\eta) \Big|\Big|\Big|_2 d\omega \leq \tilde{C} \int_a^b \mathbb{E}\Big(\sup_{\eta\in[0,1]} \sum_{i=2}^{4} \eta |||\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X|||_2^i \Big) d\omega$$
$$\leq \tilde{C} \sum_{i=2}^{4} \int_a^b \mathbb{E}\Big(\sup_{\eta\in[0,1]} |||\eta^{1/i} \big(\hat{\mathscr{F}}^{(\omega)}_X(\eta) - \mathscr{F}^{(\omega)}_X\big)|||_2 \Big)^i d\omega.$$
$$= O\Big(\frac{\log^{2/\gamma}(T)}{b_T T} \Big) + o_p\big(\frac{1}{b_T T}\big) = o\Big(\frac{1}{\sqrt{b_{T_1} T_1 + b_{T_2} T_2}} \Big).$$

This proves (i). The proof of (ii) follows along the same lines and is therefore omitted. □

Proof of Theorem 3.4. Using Lemma 3.2, it suffices to show that

$$\int_a^b \Big\langle \widetilde{M}_{\Pi,k}(\eta,\omega), \eta M_{\Pi,k}(\omega) \Big\rangle_{S_2} d\omega + \int_a^b \overline{\Big\langle \widetilde{M}_{\Pi,k}(\eta,\omega), \eta M_{\Pi,k}(\omega) \Big\rangle_{S_2}} d\omega + o_P\Big(\frac{1}{\sqrt{b_1 T_1 + b_2 T_2}} \Big)$$
$$= \int_a^b \Big(\frac{1}{\sqrt{b_1 T_1}} \Re\Big(\Big\langle \mathcal{Z}^{X,\omega}_{T,\eta}, \tilde{\Pi}^{(\omega)}_{X,Y,k} \Big\rangle_{S_2} \Big) + \frac{1}{\sqrt{b_2 T_2}} \Re\Big(\Big\langle \mathcal{Z}^{Y,\omega}_{T,\eta}, \tilde{\Pi}^{(\omega)}_{X,Y,k} \Big\rangle_{S_2} \Big) \Big) d\omega + o_P\Big(\frac{1}{\sqrt{b_1 T_1 + b_2 T_2}} \Big).$$

To ease notation, denote

$$U^{X,\omega}_{T,\eta} = \mathscr{F}^{(\omega)}_X \widetilde{\otimes} Z^{X,\omega}_{T,\eta} + Z^{X,\omega}_{T,\eta} \widetilde{\otimes} \mathscr{F}^{(\omega)}_X \quad \text{and} \quad U^{Y,\omega}_{T,\eta} = \mathscr{F}^{(\omega)}_Y \widetilde{\otimes} Z^{Y,\omega}_{T,\eta} + Z^{Y,\omega}_{T,\eta} \widetilde{\otimes} \mathscr{F}^{(\omega)}_Y. \tag{A.5}$$

Using orthogonality of the eigenfunctions, we can write

$$\sqrt{b_{T_1} T_1 + b_{T_2} T_2} \int_a^b \Big\langle \widetilde{M}_{\Pi,k}(\eta,\omega), \eta M^{(k)}(\omega) \Big\rangle_{S_2} d\omega$$
$$= \sqrt{b_{T_1} T_1 + b_{T_2} T_2} \int_a^b \Big\langle \frac{1}{\sqrt{b_{T_1} T_1}} \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty} \frac{1}{\lambda^{(\omega)}_{X,j}\lambda^{(\omega)}_{X,j'} - (\lambda^{(\omega)}_{X,k})^2} \Big\langle U^{X,\omega}_{T,\eta} \Pi^{(\omega)}_{X,k}, \Pi^{(\omega)}_{X,jj'} \Big\rangle_{S_2} \Pi^{(\omega)}_{X,jj'}, \eta\Big(\Pi^{(\omega)}_{X,k} - \Pi^{(\omega)}_{Y,k}\Big) \Big\rangle_{S_2} d\omega$$
$$- \sqrt{b_{T_1} T_1 + b_{T_2} T_2} \int_a^b \Big\langle \frac{1}{\sqrt{b_{T_2} T_2}} \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty} \frac{1}{\lambda^{(\omega)}_{Y,j}\lambda^{(\omega)}_{Y,j'} - (\lambda^{(\omega)}_{Y,k})^2} \Big\langle U^{Y,\omega}_{T,\eta} \Pi^{(\omega)}_{Y,k}, \Pi^{(\omega)}_{Y,jj'} \Big\rangle_{S_2} \Pi^{(\omega)}_{Y,jj'}, \eta\Big(\Pi^{(\omega)}_{X,k} - \Pi^{(\omega)}_{Y,k}\Big) \Big\rangle_{S_2} d\omega$$

$$= \eta \sqrt{b_{T_1} T_1 + b_{T_2} T_2} \frac{-1}{\sqrt{b_{T_1} T_1}} \int_a^b \Big\langle \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty} \frac{\Big\langle U^{X,\omega}_{T,\eta} \Pi^{(\omega)}_{X,k}, \Pi^{(\omega)}_{X,jj'} \Big\rangle_{S_2}}{\lambda^{(\omega)}_{X,j}\lambda^{(\omega)}_{X,j'} - (\lambda^{(\omega)}_{X,k})^2} \Pi^{(\omega)}_{X,jj'}, \Pi^{(\omega)}_{Y,k} \Big\rangle_{S_2} d\omega \tag{A.6}$$

$$\eta \sqrt{b_{T_1} T_1 + b_{T_2} T_2} \frac{-1}{\sqrt{b_{T_2} T_2}} \int_a^b \Big\langle \sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty} \frac{\Big\langle U^{Y,\omega}_{T,\eta} \Pi^{(\omega)}_{Y,k}, \Pi^{(\omega)}_{Y,jj'} \Big\rangle_{S_2}}{\lambda^{(\omega)}_{Y,j}\lambda^{(\omega)}_{Y,j'} - (\lambda^{(\omega)}_{Y,k})^2} \Pi^{(\omega)}_{Y,jj'}, \Pi^{(\omega)}_{X,k} \Big\rangle_{S_2} d\omega. \tag{A.7}$$

To simplify the expression, observe that the properties of the Kronecker product and the Hilbert-Schmidt inner product together with orthogonality of the eigenfunctions yield

$$\begin{aligned}\left\langle (\mathscr{F}_X^{(\omega)}\widetilde{\otimes} Z_{T,\eta}^{X,\omega})\Pi_{X,k}^{(\omega)}, \Pi_{X,jj'}^{(\omega)}\right\rangle_{S_2} &= \left\langle \mathscr{F}_X^{(\omega)}(\phi_{X,k}^{(\omega)})\otimes Z_{T,\eta}^{X,\omega}(\phi_{X,k}^{(\omega)}), \Pi_{X,jj'}^{(\omega)}\right\rangle_{S_2}\\ &= \lambda_k^{(\omega)}\left\langle \phi_{X,k}^{(\omega)},\phi_{X,j}^{(\omega)}\right\rangle\left\langle \phi_{X,j'}^{(\omega)}, Z_{T,\eta}^{X,\omega}(\phi_{X,k}^{(\omega)})\right\rangle = \lambda_{X,k}^{(\omega)}\left\langle \phi_{X,j'}^{(\omega)}, Z_{T,\eta}^{X,\omega}(\phi_{X,k}^{(\omega)})\right\rangle\end{aligned}$$

for $j = k$ and zero otherwise. Similarly,

$$\left\langle (Z_{T,\eta}^{X,\omega}\,\widetilde{\otimes}\mathscr{F}_X^{(\omega)})\Pi_{X,k}^{(\omega)}, \Pi_{X,jj'}^{(\omega)}\right\rangle_{S_2} = \lambda_{X,k}^{(\omega)}\left\langle Z_{T,\eta}^{X,\omega}(\phi_{X,k}^{(\omega)}),\phi_{X,j}^{(\omega)}\right\rangle$$

for $j' = k$ and zero otherwise. From this and (A.5), we obtain

$$\begin{aligned}&\sum_{\substack{j,j'=1,\\ \{j,j'=k\}^{\complement}}}^{\infty}\frac{1}{\lambda_{X,j}^{(\omega)}\lambda_{X,j'}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left\langle U_{T,\eta}^{X,\omega}\Pi_{X,k}^{(\omega)},\Pi_{X,jj'}^{(\omega)}\right\rangle_{S_2}\Pi_{X,jj'}^{(\omega)}\\ &=\sum_{j'\neq k}\frac{\lambda_k^{(\omega)}\langle\phi_{j'}^{(\omega)},Z_{T,\eta}^{X,\omega}\phi_k^{(\omega)}\rangle}{\lambda_{X,k}^{(\omega)}\lambda_{X,j'}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\Pi_{X,kj'}^{(\omega)}+\sum_{j\neq k}\frac{\lambda_k^{(\omega)}\langle Z_{T,\eta}^{X,\omega}\phi_k^{(\omega)},\phi_j^{(\omega)}\rangle}{\lambda_{X,j}^{(\omega)}\lambda_{X,k}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\Pi_{X,jk}^{(\omega)}\end{aligned}$$

which means that the integrand of (A.6) becomes

$$\begin{aligned}&\sum_{j'\neq k}\frac{\lambda_k^{(\omega)}\langle\phi_{j'}^{(\omega)},Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)})\rangle}{\lambda_{X,k}^{(\omega)}\lambda_{X,j'}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left\langle\Pi_{X,kj'}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}+\sum_{j\neq k}\frac{\lambda_k^{(\omega)}\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\rangle}{\lambda_{X,j}^{(\omega)}\lambda_{X,k}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left\langle\Pi_{X,jk}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}=\\ &\sum_{j\neq k}\frac{\lambda_k^{(\omega)}}{\lambda_{X,k}^{(\omega)}\lambda_{X,j}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left[\left\langle\phi_j^{(\omega)},Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)})\right\rangle\left\langle\Pi_{X,kj}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}+\left\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\right\rangle\left\langle\Pi_{X,jk}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}\right]\end{aligned}$$

whereas for its conjugate, which arises in $\overline{\left\langle \widetilde{M}_{\Pi,k}(\eta,\omega),\eta M^{(k)}(\omega)\right\rangle_{S_2}}$, we obtain

$$\sum_{j\neq k}\frac{\lambda_k^{(\omega)}}{\lambda_{X,k}^{(\omega)}\lambda_{X,j}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left[\overline{\left\langle\phi_j^{(\omega)},Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)})\right\rangle\left\langle\Pi_{X,kj}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}}+\overline{\left\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\right\rangle\left\langle\Pi_{X,jk}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}}\right].$$

Next, recall that for complex numbers $c_1, c_2 \in \mathbb{C}$, $c_1+\overline{c_1} = \Re(c_1)+\mathrm{i}\Im(c_1)+\Re(c_1)-\mathrm{i}\Im(c_1) = 2\Re(c_1)$ and $\Re(c_1c_2) = \Re(c_1)\Re(c_2)-\Im(c_1)\Im(c_2)$ and $\Re(\overline{c_1}c_2) = \Re(c_1)\Re(c_2)+\Im(c_1)\Im(c_2)$. Therefore, summing the respective integrands of $\left\langle \widetilde{M}_{\Pi,k}(\eta,\omega),\eta M^{(k)}(\omega)\right\rangle_{S_2}$ and $\overline{\left\langle \widetilde{M}_{\Pi,k}(\eta,\omega),\eta M^{(k)}(\omega)\right\rangle_{S_2}}$ yields

$$\begin{aligned}&\sum_{j\neq k}\frac{\lambda_{X,k}^{(\omega)}}{\lambda_{X,k}^{(\omega)}\lambda_{X,j}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left[2\Re\Big(\left\langle\phi_j^{(\omega)},Z_{T,\eta}^{X,\omega}\phi_k^{(\omega)}\right\rangle\left\langle\Pi_{X,kj}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}\Big)+2\Re\Big(\left\langle Z_{T,\eta}^{X,\omega}\phi_k^{(\omega)},\phi_j^{(\omega)}\right\rangle\left\langle\Pi_{X,jk}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}\Big)\right]\\ &=\sum_{j\neq k}\frac{\lambda_{X,k}^{(\omega)}}{\lambda_{X,k}^{(\omega)}\lambda_{X,j}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left[2\Re\Big(\overline{\left\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\right\rangle}\left\langle\Pi_{X,kj}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}\Big)+2\Re\Big(\left\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\right\rangle\left\langle\Pi_{X,jk}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}\Big)\right]\\ &=\sum_{j\neq k}\frac{\lambda_{X,k}^{(\omega)}}{\lambda_{X,k}^{(\omega)}\lambda_{X,j}^{(\omega)}-(\lambda_{X,k}^{(\omega)})^2}\left[4\Re\Big(\left\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\right\rangle\Big)\Re\Big(\left\langle\Pi_{X,kj}^{(\omega)},\Pi_{Y,k}^{(\omega)}\right\rangle_{S_2}\Big)\right].\end{aligned}$$

The result now follows from doing the same for the integrand in (A.7) and noting that $\langle Z_{T,\eta}^{X,\omega}(\phi_k^{(\omega)}),\phi_j^{(\omega)}\rangle = \left\langle Z_{T,\eta}^{X,\omega},\Pi_{X,jk}^{(\omega)}\right\rangle_{S_2}$ and doing the same for the integrand in (A.7). □

Proof of Theorem 3.6. Similar to the processes $\{\hat{Z}^{[a,b]}_{\mathscr{F},T_1,T_2}(\eta)\}_{\eta\in[0,1]}$ and $\{\hat{Z}^{[a,b],(k)}_{\Pi,T_1,T_2}(\eta)\}_{\eta\in[0,1]}$, we may write

$$\hat{\mathscr{Z}}^{[a,b],(k)}_{\lambda,T_1,T_2}(\eta) = \int_a^b \left|\widehat{M}_{\lambda,k}(\eta,\omega)\right| - \eta^2\left|M_{\lambda,k}(\omega)\right|^2 d\omega$$

$$= \int_a^b \Big\{ \big| \widehat{M}_{\lambda,k}(\eta,\omega) - \eta M_{\lambda,k}(\omega) \big|^2 + \eta \overline{M_{\lambda,k}(\omega)} \big(\widehat{M}_{\lambda,k}(\eta,\omega) - \eta M_{\lambda,k}(\omega) \big)$$

$$+ \eta M_{\lambda,k}(\omega) \overline{\big(\widehat{M}_{\lambda,k}(\eta,\omega) - \eta M_{\lambda,k}(\omega) \big)} \Big\} d\omega. \tag{A.8}$$

We can therefore apply Proposition 3.3 and Lemma A.1 below to find the result. □

Lemma A.1. *Suppose Assumptions 3.1-3.4 are satisfied. Then,*

$$\text{(i)} \sup_{\eta \in [0,1]} \int_a^b \Big| \widehat{M}_{\lambda,k}(\eta,\omega) - \eta M_{\lambda,k}(\omega) - \widetilde{M}_{\lambda,k}(\eta,\omega) \Big| d\omega = o_P\Big(\frac{1}{\sqrt{b_1 T_1 + b_2 T_2}} \Big);$$

$$\text{(ii)} \sup_{\eta \in [0,1]} \int_a^b \Big| \widetilde{M}_{\lambda,k}(\eta,\omega) \Big|^2 d\omega = o_P\Big(\frac{1}{\sqrt{b_1 T_1 + b_2 T_2}} \Big),$$

where

$$\widetilde{M}_{\lambda,k}(\eta,\omega) = \int_a^b \frac{1}{\sqrt{b_1 T_1}} \Big\langle \mathcal{Z}^{X,\omega}_{T,\eta}, \Pi^{(\omega)}_{X,k} \Big\rangle_{S_2} - \frac{1}{\sqrt{b_2 T_2}} \Big\langle \mathcal{Z}^{Y,\omega}_{T,\eta}, \Pi^{(\omega)}_{Y,k} \Big\rangle_{S_2} d\omega$$

with $\mathcal{Z}^{X,\omega}_{T,\eta}$ *and* $\mathcal{Z}^{Y,\omega}_{T,\eta}$ *respectively given by* (3.2) *and* (3.3).

Proof of Lemma A.1. Similar to the proof of Lemma 3.2, we can show (in this case using Proposition 3.3 and Lemma B.5) that

$$\sqrt{b_1 T_1 + b_2 T_2} \sup_{\eta \in [0,1]} \int_a^b |||E^{(\omega)}_{\lambda,k,b_T}(\eta)|||_2 d\omega$$

converges to zero in probability using as $T_1, T_2 \to \infty$. The result then follows from Lemma B.2. The details are omitted for the sake of brevity. □

A.2 Perturbations of eigenelements - proofs of Proposition 3.2 and 3.3

Proof of Proposition 3.2. We write

$$\hat{\mathscr{F}}^{\omega}(\eta) \widetilde{\otimes} \hat{\mathscr{F}}^{\omega}(\eta) = \mathscr{F}^{\omega} \widetilde{\otimes} \mathscr{F}^{\omega} + \underbrace{(\hat{\mathscr{F}}^{\omega}(\eta) \widetilde{\otimes} \hat{\mathscr{F}}^{\omega}(\eta) - \mathscr{F}^{\omega} \widetilde{\otimes} \mathscr{F}^{\omega})}_{= \Delta_\eta \mathscr{F}^{\omega}_{\widetilde{\otimes}}}$$

and to ease notation we shall moreover denote $\hat{\mathscr{F}}^{\omega}_{\widetilde{\otimes}}(\eta) = \hat{\mathscr{F}}^{\omega}(\eta) \widetilde{\otimes} \hat{\mathscr{F}}^{\omega}(\eta)$ and $\mathscr{F}^{\omega}_{\widetilde{\otimes}} = \mathscr{F}^{\omega} \widetilde{\otimes} \mathscr{F}^{\omega}$. Observe that we have the following decomposition

$$\mathscr{F}^{\omega}_{\widetilde{\otimes}} = \sum_{j,j'} \lambda^{(\omega)}_j \lambda^{(\omega)}_{j'} \Pi^{(\omega)}_j \widetilde{\otimes} \Pi^{(\omega)}_{j'},$$

and note that

$$\mathscr{F}^{\omega}_{\widetilde{\otimes}} \Pi^{(\omega)}_{k,l} = \sum_{j,j'} \lambda^{(\omega)}_j \lambda^{(\omega)}_{j'} \Big(\Pi^{(\omega)}_j \widetilde{\otimes} \Pi^{(\omega)}_{j'} \Big) \Pi^{(\omega)}_{k,l} = \lambda^{(\omega)}_k \lambda^{(\omega)}_l \Pi^{(\omega)}_{k,l}$$

Hence $\{\Pi^{(\omega)}_{k,l}\}_{k,l}$ are the eigenvectors of $\mathscr{F}^{\omega}_{\widetilde{\otimes}}$. We would like to obtain expressions for $\Delta_\eta \Pi^{(\omega)}_{k,l}$ and $\Delta_\eta \lambda^{(\omega)}_{k,l}$ by solving the equation

$$\big(\mathscr{F}^{\omega}_{\widetilde{\otimes}} + \Delta_\eta \mathscr{F}^{\omega}_{\widetilde{\otimes}} \big) (\Pi^{(\omega)}_{k,l} + \Delta_\eta \Pi^{(\omega)}_{k,l}) = (\lambda^{(\omega)}_{k,l} + \Delta_\eta \lambda^{(\omega)}_{k,l}) (\Pi^{(\omega)}_{k,l} + \Delta_\eta \Pi^{(\omega)}_{k,l})$$

$$\Leftrightarrow$$

$$\mathscr{F}^{\omega}_{\widetilde{\otimes}} \Pi^{(\omega)}_{k,l} + \Delta_\eta \mathscr{F}^{\omega}_{\widetilde{\otimes}} \Pi^{(\omega)}_{k,l} + \mathscr{F}^{\omega}_{\widetilde{\otimes}} \Delta_\eta \Pi^{(\omega)}_{k,l} + \Delta_\eta \mathscr{F}^{\omega}_{\widetilde{\otimes}} \Delta_\eta \Pi^{(\omega)}_{k,l} = \lambda^{(\omega)}_{k,l} \Pi^{(\omega)}_{k,l} + \Delta_\eta \lambda^{(\omega)}_{k,l} \Pi^{(\omega)}_{k,l} + \lambda^{(\omega)}_{k,l} \Delta_\eta \Pi^{(\omega)}_{k,l} + \Delta_\eta \lambda^{(\omega)}_{k,l} \Delta_\eta \Pi^{(\omega)}_{k,l}$$

$$\Leftrightarrow$$

$$\Delta_\eta \mathscr{F}^{\omega}_{\widetilde{\otimes}} \Pi^{(\omega)}_{k,l} + \mathscr{F}^{\omega}_{\widetilde{\otimes}} \Delta_\eta \Pi^{(\omega)}_{k,l} = \Delta_\eta \lambda^{(\omega)}_{k,l} \Pi^{(\omega)}_{k,l} + \lambda^{(\omega)}_{k,l} \Delta_\eta \Pi^{(\omega)}_{k,l} + (\Delta_\eta \lambda^{(\omega)}_{k,l} - \Delta_\eta \mathscr{F}^{\omega}_{\widetilde{\otimes}}) \Delta_\eta \Pi^{(\omega)}_{k,l}$$

since $\mathscr{F}^{\omega}_{\widetilde{\otimes}}\Pi^{(\omega)}_{k,l} = \lambda^{(\omega)}_{k,l}\Pi^{(\omega)}_{k,l}$. First note that $\Delta_\eta\Pi^{(\omega)}_{k,l}$ is a well-defined element of $S_2(\mathscr{H})$. we can therefore use a basis expansion to write

$$\Delta_\eta\Pi^{(\omega)}_{k,l} = \hat{\Pi}^{(\omega)}_{X,k,l}(\eta) - \Pi^{(\omega)}_{X,k,l} \overset{S_2}{=} \sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm}$$

where $a^{\eta}_{n,m}$ is a set of coefficients. Plugging this into the second term on the left and right hand side of the above equation we have

$$\Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}}\Pi^{(\omega)}_{k,l} + \mathscr{F}^{\omega}_{\widetilde{\otimes}}\sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm} = \Delta_\eta\lambda^{(\omega)}_{k,l}\Pi^{(\omega)}_{k,l} + \lambda^{(\omega)}_{k,l}\sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm} + (\Delta_\eta\lambda^{(\omega)}_{k,l} - \Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}})\Delta_\eta\Pi^{(\omega)}_{k,l}. \tag{A.9}$$

Observe that orthogonality of the eigenfunctions yields

$$\mathscr{F}^{\omega}_{\widetilde{\otimes}}\sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm} = \sum_{r,s=1}^{\infty}\lambda^{(\omega)}_{r,s}\Pi^{(\omega)}_{r}\widetilde{\otimes}\Pi^{(\omega)}_{s}\sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm} = \sum_{r,s=1}^{\infty}\lambda^{(\omega)}_{r,s}a^{\eta}_{r,s}\Pi^{(\omega)}_{r,s}.$$

and therefore (A.9) becomes

$$\Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}}\Pi^{(\omega)}_{k,l} + \sum_{r,s=1}^{\infty}\lambda^{(\omega)}_{r,s}a^{\eta}_{r,s}\Pi^{(\omega)}_{r,s} = \Delta_\eta\lambda^{(\omega)}_{k,l}\Pi^{(\omega)}_{k,l} + \lambda^{(\omega)}_{k,l}\sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm} + (\Delta_\eta\lambda^{(\omega)}_{k,l} - \Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}})\Delta_\eta\Pi^{(\omega)}_{k,l}.$$

Taking the Hilbert-Schmidt inner product with $\Pi^{(\omega)}_{jj'}$, $j \neq k, j' \neq l$

$$\begin{aligned}\Big\langle\Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}}\Pi^{(\omega)}_{k,l},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2} + \Big\langle\sum_{r,s=1}^{\infty}\lambda^{(\omega)}_{r,s}a^{\eta}_{r,s}\Pi^{(\omega)}_{r,s},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2} &= \Big\langle\Delta_\eta\lambda^{(\omega)}_{k,l}\Pi^{(\omega)}_{k,l},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2} + \Big\langle\lambda^{(\omega)}_{k,l}\sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\\ &\quad + \Big\langle(\Delta_\eta\lambda^{(\omega)}_{k,l} - \Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}})\Delta_\eta\Pi^{(\omega)}_{k,l},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\end{aligned}$$

which becomes

$$\Big\langle\Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}},\Pi^{(\omega)}_{jj'}\widetilde{\otimes}\Pi^{(\omega)}_{k,l}\Big\rangle_{S_2} + \lambda^{(\omega)}_{j,j'}a^{\eta}_{j,j'} = 0 + \lambda^{(\omega)}_{j,j'}a^{\eta}_{j,j'} + \Big\langle(\Delta_\eta\lambda^{(\omega)}_{k,l} - \Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}})\Delta_\eta\Pi^{(\omega)}_{k,l},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}$$

rearranging, we find the coefficients are given by

$$a^{\eta}_{j,j'} = \frac{1}{\lambda^{(\omega)}_{j,j'} - \lambda^{(\omega)}_{k,l}}\Big[-\Big\langle\Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}},\Pi^{(\omega)}_{jj'}\widetilde{\otimes}\Pi^{(\omega)}_{k,l}\Big\rangle_{S_2} + \Big\langle(\Delta_\eta\lambda^{(\omega)}_{k,l} - \Delta_\eta\mathscr{F}^{\omega}_{\widetilde{\otimes}})\Delta_\eta\Pi^{(\omega)}_{k,l},\Pi^{(\omega)}_{jj'}\Big\rangle_{S_2}\Big].$$

If $j = k, l = j'$, we set $a^{\eta}_{k,l} = \Big\langle\hat{\Pi}^{\omega}_{k,l} - \Pi^{\omega}_{k,l},\Pi^{\omega}_{k,l}\Big\rangle_{S_2}$. The statement now follows. □

Proof of Proposition 3.3. In order to obtain an expression for $\Delta_\eta\lambda^{(\omega)}_{k}$, write

$$\hat{\mathscr{F}}^{\omega}(\eta) = \mathscr{F}^{\omega} + \underbrace{(\hat{\mathscr{F}}^{\omega}(\eta) - \mathscr{F}^{\omega})}_{=\Delta_\eta\mathscr{F}^{\omega}}.$$

Since $\mathscr{F}^{\omega}\Pi^{(\omega)}_{k} = \lambda^{(\omega)}_{k}\Pi^{(\omega)}_{k}$, we obtain similar to the proof of Proposition 3.2 that

$$\begin{gathered}\big(\mathscr{F}^{\omega} + \Delta_\eta\mathscr{F}^{\omega}\big)(\Pi^{(\omega)}_{k} + \Delta_\eta\Pi^{(\omega)}_{k}) = (\lambda^{(\omega)}_{k} + \Delta_\eta\lambda^{(\omega)}_{k})(\Pi^{(\omega)}_{k} + \Delta_\eta\Pi^{(\omega)}_{k})\\ \Leftrightarrow\\ \Delta_\eta\mathscr{F}^{\omega}\Pi^{(\omega)}_{k} + \mathscr{F}^{\omega}\Delta_\eta\Pi^{(\omega)}_{k} = \Delta_\eta\lambda^{(\omega)}_{k}\Pi^{(\omega)}_{k} + \lambda^{(\omega)}_{k}\Delta_\eta\Pi^{(\omega)}_{k} + (\Delta_\eta\lambda^{(\omega)}_{k} - \Delta_\eta\mathscr{F}^{\omega})\Delta_\eta\Pi^{(\omega)}_{k}\end{gathered}$$

And using again that $\Delta_\eta\Pi^{(\omega)}_{k} = \sum_{n,m=1}^{\infty} a^{\eta}_{n,m}\Pi^{(\omega)}_{nm}$ and taking the inner product with $\Pi^{(\omega)}_{k}$

$$\Big\langle\Delta_\eta\mathscr{F}^{\omega}\Pi^{(\omega)}_{k},\Pi^{(\omega)}_{k}\Big\rangle_{S_2} + \Big\langle\sum_{r,m=1}^{\infty}\lambda^{(\omega)}_{r}a^{\eta}_{r,m}\Pi^{(\omega)}_{rm},\Pi^{(\omega)}_{k}\Big\rangle_{S_2}$$

$$= \left\langle \Delta_\eta \lambda_k^{(\omega)} \Pi_k^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2} + \left\langle \lambda_k^{(\omega)} \sum_{n,m=1}^{\infty} a_{n,m}^{\eta} \Pi_{nm}^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2} + \left\langle (\Delta_\eta \lambda_k^{(\omega)} - \Delta_\eta \mathscr{F}^{\omega}) \Delta_\eta \Pi_k^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2}$$

which becomes, due to orthogonality

$$\left\langle \Delta_\eta \mathscr{F}^{\omega} \Pi_k^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2} + \lambda_k^{(\omega)} a_{k,k}^{\eta} = \Delta_\eta \lambda_k^{(\omega)} + \lambda_k^{(\omega)} a_{k,k}^{\eta} + \left\langle (\Delta_\eta \lambda_k^{(\omega)} - \Delta_\eta \mathscr{F}^{\omega}), \Delta_\eta \Pi_k^{(\omega)} \right\rangle_{S_2}$$

rearranging yields

$$\begin{aligned}
\Delta_\eta \lambda_k^{(\omega)} &= \left\langle \Delta_\eta \mathscr{F}^{\omega} \Pi_k^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2} - \left\langle (\Delta_\eta \lambda_k^{(\omega)} - \Delta_\eta \mathscr{F}^{\omega}) \Delta_\eta \Pi_k^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2} \\
&= \left\langle \Delta_\eta \mathscr{F}^{\omega}, \Pi_k^{(\omega)} \right\rangle_{S_2} - \left\langle (\Delta_\eta \lambda_k^{(\omega)} - \Delta_\eta \mathscr{F}^{\omega}) \Delta_\eta \Pi_k^{(\omega)}, \Pi_k^{(\omega)} \right\rangle_{S_2}.
\end{aligned}$$

□

A.3 Proof of Theorem 5.2

Proof of Theorem 5.2. First, observe that $\{N_{m,T,t}^{(\omega)}, 1 \le t \le T\}$ forms a (square integrable) ergodic complex-valued martingale difference sequence for each $\omega \in [-\pi, \pi]$. To ease notation, we shall sometimes denote its integral in frequency direction over $[a, b]$ by

$$N_{m,T,t} = \int_a^b N_{m,T,t}^{(\omega)} d\omega, \quad 2 \le t \le T. \tag{A.10}$$

We derive the result by verifying the conditions of Corollary 3.8 of McLeish (1974). Firstly, observe that by Cauchy's inequality and Lemma B.6

$$\begin{aligned}
\int_a^b \mathbb{E}\big|N_{m,T,t}^{(\omega)}| d\omega &\le 2 \sup_{\omega} \|\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\|_{\mathcal{H}} \int_a^b \mathbb{E}\Big\| \sum_{s=1}^{t-1} \tilde{w}_{b_T,t,s}^{(\omega)} \mathscr{D}_{XY,t,s}^{(\omega)} \Big\|_{\mathcal{H}} d\omega \\
&\le 2 \sup_{\omega} \big(\|\!|\mathscr{U}_{X,Y}^{(\omega)}\|\!|_2 + \|\!|\mathscr{U}_{Y,X}^{(\omega)}\|\!|_2 \big) \\
&\times \int_a^b \mathbb{E}\Big(\Big\|\!\Big| \sum_{s=1}^{t-1} \tilde{w}_{b_T,t,s}^{(\omega)} (D_{X,m,t}^{(\omega)} \otimes D_{X,m,s}^{(\omega)}) \Big\|\!\Big|_2 + \Big\|\!\Big| \sum_{s=1}^{t-1} \tilde{w}_{b_T,t,s}^{(\omega)} (D_{Y,m,t}^{(\omega)} \otimes D_{Y,m,s}^{(\omega)}) \Big\|\!\Big|_2 \Big) d\omega \\
&\le 2 \sup_{\omega} \big(\|\!|\mathscr{U}_{X,Y}^{(\omega)}\|\!|_2 + \|\!|\mathscr{U}_{Y,X}^{(\omega)}\|\!|_2 \big) \times \\
&\int_a^b \|D_{X,m,t}^{(\omega)}\|_{\mathbb{H},2} \Big(\mathbb{E}\Big\| \sum_{s=1}^{t-1} \tilde{w}_{b_T,t,s}^{(\omega)} D_{X,m,s}^{(\omega)}) \Big\|_2^2 \Big)^{1/2} + \|D_{Y,m,t}^{(\omega)}\|_{\mathbb{H},2} \Big(\mathbb{E}\Big\| \sum_{s=1}^{t-1} \tilde{w}_{b_T,t,s}^{(\omega)} D_{Y,m,s}^{(\omega)}) \Big\|_2^2 \Big)^{1/2} d\omega \\
&\le C \max_t \Big(\sum_{s=1}^{t-1} |w_{b_T,t,s}|^2 \Big)^{1/2} = O(b_T^{-1/2}).
\end{aligned} \tag{A.11}$$

for some constant C. Therefore, since T is fixed, Fubini's theorem and the tower property imply that, for any $G \in \mathscr{G}_{t-1}$,

$$\begin{aligned}
\sum_{t=1}^{\lfloor \eta T \rfloor} \Big| \mathbb{E}\big[1_G \int_a^b N_{m,T,t}^{(\omega)} d\omega \big] \Big| &= \sum_{t=1}^{\lfloor \eta T \rfloor} \Big| \mathbb{E}\big[\int_a^b 1_G N_{m,T,t}^{(\omega)} d\omega \big] \Big| \\
&= \sum_{t=1}^{\lfloor \eta T \rfloor} \Big| \int_a^b \mathbb{E}\Big[\mathbb{E}[1_G N_{m,T,t}^{(\omega)} | \mathscr{G}_{t-1}] \Big] d\omega \Big| \\
&= \sum_{t=1}^{\lfloor \eta T \rfloor} \Big| \int_a^b \mathbb{E}\Big[1_G \mathbb{E}[N_{m,T,t}^{(\omega)} | \mathscr{G}_{t-1}] \Big] d\omega \Big| = 0 \quad \forall \eta \in [0,1],
\end{aligned}$$

where the last equality follows from the fact that $\{N_{m,T,t}^{(\omega)}, 1 \le t \le T\}$ forms a martingale difference sequence with respect to the filtration $\{\mathscr{G}_t\}$. We therefore obtain

$$\sum_{t=1}^{\lfloor \eta T \rfloor} \Big| \mathbb{E}\big[\int_a^b N_{m,T,t}^{(\omega)} d\omega \big] \big| \mathscr{G}_{t-1} \Big| = 0 \quad \forall T \in \mathbb{N}, \forall \eta \in [0,1],$$

showing that condition (3.11) of McLeish (1974) is satisfied. Next, we verify that the conditional Lindeberg condition is satisfied. This is implied if we show that the Lindeberg condition is satisfied, which follows almost along the same lines as in the proof of Theorem 3.2 of van Delft (2020). Therefore, we only give the main steps. From Jensen's inequality, Tonelli's theorem and Lemma B.1 of van Delft (2020), we obtain

$$\begin{aligned}
&\sum_{t=2}^{\lfloor \eta T\rfloor}\mathbb{E}\Big|\int_a^b \Big\langle \sum_{s=1}^{t-1}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}} d\omega\Big|^2\\
&=\sum_{t=2}^{\lfloor \eta T\rfloor}\mathbb{E}\Big|\int_a^b \Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s}+\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}} d\omega\Big|^2\\
&\le 2\sum_{t=2}^{\lfloor \eta T\rfloor}\mathbb{E}\Big|\int_a^b \Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}\Big|^2 d\omega\\
&+(b-a)\int_a^b 2\mathbb{E}\sum_{t=2}^{\lfloor \eta T\rfloor}\Big|\Big\langle \sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}\Big|^2 d\omega\\
&=2\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\mathbb{E}\Big|\int_a^b \Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}\Big|^2 d\omega+o(\mathscr{W}^2_{b_T}).
\end{aligned}$$

where we used in the last equation that $o(\frac{\eta T}{b_T})=o(\mathscr{W}^2_{b_T})$. It is therefore sufficient to focus on

$$\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\int_a^b \tilde{N}^{(\omega)}_{m,T,t}d\omega:=\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\int_a^b \Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}+\Big\langle \sum_{s=1}^{t-4m}\big(\tilde{w}^{(\omega)}_{b_T,s,t}\mathscr{D}^{(\omega)}_{XY,s,t}\big)^{\dagger},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}} d\omega \quad \text{(A.12)}$$

Now,

$$\begin{aligned}
\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\mathbb{E}\Big\{|\tilde{N}_{m,T,t}|^2 1_{|N_{m,T,t}|>\epsilon}\Big\}\le{}& 2\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\mathbb{E}\Big\{\Big|\int_a^b \Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}} d\omega\Big|^2 1_{|N_{m,T,t}|>\epsilon}\Big\}\\
&+2\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\mathbb{E}\Big\{\Big|\int_a^b \Big\langle \sum_{s=1}^{t-4m}\big(\tilde{w}^{(\omega)}_{b_T,s,t}\mathscr{D}^{(\omega)}_{XY,s,t}\big)^{\dagger},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}} d\omega\Big|^2 1_{|N_{m,T,t}|>\epsilon}\Big\}.
\end{aligned}$$

where we verify only the first term as the second is of the same order. Jensen's inequality and Lemma B.6 in turn yield under Assumption 3.1

$$\begin{aligned}
&\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\mathbb{E}\Big\{\Big|\mathscr{W}^{-1}_{b_T}\int_a^b \Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}} d\omega\Big|^2 1_{|\mathscr{W}^{-1}_{b_T}N_{m,T,t}|>\epsilon}\Big\}\\
&\le \frac{C}{\mathscr{W}^4_{b_T}}\int_a^b \sum_{t=1+4m}^{\lfloor \eta T\rfloor}\mathbb{E}\Big|\Big\langle \sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\mathscr{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}\Big|^4 d\omega\\
&=O\Big(\frac{b_T^2}{\kappa^2\lfloor \eta T\rfloor^2}\Big)O\Big(\frac{\lfloor \eta T\rfloor \kappa^2}{b_T^2}\Big)=o(1)
\end{aligned}$$

for some constant C. Using that $|x+\mathrm{i}y|=\sqrt{|x+\mathrm{i}y|^2}=\sqrt{x^2+y^2}$ and that $\{1_{|\Re(N_{m,T,t})|>\epsilon}\}\subset\{1_{|N_{m,T,t}|>\epsilon}\}$, we thus obtain from the above that

$$\lim_{T\to\infty}\frac{1}{\mathscr{W}^2_{b_T}}\sum_{t=1}^{\lfloor \eta T\rfloor}\mathbb{E}\Big\{\big(\Re(N_{m,T,t})\big)^2 1_{|\Re(\mathscr{W}^{-1}_{b_T}N_{m,T,t})|>\epsilon}\Big\}=0,$$

Finally, we verify condition (3.10) of McLeish (1974). Observe first that for the conditional variance, we obtain

$$\begin{aligned}
\frac{1}{\mathscr{W}^2_{b_T}}\sum_{t=1}^{\lfloor \eta T\rfloor}\mathbb{E}[|N_{m,T,t}|^2|\mathscr{G}_{t-1}]&=\frac{1}{\mathscr{W}^2_{b_T}}\sum_{t=1}^{\lfloor \eta T\rfloor}\mathbb{E}\Big[\int_a^b N^{(\omega)}_{m,T,t}d\omega\overline{\int_a^b N^{(\lambda)}_{m,T,t}d\lambda}\Big|\mathscr{G}_{t-1}\Big]\\
&=\frac{1}{\mathscr{W}^2_{b_T}}\sum_{t=1}^{\lfloor \eta T\rfloor}\lim_{n\to\infty}\Big(\sum_{i_1=1}^{n}\frac{(b-a)}{n}\mathbb{E}\Big[|N^{\omega_{i_1}}_{m,T,t}|^2\Big|\mathscr{G}_{t-1}\Big]
\end{aligned}$$

$$+\frac{(b-a)^2}{n^2}\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n}\mathbb{E}\Big[N^{\omega_{i_1}}_{m,T,t}\overline{N^{\omega_{i_2}}_{m,T,t}}\Big|\mathcal{G}_{t-1}\Big]\Big),$$

where Fubini's theorem justifies, via an analoguous reasoning to the above, the interchange of integrals. The conditions for Fubini's theorem can be verified via a similar derivation as in (A.11), More specifically, using the Cauchy-Schwarz inequality and exploiting uncorrelatedness of the increments we obtain in this case

$$\begin{aligned}\int_a^b\int_a^b\mathbb{E}\Big|N^{(\omega)}_{m,T,t}\overline{N^{(\lambda)}_{m,T,t}}\Big|d\omega d\lambda &\le \int_a^b\sqrt{\mathbb{E}|N^{(\omega)}_{m,T,t}|^2}d\omega\int_a^b\sqrt{\mathbb{E}|N^{(\lambda)}_{m,T,t}|^2}d\lambda\\ &\le\Big(C\pi\max_t\big(\sum_{s=1}^{t-1}|w_{b_T,t,s}|^2\big)^{1/2}\Big)^2=O(b_T^{-1}).\end{aligned}$$

Let $\mathcal{V}^{(\omega)}_X$ and $\mathcal{V}^{(\omega)}_Y$ be arbitrary elements of $S_2(\mathcal{H})$. These may be written in their canonical form, i.e., in the form $\sum_i s^{(\omega)}_{l,i}(u^{(\omega)}_{l,i}\otimes v^{(\omega)}_{l,i})$ where, for fixed ω, $\{u^{(\omega)}_{l,i}\}$ and $\{v^{(\omega)}_{l,i}\}$ are orthonormal bases of $\mathcal{H}$ and $\{s^{(\omega)}_{l,i}\}$ is a non-decreasing sequence of non-negative numbers converging to zero. Hence,

$$\mathcal{V}^{(\omega)}_{\boldsymbol{X}}:=(\mathcal{V}^{(\omega)}_X,\mathcal{V}^{(\omega)}_Y)^\top=(\sum_i s^{(\omega)}_{X,i}u^{(\omega)}_{X,i}\otimes v^{(\omega)}_{X,i},\sum_i s^{(\omega)}_{Y,i}u^{(\omega)}_{Y,i}\otimes v^{(\omega)}_{Y,i})^\top. \tag{A.13}$$

Using the definition of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and of $\langle\cdot,\cdot\rangle_{S_2}$, and the fact that the latter is continuous with respect to the $S_2(\mathcal{H})$-norm topology, we can write (A.12)

$$\begin{aligned}\tilde{N}^{(\omega)}_{m,T,t}&=\sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\Big\langle\mathcal{D}^{(\omega)}_{XY,t,s},\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}+\Big\langle\big(\tilde{w}^{(\omega)}_{b_T,s,t}\mathcal{D}^{(\omega)}_{XY,s,t}\big)^\dagger,\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\Big\rangle_{\mathcal{H}}\\ &=\sum_{l\in\{X,Y\}}\sum_{i=1}^{\infty}s^{(\omega)}_{l,i}\Big(D^{(\omega)}_{l,m,t}(u_{l,i})\overline{\tilde{J}^{(\omega)}_{m,b_T,t}(v_{l,i})}+\overline{D^{(\omega)}_{l,m,t}(v_{l,i})}\tilde{J}^{(\omega)}_{m,b_T,t}(u_{l,i})\Big)\end{aligned}$$

where we abbreviated

$$D^{(\omega)}_{l,m,t}(u_{l,i}):=\langle D^{(\omega)}_{l,m,t},u_{l,i}\rangle\ \text{ and }\ \tilde{J}^{(\omega)}_{m,b_T,t}(v_{l,i}):=\sum_{s=1}^{t-4m}\tilde{w}^{(\omega)}_{b_T,t,s}\langle D^{(\omega)}_{l,m,s},v_{l,i}\rangle\quad l\in\{X,Y\}. \tag{A.14}$$

Then, from the previous and the fact that $D^{(\omega)}_{j,m,t}(\cdot)$ is $\mathcal{G}_{t,t-m}$-measurable and $N^{\omega}_{m,T,t}$ is $\mathcal{G}_{t-4m}$-measurable, we obtain

$$\begin{aligned}&\frac{1}{\mathcal{W}^2_{b_T}}\sum_{t=1}^{\lfloor\eta T\rfloor}\lim_{n\to\infty}\Big(\sum_{r=1}^{n}\frac{(b-a)}{n}\mathbb{E}\Big[N^{\omega_r}_{m,T,t}\overline{N^{\omega_r}_{m,T,t}}\Big|\mathcal{G}_{t-1}\Big]+\frac{(b-a)^2}{n^2}\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{n}\mathbb{E}\Big[N^{\omega_{r_1}}_{m,T,t}\overline{N^{\omega_{r_2}}_{m,T,t}}\Big|\mathcal{G}_{t-1}\Big]\Big)\\
&=\frac{1}{\mathcal{W}^2_{b_T}}\sum_{t=1+4m}^{\lfloor\eta T\rfloor}\lim_{n\to\infty}\Big(\sum_{r=1}^{n}\frac{(b-a)}{n}\mathbb{E}\Big[\tilde{N}^{\omega_r}_{m,T,t}\overline{\tilde{N}^{\omega_r}_{m,T,t}}\Big|\mathcal{G}_{t-1}\Big]+\frac{(b-a)^2}{n^2}\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{n}\mathbb{E}\Big[\tilde{N}^{\omega_{r_1}}_{m,T,t}\overline{\tilde{N}^{\omega_{r_2}}_{m,T,t}}\Big|\mathcal{G}_{t-m-1}\Big]\Big)+o_p(1)\\
&=\frac{1}{\mathcal{W}^2_{b_T}}\sum_{t=1+4m}^{\lfloor\eta T\rfloor}\lim_{n\to\infty}\Big\{\frac{(b-a)}{n}\sum_{r=1}^{n}\sum_{j,l\in\{X,Y\}}\sum_{k,i=0}^{\infty}s^{(\omega_r)}_{l,i}s^{(\omega_r)}_{j,k}\Big(\mathbb{E}D^{(\omega_r)}_{l,m,t}(u_{l,i})\overline{D^{(\omega_r)}_{j,m,t}(u_{j,k})}\mathbb{E}\overline{\tilde{J}^{(\omega_r)}_{m,b_T,t}(v_{l,i})}\tilde{J}^{(\omega_r)}_{m,b_T,t}(v_{j,k})\\
&\quad+\mathbb{E}\Big[D^{(\omega_r)}_{l,m,t}(u_{l,i})D^{(\omega_r)}_{j,m,t}(v_{j,k})\Big]\mathbb{E}\Big[\overline{\tilde{J}^{(\omega_r)}_{m,b_T,t}(v_{l,i})\tilde{J}^{(\omega_r)}_{m,b_T,t}(u_{j,k})}\Big]+\mathbb{E}\Big[\overline{D^{(\omega_r)}_{l,m,t}(v_{l,i})D^{(\omega_r)}_{j,m,t}(u_{j,k})}\Big]\mathbb{E}\Big[\tilde{J}^{(\omega_r)}_{m,b_T,t}(u_{l,i})\tilde{J}^{(\omega_r)}_{m,b_T,t}(v_{j,k})\Big]\\
&\quad+\mathbb{E}\Big[\overline{D^{(\omega_r)}_{l,m,t}(v_{l,i})}D^{(\omega_r)}_{j,m,t}(v_{j,k})\Big]\mathbb{E}\Big[\tilde{J}^{(\omega_r)}_{m,b_T,t}(u_{l,i})\overline{\tilde{J}^{(\omega_r)}_{m,b_T,t}(u_{j,k})}\Big)\Big]+o_p(1)\end{aligned} \tag{A.15}$$

Furthermore, using Lemma B.8 and

$$\mathbb{E}\langle D^{(\omega)}_{l,m,t},x\rangle\overline{\langle D^{(\omega)}_{j,m,t},z\rangle}=\sum_{|k|\le m}\langle C^{(m)}_{lj,k}(z),x\rangle e^{-\mathrm{i}\lambda k}=2\pi\langle\mathcal{F}^{(\lambda)}_{l,j,m}(z),x\rangle,$$

where $C^{(m)}_{lj,k} = \mathbb{E}(l^{(m)}_k \otimes j^{(m)}_0)$ for $l, j \in \{X, Y\}$ (see proof of Proposition 3.2 of van Delft , 2020), (A.15) becomes

$$\frac{4\pi^2 \sum_{t=1+4m}^{\lfloor \eta T \rfloor} \sum_{s=1}^{t-4m} w(b_T(t-s))^2}{\sum_{t=1}^{T} \sum_{s=1}^{T} w(b_T(t-s))^2} \lim_{n\to\infty} \Big\{ \frac{(b-a)}{n} \sum_{r=1}^{n} \sum_{j,l\in\{X,Y\}} \sum_{k,i=1}^{\infty} s^{(\omega)}_{l,i} s^{(\omega)}_{j,k} \Big(\Big\langle \big(\mathscr{F}^{(\omega_r)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega_r)}_{j,l,m}\big)(u_{j,k} \otimes v_{j,k}), u_{l,i} \otimes v_{l,i} \Big\rangle_{S_2}$$
$$+ 1_{\omega_r \in \{0,\pi\}} \Big\langle \big(\mathscr{F}^{(\omega_r)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega_r)}_{j,l,m}\big)(u_{j,k} \otimes v_{j,k}), \overline{v_{l,i}} \otimes \overline{u_{l,i}} \Big\rangle_{S_2}$$
$$+ \Big\langle \big(\mathscr{F}^{(\omega_r)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega_r)}_{j,l,m}\big)(v_{l,i} \otimes u_{l,i}), v_{j,k} \otimes u_{j,k} \Big\rangle_{S_2}$$
$$+ 1_{\omega_r \in \{0,\pi\}} \Big\langle \big(\mathscr{F}^{(\omega_r)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega_r)}_{j,l,m}\big)(u_{j,k} \otimes v_{j,k}), \overline{v_{l,i}} \otimes \overline{u_{l,i}} \Big\rangle_{S_2} \Big) \Big\} + o_p(1)$$
$$= \frac{4\pi^2 \sum_{t=1+4m}^{\lfloor \eta T \rfloor} \sum_{s=1}^{t-4m} w(b_T(t-s))^2}{\sum_{t=1}^{T} \sum_{s=1}^{T} w(b_T(t-s))^2} \int_0^{\pi} \Gamma^{\omega}_{m,\boldsymbol{X}}\big(\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\big) + o_p(1)$$

where, by continuity of the inner product $\langle \cdot, \cdot \rangle_{S_2}$ with respect to the S_2-norm topology,

$$\Gamma^{\omega}_{\boldsymbol{X},m}\big(\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\big) = 4\pi^2 \sum_{l,j\in\{X,Y\}} \Big\langle \big(\mathscr{F}^{(\omega)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{j,l,m}\big)(\mathcal{V}^{(\omega)}_j), \mathcal{V}^{(\omega)}_l \Big\rangle_{S_2} + 1_{\omega\in\{0,\pi\}} \Big\langle \big(\mathscr{F}^{(\omega)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{j,l,m}\big)(\mathcal{V}^{(\omega)}_j), \overline{\mathcal{V}^{\dagger(\omega)}_l} \Big\rangle_{S_2}$$
$$+ \Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{\dagger(\omega)}_l), \mathcal{V}^{\dagger(\omega)}_j \Big\rangle_{S_2} + 1_{\omega\in\{0,\pi\}} \Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{(\omega)}_j), \overline{(\mathcal{V}^{\dagger(\omega)}_j)} \Big\rangle_{S_2}.$$

For the conditional pseudo-covariance, Fubini's theorem yields

$$\frac{1}{\mathscr{W}^2_{b_T}} \sum_{t=1}^{\lfloor \eta T \rfloor} \mathbb{E}[(N_{m,T,t})^2 | \mathscr{G}_{t-1}] = \frac{1}{\mathscr{W}^2_{b_T}} \sum_{t=1}^{\lfloor \eta T \rfloor} \mathbb{E}\Big[\int_a^b N^{(\omega)}_{m,T,t} d\omega \int_a^b N^{\lambda}_{m,T,t} d\lambda \Big| \mathscr{G}_{t-1} \Big]$$
$$= \frac{1}{\mathscr{W}^2_{b_T}} \sum_{t=1}^{\lfloor \eta T \rfloor} \int_a^b \int_a^b \mathbb{E}\Big[N^{(\omega)}_{m,T,t} N^{\lambda}_{m,T,t} \Big| \mathscr{G}_{t-1} \Big] d\omega d\lambda$$
$$= \frac{1}{\mathscr{W}^2_{b_T}} \sum_{t=1}^{\lfloor \eta T \rfloor} \lim_{n\to\infty} \Big(\frac{(b-a)}{n} \sum_{i_1=1}^{n} \mathbb{E}\Big[(N^{\omega_{i_1}}_{m,T,t})^2 \Big| \mathscr{G}_{t-1} \Big]$$
$$+ \frac{(b-a)^2}{n^2} \sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{n} \mathbb{E}\Big[N^{\omega_{i_1}}_{m,T,t} N^{\omega_{i_2}}_{m,T,t} \Big| \mathscr{G}_{t-1} \Big] \Big).$$

Similar to the conditional covariance, we obtain for the conditional pseudo-covariance

$$\frac{1}{\mathscr{W}^2_{b_T}} \sum_{t=1}^{\lfloor \eta T \rfloor} \mathbb{E}[(N_{m,T,t})^2 | \mathscr{G}_{t-1}] = \frac{\sum_{t=1+4m}^{\lfloor \eta T \rfloor} \sum_{s=1}^{t-4m} w(b_T(t-s))^2}{\sum_{t=1}^{T} \sum_{s=1}^{T} w(b_T(t-s))^2} \int_a^b \Sigma^{\omega}_{X,m}\big(\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\big) d\omega + o_p(1)$$

where now,

$$\Sigma^{\omega}_{\boldsymbol{X},m}\big(\mathcal{V}^{(\omega)}_{\boldsymbol{X}}\big) = 4\pi^2 \sum_{l,j\in\{X,Y\}} \Big(\Big\langle \big(\mathscr{F}^{(\omega)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{j,l,m}\big)(\mathcal{V}^{\dagger(\omega)}_j), \mathcal{V}^{(\omega)}_l \Big\rangle_{S_2} + \Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{\dagger(\omega)}_l), \mathcal{V}^{(\omega)}_j \Big\rangle_{S_2}$$
$$1_{\omega\in\{0,\pi\}} \Big\langle \big(\mathscr{F}^{(\omega)}_{l,j,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{j,l,m}\big)(\overline{\mathcal{V}^{(\omega)}_j}), \mathcal{V}^{(\omega)}_l \Big\rangle_{S_2} + 1_{\omega\in\{0,\pi\}} \Big\langle \big(\mathscr{F}^{(\omega)}_{j,l,m} \widetilde{\otimes} \mathscr{F}^{\dagger(\omega)}_{l,j,m}\big)(\mathcal{V}^{\dagger(\omega)}_l), \overline{\mathcal{V}^{\dagger(\omega)}_j} \Big\rangle_{S_2} \Big).$$

By a change of variables and symmetry of the weight function in zero;

$$\frac{\sum_{t=1}^{\lfloor \eta T \rfloor} \sum_{s=1}^{\lfloor \eta T \rfloor} w(b_T(t-s))^2}{\sum_{t=1}^{T} \sum_{s=1}^{T} w(b_T(t-s))^2} = \frac{\lfloor \eta T \rfloor}{T} \frac{\sum_{|h|<\lfloor \eta T \rfloor} w(b_T h)^2}{\sum_{|h|<T}^{T} w(b_T h)^2}$$
$$= \frac{\lfloor \eta T \rfloor}{T} \frac{\sum_{|h|<\lfloor \eta T \rfloor} w(b_T h)^2}{\sum_{|h|<\lfloor \eta T \rfloor} w(b_T h)^2 + \sum_{|h|\geq\lfloor \eta T \rfloor} w(b_T h)^2} \to \eta,$$

where we used that $\sum_{|h|\geq \frac{\lfloor \eta T\rfloor b_T}{b_T}} w(b_T h)^2 = o(1/b_T)$, which follows from Assumption 3.2 together with Assumption 3.4, where the latter implies $\lfloor \eta T\rfloor b_T \to \infty$ as $T\to\infty$ for any $\eta \in [0,1]$. Therefore, we obtain for fixed m that

$$\frac{\sum_{t=1+4m}^{\lfloor \eta T\rfloor}\sum_{s=1}^{t-4m} w(b_T(t-s))^2}{\sum_{t=1}^{T}\sum_{s=1}^{T} w(b_T(t-s))^2} = \frac{\kappa\lfloor \eta T\rfloor/b_T}{2\kappa T/b_T} + o(1).$$

Observe then that

$$\sum_{t=1}^{\lfloor \eta T\rfloor}\mathbb{E}[\big(\Re(N_{m,T,t})\big)^2|\mathscr{G}_{t-1}] = \frac{1}{2}\sum_{t=1}^{\lfloor \eta T\rfloor}\Re\Big(\mathbb{E}[|N_{m,T,t}|^2|\mathscr{G}_{t-1}]\Big) + \frac{1}{2}\sum_{t=1}^{\lfloor \eta T\rfloor}\Re\Big(\mathbb{E}[(N_{m,T,t})^2|\mathscr{G}_{t-1}]\Big)$$

which, together with the above, yields

$$\frac{1}{\mathscr{W}_{b_T}^2}\sum_{t=1}^{\lfloor \eta T\rfloor}\mathbb{E}[\big(\Re(N_{m,T,t})\big)^2|\mathscr{G}_{t-1}] \to \eta\cdot\frac{1}{4}\Re\Big(\int_a^b \Gamma^{\omega}_{\boldsymbol{X},m}\big(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\big) + \Sigma^{\omega}_{\boldsymbol{X},m}\big(\mathcal{V}_{\boldsymbol{X}}^{(\omega)}\big)d\omega\Big)$$

in probability as $T\to\infty$. The result now follows. $\square$

B Some technical results and auxiliary statements

Lemma B.1 (Maximum of partial sums). *Let $\{X_t\}_{t\in\mathbb{Z}}$ satisfy Conditions **A.1**-**A.2** with $p = 4+\epsilon$ and let the complex-valued array $\{\tilde{w}^{(\omega)}_{b_T,s,t}\}_{s,t\in\mathbb{N}}$ be defined through* (3.4). *Furthermore, suppose Assumption 3.2 and Assumption 3.3 for some $\ell \geq 1$ are satisfied. Denote the spectral density operators of $\{X_t\}$ by $\mathscr{F}^{(\omega)}$ and consider the partial sum process*

$$S_k^{\omega} = \sum_{s=1}^{k}\sum_{t=1}^{k}\tilde{w}^{(\omega)}_{b_T,s,t}\big(X_s\otimes X_t\big) \quad 1\leq k\leq T.$$

Then

$$\max_{1\leq k\leq T}\frac{b_T^{1-i}}{k^i}|||S_k^{\omega}-k\mathscr{F}^{\omega}|||_2 = \begin{cases} O_p(\log^{1/\gamma}(T)) & \text{if } i = 1/2.\\ o_p(1) & \text{if } i>1/2,\end{cases}$$

where $\gamma = 2+\epsilon/2$.

Proof. Proposition 3.1 implies

$$\frac{1}{k}|||S_k^{\omega}-\mathbb{E}S_k^{\omega}|||_2 = O_p(\frac{1}{\sqrt{b_T k}}) \quad\leftrightarrow\quad |||S_k^{\omega}-\mathbb{E}S_k^{\omega}|||_2 = O_p(\frac{\sqrt{k}}{\sqrt{b_T}}),\quad 1\leq k\leq T$$

and also $\mathbb{E}|||S_k^{\omega}-\mathbb{E}S_k^{\omega}|||_2^{2+\epsilon} = \|S_k^{\omega}-\mathbb{E}S_k^{\omega}\|_{S_2,2+\epsilon}^{2+\epsilon} = O(b_T^{-1-\epsilon/2}k^{1+\epsilon/2})$, i.e.,

$$|||S_k^{\omega}-\mathbb{E}S_k^{\omega}|||_2^{2+\epsilon} = O_p(b_T^{-1-\epsilon/2}k^{1+\epsilon/2}).$$

Note moreover that, under the conditions of Proposition 3.1, $|||\mathbb{E}S_k^{\omega}-k\mathscr{F}^{\omega}|||_2 = O(b_T^{\ell}k)$ and thus

$$|||\mathbb{E}S_k^{\omega}-k\mathscr{F}^{\omega}|||_2^{2+\epsilon} = O(b_T^{\ell(2+\epsilon)}k^{2+\epsilon})$$

Let $\gamma = 2+\epsilon$ and $\alpha = 1+\epsilon/2$. Since, for any $1\leq k<k+l\leq T$,

$$(k)^{\alpha}+(l)^{\alpha}\leq (l+k)^{\alpha},$$

Condition (1.1) of Theorem 1 of Mòricz (1976) is satisfied, from which we obtain

$$\mathbb{E}(\max_{1\leq k\leq T}|||S_k^{\omega}-\mathbb{E}S_k^{\omega}|||_2)^{\gamma} = O(b_T^{-1-\epsilon/2}T^{1+\epsilon/2}).$$

By standard arguments, we have for $1/2 \le i \le 1$

$$\max_{1\le k\le T} \frac{b_T^{1-i}}{k^i} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2 \le b_T^{1-i} \max_{1\le j\le c\log(T)} \max_{\beta^{j-1}<k\le\beta^j} \frac{1}{k^i} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2$$
$$\le b_T^{1-i} \max_{1\le j\le c\log(T)} \beta^{-(j-1)i} \max_{\beta^{j-1}<k\le\beta^j} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2,$$

since $x^{-1/2}$ is a decreasing function of x. Hence, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\max_{1\le k\le T} \frac{b_T^{1-i}}{k^i} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2 > \varepsilon\big) \\
&\le b_T^{p(1-i)} \sum_{j=1}^{c\log(T)} \beta^{-(j-1)pi} \varepsilon^{-p} \mathbb{E}\Big[\big(\max_{1\le k\le\beta^j} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2\big)^p\Big] \\
&\le b_T^{p(1-i)} \sum_{j=1}^{c\log(T)} \beta^{-(j-1)pi} \varepsilon^{-p} 2^{p-1} \Big[C_1 \beta^{jp/2} b_T^{-p/2} + C_2 \beta^{jp} b_T^{\ell p}\Big] \\
&\le \beta^{pi} \sum_{j=1}^{c\log(T)} \varepsilon^{-p} 2^{p-1} C_3 \Big[\beta^{jp(1/2-i)} b_T^{p(1/2-i)} + \beta^{jp(1-i)} b_T^{p(1-i+\ell)}\Big].
\end{aligned}$$

Plugging in $b_T = T^{-\kappa}$ and noting that $\beta^j \le T$ for all $1 \le j \le c\log(T)$, this is bounded by

$$\beta^{pi} \sum_{j=1}^{c\log(T)} \varepsilon^{-p} 2^{p-1} C_3 T^{p(1/2(1-\kappa)-i(1-\kappa))} + \beta^{pi} \sum_{j=1}^{c\log(T)} \varepsilon^{-p} 2^{p-1} C_3 T^{p\left(1-i-\kappa(1-i+\ell)\right)} \tag{B.1}$$

Observe that if $i = 1/2$, we can bound the first term by choosing $\varepsilon = c_2 \log^{-1/p}(T)$ for some sufficiently large constant c_2. The second term can in this case be bounded by a constant if $b_T^{1+2\ell} T = O(1)$, i.e., if $\kappa = 1/(1+2\ell)$ and is of lower order if $\kappa > 1/(1+2\ell)$. It follows therefore that under the conditions of Lemma B.1.

$$\max_{1\le k\le T} \frac{\sqrt{b_T}}{\sqrt{k}} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2 = O_p(\log^{1/\gamma}(T)).$$

Consider then the case $i > 1/2$. Let us first look at the second term of (B.1). Observe that $b_T^{1-i+\ell} T^{1-i} = O(1)$, i.e., if $\kappa = (1-i)/(1-i+\ell)$ and is of lower order if $\kappa > (1-i)/(1-i+\ell)$. Since $\frac{1/2}{(1/2+\ell)} > \frac{1-i}{(1-i+\ell)}$ for any $1/2 < i \le 1$, we obtain $(1-i) < \kappa(1-i+\ell)$ for any $1/2 < i \le 1$ under the conditions of Lemma B.1 since these require $\kappa \ge \frac{1/2}{(1/2+\ell)}$. Observe also that the exponent of T in the first term of (B.1)is negative for $1/2 < i \le 1$. Recall then that an application of Hôpital's rule yields

$$\lim_{T\to\infty} \frac{\log(T)}{T^{\rho}} = \lim_{T\to\inf} \frac{1/T}{\rho T^{\rho-1}} = \lim_{T\to\infty} \frac{1}{\rho T^{\rho}} = 0 \quad \rho > 0.$$

It follows thus that, for any ε, both terms converge to zero under the conditions of Lemma B.1. Hence, $\max_{1\le k\le T} \frac{b_T^{1-i}}{k^i} |||S_k^{\omega} - k\mathscr{F}^{\omega}|||_2 = o_p(1)$, for $i > 1/2$.

□

Theorem B.1. *Suppose Assumptions 3.1-3.3 are satisfied. Then,*

$$\sup_{\eta\in[0,1]} \int_a^b \Big\|\Big|\eta^{1/r}(\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)})\Big\|\Big|_2^r d\omega = \begin{cases} O_p(\log^{r/\gamma}(T) b_T^{-1} T^{-1}) & \text{if } r = 2. \\ o_P(b_T^{-1} T^{-1}) & \text{if } r > 2, \end{cases}$$

for some $\gamma > 2$.

Proof. We have

$$\mathbb{E} \sup_{\eta\in[0,1]} \int_a^b \Big\|\Big|\eta^{1/r}(\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)})\Big\|\Big|_2^r d\omega \le \mathbb{E} \int_a^b \sup_{\eta\in[0,1]} \Big\|\Big|\eta^{1/r}(\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)})\Big\|\Big|_2^r d\omega$$

$$\leq \int_a^b \mathbb{E}\Big(\sup_{\eta\in[0,1]} \Big|\Big|\Big|\eta^{1/r}(\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)})\Big|\Big|\Big|_2\Big)^r d\omega,$$

where we applied Tonelli's theorem and Jensen's inequality in the second inequality. We remark that the integrand on the right-hand side is measurable. Observe then that

$$\frac{\eta^{1/r}}{\lfloor \eta T\rfloor} \approx \frac{1}{b_T^{1/r}T^{1/r}} \frac{(\eta T)^{1/r}}{(\lfloor \eta T\rfloor)^{1/r}} \frac{b_T^{1/r}}{(\lfloor \eta T\rfloor)^{1-1/r}}$$

where, similar to the proof of Lemma B.2, we can set $\frac{\eta T}{\lfloor \eta T\rfloor} = 0$ for $\eta \leq 1/T$ as otherwise the term is zero. Then using (B.5) and a change of variables $i = 1-1/r$, therefore yields for $\eta \geq 1/T$,

$$\frac{\eta^{1-i}}{\lfloor \eta T\rfloor} = \frac{1}{b_T^{1-i}T^{1-i}} \frac{(\eta T)^{1-i}}{(\eta T)^{1-i}} \frac{b_T^{1-i}}{(\lfloor \eta T\rfloor)^{i}} + \frac{1}{b_T^{1-i}T^{1-i}} O(T^{-1+i}) \frac{b_T^{1-i}}{(\lfloor \eta T\rfloor)^{i}}.$$

The result then follows from the proof of Lemma B.1 and Minkowski's inequality. □

Lemma B.2. *Let $\mathcal{Z}_{T,\eta}^{X,\omega}$ and $\mathcal{Z}_{T,\eta}^{Y,\omega}$ be defined as in Theorem 3.1 and let $\mathscr{U}_{XY}^{(\omega)} \in S_2(\mathscr{H})$, then*

$$\sup_{\eta\in[0,1]} \Big|\int_a^b \Big\langle \eta(\hat{\mathscr{F}}_X^{(\omega)}(\eta) - \mathscr{F}_X^{(\omega)}), \eta\mathscr{U}_{XY}^{(\omega)}\Big\rangle d\omega - \int_a^b \Big\langle \frac{\mathcal{Z}_{T,\eta}^{X,\omega}}{\sqrt{b_1T_1}}, \eta\mathscr{U}_{XY}^{(\omega)}\Big\rangle d\omega\Big| = o_P\Big(\frac{1}{\sqrt{b_1T_1 + b_2T_2}}\Big); \tag{B.2}$$

$$\sup_{\eta\in[0,1]} \Big|\int_a^b \Big\langle \eta(\hat{\mathscr{F}}_Y^{(\omega)}(\eta) - \mathscr{F}_Y^{(\omega)}), \eta\mathscr{U}_{XY}^{(\omega)}\Big\rangle d\omega - \int_a^b \Big\langle \frac{\mathcal{Z}_{T,\eta}^{Y,\omega}}{\sqrt{b_2T_2}}, \eta\mathscr{U}_{XY}^{(\omega)}\Big\rangle d\omega\Big| = o_P\Big(\frac{1}{\sqrt{b_1T_1 + b_2T_2}}\Big). \tag{B.3}$$

Proof. We prove only (B.4) as (B.3) follows similarly. Note from the definition of $\mathcal{Z}_{T,\eta}^{X,\omega}$ and the triangle inequality, it suffices to show that

$$\sup_{\eta\in[0,1]} \int_a^b \Big|\frac{\eta T_1}{\lfloor \eta T_1\rfloor} - 1\Big|\Big|\Big\langle \frac{1}{\sqrt{b_1T_1}}\mathcal{Z}_{T,\eta}^{X,\omega}, \eta\mathscr{U}_{XY}^{(\omega)}\Big\rangle\Big| d\omega = o_P\Big(\frac{1}{\sqrt{b_1T_1 + b_2T_2}}\Big). \tag{B.4}$$

The Cauchy-Schwarz inequality implies

$$\begin{aligned}
&\mathbb{E}\sup_{\eta\in[0,1]} \int_a^b \Big|\frac{\eta T}{\lfloor \eta T\rfloor} - 1\Big|\Big|\Big\langle \frac{1}{T_1}\sum_{s=1}^{\lfloor \eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_1\rfloor} \tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t) - \mathscr{F}_X^{(\omega)}\Big), \eta\mathscr{U}_{XY}^{(\omega)}\Big\rangle\Big| d\omega \\
&\leq \sup_{\eta\in[0,1]} \Big|\frac{\eta T}{\lfloor \eta T\rfloor} - 1\Big| \mathbb{E}\int_a^b \sup_{\eta\in[0,1]} \Big|\Big|\Big|\frac{1}{T_1}\sum_{s=1}^{\lfloor \eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_1\rfloor} \tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t) - \mathscr{F}_X^{(\omega)}\Big)\Big|\Big|\Big|_2 \Big|\Big|\Big|\eta\mathscr{U}_{XY}^{(\omega)}\Big|\Big|\Big|_2 d\omega
\end{aligned}$$

Observe that for $\eta \leq 1/T$, the summation is identically zero, while the approximation to the floor function is identically zero for $\eta = 1/T$. Hence, we only we only have to consider the case where $\eta > 1/T$. The approximation error of the floor function is in this case given by

$$\sup_{\eta\in(1/T,1]} \Big|\frac{\eta T}{\lfloor \eta T\rfloor} - 1\Big| \leq C \sup_{\eta\in(1/T,1]} \Big|\frac{1}{\lfloor \eta T\rfloor}\Big| = O(1/T). \tag{B.5}$$

Then, using additionally that $\sup_\eta |||\eta\mathscr{U}_{XY}^{(\omega)}|||_2 \leq C$ for some constant C, we find

$$\begin{aligned}
&\leq \big(\frac{C}{T_1}\big)\int_a^b \mathbb{E}\sup_{\eta\in[0,1]} \Big|\Big|\Big|\frac{1}{T_1}\sum_{s=1}^{\lfloor \eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_1\rfloor} \tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t) - \mathscr{F}_X^{(\omega)}\Big)\Big|\Big|\Big|_2 d\omega \\
&\leq \big(\frac{C}{T_1}\big)\int_a^b \Big(\mathbb{E}\big(\sup_{\eta\in[0,1]} \Big|\Big|\Big|\frac{1}{T_1}\sum_{s=1}^{\lfloor \eta T_1\rfloor}\Big(\sum_{t=1}^{\lfloor \eta T_1\rfloor} \tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t) - \mathscr{F}_X^{(\omega)}\Big)\Big|\Big|\Big|_2\big)^\gamma\Big)^{1/\gamma} d\omega \\
&= O\big(\frac{1}{T}\big)O\big(T_1^{-1}b_1^{-\gamma/2}T_1^{\gamma/2} + b_1^{2\gamma\ell}\big)^{1/\gamma}
\end{aligned}$$

where we used Jensen's inequality in the last inequality and where the last line follows from Lemma B.1 and Assumption 3.4. □

Lemma B.3. *Suppose Assumptions 3.1-3.3 are satisfied. Denote* $\hat{\mathscr{F}}_k^{(\omega)} = k^{-1}\sum_{s,t=1}^k \tilde{w}_{b_T,s,t}^{(\omega)}(X_s \otimes X_t)$ *and let* $\mathscr{M}_{X,k,m}^{(\omega)} = k^{-1}\sum_{s=1}^k\sum_{t=0}^{s-1}\tilde{w}_{b_T,s,t}^{(\omega)}(D_{X,m,s}^{(\omega)} \otimes D_{X,m,t}^{(\omega)})$ *where* $\{D_{X,m,s}^{(\omega)}\}$ *is given by* (5.2)*. Then, for any* $1 \le k \le T$

$$\lim_{m\to\infty}\limsup_{T\to\infty}\frac{1}{\mathscr{W}_{b_T}^{\gamma}}\mathbb{E}\Big(\max_{1\le k\le T}\Big|\Big|\Big|k(\hat{\mathscr{F}}_k^{(\omega)} - \mathbb{E}\hat{\mathscr{F}}_k^{(\omega)}) - \mathscr{M}_{k,m}^{(\omega)} - \mathscr{M}_{k,m}^{\dagger(\omega)}\Big|\Big|\Big|_2\Big)^{\gamma} = 0,$$

for $\gamma > 2$.

Proof of Lemma B.3. From the proof of Lemma 3.3 of van Delft (2020), applied to the sequential spectral density estimators, we have

$$\begin{aligned}
&\Big(\mathbb{E}\Big|\Big|\Big|k(\hat{\mathscr{F}}_k^{(\omega)} - \mathbb{E}\hat{\mathscr{F}}_k^{(\omega)}) - \mathscr{M}_{k,m}^{(\omega)} - \mathscr{M}_{k,m}^{\dagger(\omega)}\Big|\Big|\Big|_2^{\gamma}\Big)^{1/\gamma}\\
&\le \frac{\sqrt{k}}{\sqrt{b_T}}\kappa K_{2p}\Upsilon_{2p,m}\sum_{t=0}^{\infty}\nu_{\mathbb{H},2p}(X_t) + |w(0)|\sqrt{k}\sum_{t=0}^{\infty}\nu_{\mathbb{H},2p}(X_t)\\
&+\|X_0\|_{\mathbb{H},2p}^2 m^2\sqrt{k}\big(\max_t|w(b_T t)|^2 + m\sum_{h=1}^{k}|w(b_T h) - w(b_T(h-1))|^2\big)^{1/2},
\end{aligned}$$

where $\Upsilon_{2p,m} = 2\sum_{t=0}^{\infty}\min\big(\nu_{\mathbb{H},2p}(X_t), \sum_{i=m}^{\infty}\nu_{\mathbb{H},2p}^2(X_i)\big)$. Observe that the conditions on the weight function (See also Theorem 4.2 of van Delft , 2020)imply that we obtain

$$\mathbb{E}\Big|\Big|\Big|k(\hat{\mathscr{F}}_k^{(\omega)} - \mathbb{E}\hat{\mathscr{F}}_k^{(\omega)}) - \mathscr{M}_{k,m}^{(\omega)} - \mathscr{M}_{k,m}^{\dagger(\omega)}\Big|\Big|\Big|_2^{\gamma} = C_{\gamma}(k^{\gamma/2}b_T^{\gamma/2}\Upsilon_{2p,m} + k^{\gamma/2} + m^2k^{\gamma/2}\big(o(\frac{1}{b_T})(1+m)\big)^{1/2})^{\gamma},$$

for some positive constant C_{γ}. Since m and b_T are fixed, it is straightforward to see that the condition (1.1) of Móricz, (1976) is satisfied for any $\gamma > 2$. Therefore, Theorem 1 of Móricz, (1976) implies for $\gamma > 2$,

$$\mathbb{E}\Big(\max_{1\le k\le T}\Big|\Big|\Big|k(\hat{\mathscr{F}}_k^{(\omega)} - \mathbb{E}\hat{\mathscr{F}}_k^{(\omega)}) - \mathscr{M}_{k,m}^{(\omega)} - \mathscr{M}_{k,m}^{\dagger(\omega)}\Big|\Big|\Big|_2\Big)^{\gamma} = O(T^{\gamma/2}b_T^{-\gamma/2}\Upsilon_{2p,m}^{\gamma} + T^{\gamma/2} + m^{2\gamma}T^{\gamma/2}\big(o(\frac{1}{b_T})(1+m)\big)^{\gamma/2}).$$

Consequently, since $b_T \to 0$ as $T \to \infty$,

$$\begin{aligned}
&\lim_{m\to\infty}\limsup_{T\to\infty} b_T^{\gamma}T^{-\gamma}\mathbb{E}\Big(\max_{1\le k\le T}\Big|\Big|\Big|k(\hat{\mathscr{F}}_k^{(\omega)} - \mathbb{E}\hat{\mathscr{F}}_k^{(\omega)}) - \mathscr{M}_{k,m}^{(\omega)} - \mathscr{M}_{k,m}^{\dagger(\omega)}\Big|\Big|\Big|_2\Big)^{\gamma}\\
&= \lim_{m\to\infty}\limsup_{T\to\infty} b_T^{\gamma/2}T^{-\gamma/2}O(T^{\gamma/2}b_T^{-\gamma/2}\Upsilon_{2p,m}^{\gamma} + T^{\gamma/2} + m^{2\gamma}T^{\gamma/2}\big(o(\frac{1}{b_T})(1+m)\big)^{\gamma/2})\\
&= \lim_{m\to\infty}\limsup_{T\to\infty} O(T^{-\gamma/2}T^{\gamma/2}\Upsilon_{2p,m}^{\gamma} + b_T^{\gamma/2}T^{-\gamma/2}T^{\gamma/2})\\
&\quad+\lim_{m\to\infty}\limsup_{T\to\infty} O(m^{2\gamma}b_T^{\gamma/2}T^{-\gamma/2}T^{\gamma/2})o\Big(\frac{m}{b_T}\Big)^{\gamma/2} = 0,
\end{aligned}$$

where we used in the first line of the last equality that $\lim_{m\to\infty}\Upsilon_{2p,m}^{\gamma} = 0$. □

Lemma B.4. *Suppose Assumptions 3.1-3.3 hold. Then, for any* $1 \le k \le T_1$,

$$\lim_{T_1,T_2\to\infty}\frac{\sqrt{b_{T_1}T_1 + b_{T_2}T_2}}{\sqrt{b_{T_1}T_1}}\sup_{\omega}\max_{1\le k\le T_1}\Big\langle\frac{\sqrt{b_{T_1}}}{\sqrt{T_1}}\mathbb{E}\sum_{s=1}^{k}\Big(\sum_{t=1}^{k}\tilde{w}_{b_1,s,t}^{(\omega)}(X_s\otimes X_t) - \mathscr{F}_X^{\omega}\Big), \mathscr{U}_{X,Y}^{(\omega)}\Big\rangle_S = 0.$$

Proof of Lemma B.4. For any $1 \le k \le T_1$, we obtain using the Cauchy-Schwarz inequality and Proposition 3.1

$$\begin{aligned}
\sup_{\omega}\max_{1\le k\le T_1}\Big|\Big|\Big|\frac{\sqrt{b_{T_1}}}{\sqrt{T_1}}\mathbb{E}\sum_{s=1}^{k}\Big(\sum_{t=1}^{k}\tilde{w}_{b_{T_1},s,t}^{(\omega)}(X_s\otimes X_t) - \mathscr{F}_X^{\omega}\Big)\Big|\Big|\Big|_2\Big|\Big|\Big|\mathscr{U}_{X,Y}^{(\omega)}\Big|\Big|\Big|_2 &= \max_{1\le k\le T}O(\frac{\sqrt{b_{T_1}}}{\sqrt{T_1}})O(b_{T_1}^{\ell}k)\\
&= O(b_{T_1}^{\ell+1/2}T_1^{1/2}),
\end{aligned}$$

which goes to zero for bandwidths satisfying satisfying Assumption 3.3 The result now follows from Assumption 3.4. □

Lemma B.5. *Let $\mathscr{F}_X^{(\omega)}$ be the spectral density operator of a weakly stationary functional time series $\{X_t\}_{t\in\mathbb{Z}}$ with eigendecomposition $\sum_{k=1}^{\infty}\lambda_{X,k}^{(\omega)}\Pi_{X,k}^{\omega}$. Let $\left\{\hat{\lambda}_{X,k}^{(\omega)}(\eta)\right\}_{k\geq 1}$ $\left\{\hat{\Pi}_{X,k}^{(\omega)}(\eta)\right\}_{k\geq 1}$ be the sequence of eigenvalues and eigenprojectors, respectively, of the sequential estimators $\hat{\mathscr{F}}_X^{(\omega)}(\eta)$, $\eta\in[0,1]$ of $\mathscr{F}_X^{(\omega)}$ as defined in* (3.8). *Then, under Assumption 3.1-3.3,*

$$\text{(i) } \left|\!\left|\!\left|\hat{\Pi}_{X,k}^{(\omega)}(\eta)-\Pi_{X,k}^{(\omega)}\right|\!\right|\!\right|_2 \leq 2G_{X,k}|||\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}|||_2+\frac{8}{G_{X,k}^2}|||\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}|||_2^2$$

$$\text{(ii) } \left|(\hat{\lambda}_{X,k}^{(\omega)}(\eta))^2-(\lambda_{X,k}^{(\omega)})^2\right| \leq C|||\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}|||_2+|||\hat{\mathscr{F}}_X^{(\omega)}(\eta)-\mathscr{F}_X^{(\omega)}|||_2^2,$$

where $G_{X,k}=\inf_{l\neq k}\left|\lambda_{X,k}^{(\omega)}-\lambda_{X,l}^{(\omega)}\right|$ and were $C\geq 0$ is a bounded constant.

Proof. $\hat{\Pi}_{X,k}^{(\omega)}(\eta)$ and $\hat{\lambda}_{X,k}^{(\omega)}(\eta)$ are, respectively, perturbed eigenprojectors and eigenvalues of $\Pi_{X,k}^{(\omega)}$ and $\lambda_{X,k}^{(\omega)}$. For (ii), we note that

$$\left|(\hat{\lambda}_{X,k}^{(\omega)}(\eta))^2-(\lambda_{X,k}^{(\omega)})^2\right|\leq 2\lambda_{X,k}^{(\omega)}\left|\hat{\lambda}_{X,k}^{(\omega)}(\eta)-\lambda_{X,k}^{(\omega)}\right|+\left|\hat{\lambda}_{X,k}^{(\omega)}(\eta)-\lambda_{X,k}^{(\omega)}\right|\left|\lambda_{X,k}^{(\omega)}-\hat{\lambda}_{X,k}^{(\omega)}(\eta)\right|,$$

where we used that the eigenvalues are real. Since $\hat{\mathscr{F}}_X^{(\omega)}(\eta)$ is a sequential consistent estimator of $\mathscr{F}_X^{(\omega)}$ under Assumption 3.1-3.3, the results now follow similarly to the proof of Proposition 4.1 of van Delft (2020). □

Lemma B.6. *[(van Delft , 2020, Lemma A.1)] Let $\{M_k\}_{k=1,\ldots,n}\in\mathscr{L}_H^p$, $p>1$, be a martingale with respect to $\mathscr{G}$ with $\{D_k\}$ denoting its difference sequence and let $\{A_k\}_{k=1,\ldots,n}\in S_\infty(H)$. Then, for $q=\min(2,p)$,*

$$\left\|\sum_{k=1}^n A_k(D_k)\right\|_{\mathbb{H},p}^q \leq K_p^q\sum_{k=1}^n|||A_k|||_\infty^q\|D_k\|_{\mathbb{H},p}^q$$

where $K_p^q=(p^\star-1)^q$ with $p^\star=\max(p,\frac{p}{p-1})$.

Proof of Lemma B.6. By Burkholder's inequality,

$$\left\|\sum_{k=1}^n A_k(D_k)\right\|_{\mathbb{H},p}^q=\left(\mathbb{E}\left\|\sum_{k=1}^n A_k(D_k)\right\|_H^p\right)^{q/p}\leq(p^\star-1)^q\left(\mathbb{E}\left|\left(\sum_{k=1}^n\|A_k(D_k)\|_H^2\right)^{1/2}\right|^p\right)^{q/p}.$$

Let $p<2$. Then, applying the inequality $|\sum_k x_k|^r\leq\sum_k|x_k|^r$ for $r<1$ to $x_k=\|A_k(D_k)\|_H^2$, we obtain

$$\begin{aligned}(p^\star-1)^q\left(\mathbb{E}\left|\left(\sum_{k=1}^n\|A_k(D_k)\|_H^2\right)^{1/2}\right|^p\right)^{q/p}&\leq(p^\star-1)^q\left(\mathbb{E}\sum_{k=1}^n\|A_k(D_k)\|_H^p\right)^{q/p}\\&\leq(p^\star-1)^q\left(\sum_{k=1}^n|||A_k|||_\infty^p\mathbb{E}\|D_k\|_H^p\right)^{q/p}\\&\leq(p^\star-1)^q\sum_{k=1}^n|||A_k|||_\infty^q\|D_k\|_{\mathbb{H},p}^q,\end{aligned}$$

where the one before last inequality follows from Holder's inequality for operators and from the fact that $p=q$. For $p\geq 2$, $q=2$. Therefore, an application of Minkowski's inequality to $\|\cdot\|_{\mathbb{C},p/q}$ and Holder's inequality yield in this case

$$\begin{aligned}(p^\star-1)^q\left(\mathbb{E}\left|\left(\sum_{k=1}^n\|A_k(D_k)\|_H^q\right)^{1/q}\right|^p\right)^{q/p}&\leq(p^\star-1)^q\left(\mathbb{E}\left|\sum_{k=1}^n\|A_k(D_k)\|_H^q\right|^{p/q}\right)^{q/p}\\&\leq(p^\star-1)^q\sum_{k=1}^n|||A_k|||_\infty^q(\mathbb{E}\|D_k\|_H^p)^{q/p}\\&\leq(p^\star-1)^q\sum_{k=1}^n|||A_k|||_\infty^q\|D_k\|_{\mathbb{H},p}^q.\end{aligned}$$

□

Lemma B.7. *Let* $\{\mathscr{D}^{(\omega)}_{XY,s,t}\}$ *be defined as in* (5.3)*. Then, under Assumption 3.2*

$$\Big\|\sum_{t=2}^{T}\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,s,t}\mathscr{D}^{(\omega)}_{XY,s,t}\Big\|_{\mathcal{H},2}=o(\sqrt{T/b_T}).$$

Proof. By definition of $\mathcal{H}$, we can write

$$\begin{aligned}\Big\|\sum_{t=2}^{T}\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,s,t}\mathscr{D}^{(\omega)}_{XY,s,t}\Big\|^2_{\mathcal{H},2}&=\mathbb{E}\Big\|\sum_{t=2}^{T}\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,s,t}\mathscr{D}^{(\omega)}_{XY,s,t}\Big\|^2_{\mathcal{H}}\\&\le 2\sum_{i=1}^{2}\mathbb{E}\Big\|\sum_{t=2}^{T}\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,s,t}\big(D^{(\omega)}_{X_i,m,s}\otimes D^{(\omega)}_{X_i,m,t}\big)\Big\|^2_{\mathscr{H}_i}\end{aligned}$$

By orthogonality of the increments, Lemma B.6 and Jensen's inequality, we obtain for fixed m,

$$\begin{aligned}&\sum_{i=1}^{2}\mathbb{E}\Big\|\sum_{t=2}^{T}\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,s,t}\big(D^{(\omega)}_{X_i,m,s}\otimes D^{(\omega)}_{X_i,m,s}\big)\Big\|^2_{\mathscr{H}_i}\le\sum_{i=1}^{2}\sum_{t=2}^{T}\mathbb{E}\|D^{(\omega)}_{X_i,m,t}\|^2_{\mathscr{H}_i}\mathbb{E}\Big\|\sum_{s=t-4m+1\vee 1}^{t-1}\tilde{w}^{(\omega)}_{b_T,s,t}D^{(\omega)}_{X_i,m,s}\Big\|^2_{\mathscr{H}_i}\\&\quad=2K_4^2\sum_{i=1}^{2}\|D^{(\omega)}_{X_i,m,0}\|^4_{\mathbb{H}_i,4}\Big(\sum_{t=2}^{4m}\sum_{s=1}^{t-1}|\tilde{w}^{(\omega)}_{b_T,s,t}|^2+\sum_{t=4m+1}^{T}\sum_{s=t-4m+1}^{t-1}|\tilde{w}^{(\omega)}_{b_T,s,t}|^2\Big)=o(T/b_T).\end{aligned}$$

□

Lemma B.8. *Let* $\widetilde{J}^{(\lambda)}_{m,b_T,t}$ *be defined as in* (A.14) *and suppose Assumption 3.2 is satisfied. Furthermore, assume* $\lambda_1\pm\lambda_2\neq 0 \mod 2\pi$*. Then for any* $u,v\in\mathscr{H}$*,*

$$\sum_{t=1+4m}^{T}|\mathbb{E}\widetilde{J}^{(\lambda_1)}_{m,b_T,t}(u)\widetilde{J}^{(\lambda_2)}_{m,b_T,t}(v)|=o(T/b_T).$$

Proof. This follows from a slight adjustment of the proof of Lemma B.3 of van Delft (2020). □