Limit theorems for moving averages of discretized processes plus noise

Jean Jacod [*], Mark Podolskij [†] and Mathias Vetter [‡]

Abstract

This paper presents some limit theorems for certain functionals of moving averages of semimartingales plus noise, which are observed at high frequency. Our method generalizes the pre-averaging approach (see [13],[11]) and provides consistent estimates for various characteristics of general semimartingales. Furthermore, we prove the associated multidimensional (stable) central limit theorems. As expected, we find central limit theorems with a convergence rate $n^{-1/4}$, if n is the number of observations.

Keywords: central limit theorem, high frequency observations, microstructure noise, quadratic variation, semimartingale, stable convergence.

AMS 2000 subject classifications. Primary 60F05, 60G44, 62M09; secondary 60G42, 62G20.

1 Introduction

The last years have witnessed a considerable development of the statistics of processes observed at very high frequency, due to the recent availability of such data. This is particularly the case for market prices of stocks, currencies, and other financial instruments. Correlatively, the technology for the analysis of such data has grown rapidly. The emblematic problem is the question of how to estimate daily volatility for financial prices (in stochastic process terms, the quadratic variation of log prices).

However, those high frequency data are almost always corrupted by some noise. This may be recording or measurement errors, a situation which can be modeled by an additive white noise. For financial data we also have a different sort of "noise", due to the fact that prices are recorded as multiples of the basic currency unit, so that some rounding is necessarily performed, and the level of rounding is far from being negligible for very high frequency data in comparison with the intrinsic variability of the underlying process. For

[*]Institut de Mathématiques de Jussieu, 175 rue du Chevaleret 75 013 Paris, France (CNRS – UMR 7586, and Université Pierre et Marie Curie - P6), Email: jean.jacod@upmc.fr

[†]Department of Mathematics, ETH Zürich, HG G37.1, 8092 Zürich, Switzerland, Email: mark.podolskij@math.ethz.ch.

[‡]Ruhr-Universität Bochum, Fakultät für Mathematik, 44780 Bochum, Germany, Email: mathias.vetter@rub.de

these reasons, it is commonly acknowledged that the underlying process of interest, such as the price semimartingale, is latent rather than observed.

A large amount of work has already been devoted to the subject, especially for additive white noise, but also for some other types of noise like rounding effects. A comprehensive discussion of the noise models and the effect of noise on the inference for the underlying process may be found in [12]. And various statistical procedures for getting rid of the noise have been proposed, see for example [1], [2], [3], [15], [16] and, more closely related to the present work, [5], [13], [14], [11].

As a matter of fact, most of the afore-mentioned papers are concerned with the estimation of the integrated volatility, that is the quadratic variation, for a continuous semimartingale. Only a few consider discontinuous semimartingales, and mostly study again the quadratic variation or its continuous part. So there is a lack of more general results, allowing for example to estimate other powers of the volatility (like the "quarticity") or the sum of some powers of the jumps, for a general Itô semimartingale. These quantities have proved extremely useful for a number of estimation or testing problems in the context of high frequency data, but they have been studied so far when the process is observed without noise.

The aim of this paper is to (partly) fill this gap. This is a probabilistic paper, with no explicit statistical application, but of course the interest and motivation of the forthcoming results lie essentially in potential applications. It is done in the context of the "pre-averaging method" developed in [11] and [13] for the estimation of the integrated volatility for a continuous semimartingale.

Let us be more specific: we consider an Itô semimartingale X which is corrupted by noise. The observed process $Z = (Z_t)_{t\geq 0}$ is given as

$$Z_t = X_t + \chi_t \ , \qquad t \geq 0 \ ,$$

where $(\chi_t)_{t\geq 0}$ are errors, which are, conditionally on the process X, centered and independent. The process Z is assumed to be observed at equidistant time points $i\Delta_n$, $i = 0, 1, \ldots, [t/\Delta_n]$, with $\Delta_n \to 0$ as $n \to \infty$. This structure of noise allows for an additive white noise, but also for noise involving rounding effects since χ_t may depend on X_t, or even on the whole past of X before time t. It rules out, though, some other interesting types of noise, like an additive colored noise. Note however that the χ_t are not necessarily independent (the independence is only "conditional on X").

In the no-noise case (i.e. $\chi \equiv 0$) an extensive theory has been developed in various papers, which allows for estimating quantities like $\sum_{s\leq t} |\Delta X_s|^p$ where ΔX_s denotes the jump size of X at time s, or $\int_0^t |\sigma_s|^p ds$ where σ is the volatility. See, for instance, [4] or [10] among others. Typically, these quantities are estimated by sums of powers of the successive increments of X, that is are limits of such sums. When the noise is present, these estimators are inadequate because they converge toward some characteristics of the noise rather than toward the characteristics of the process X in which we are interested. There are currently three main approaches to overcome this difficulties, mainly for the estimation of the quadratic variation in the continuous case: the subsampling method ([16]), the realized kernel method ([5]) and the pre-averaging method ([13],[11]) (see also

[6] for a comprehensive theory in the parametric setting). All these approaches achieve the optimal rate of $\Delta_n^{1/4}$. In this paper we use the pre-averaging method to derive rather general estimators.

More precisely, we choose a (smooth) weight function g on $[0,1]$ and an appropriate sequence k_n, with which we associate the (observed) variables

$$\overline{Z}(g)_i^n = \sum_{j=1}^{k_n-1} g(j/k_n)(Z_{(i+j)\Delta_n} - Z_{(i+j-1)\Delta_n}),$$

$$\widehat{Z}(g)_i^n = \sum_{j=1}^{k_n} (g(j/k_n) - g((j-1)/k_n))^2 (Z_{(i+j)\Delta_n} - Z_{(i+j-1)\Delta_n})^2.$$

Our aim is to study the asymptotic behavior of the following functionals:

$$V(Z,g,p,r)_t^n = \sum_{i=0}^{[t/\Delta_n]-k_n} |\overline{Z}(g)_i^n|^p \ |\widehat{Z}(g)_i^n|^r$$

for suitable powers $p, r \geq 0$. The role of $\overline{Z}(g)_i^n$ is the reduction of the influence of the noise process χ, whereas $\widehat{Z}(g)_i^n$ is used for bias corrections. The asymptotic theory for the functionals $V(Z,g,p,0)_t^n$ in the absence of jumps is (partially) derived in [11] and [14], but here we extend these results to the case of general semimartingales.

Quite naturally, the asymptotic behavior of $V(Z,g,p,r)_t^n$ is different according to whether the process X is continuous or not. In particular, different scaling is required to obtain non-trivial limits for $V(Z,g,p,r)_t^n$. More precisely, we show the following ($\stackrel{\mathbb{P}}{\longrightarrow}$ means convergence in probability, and $\stackrel{\text{u.c.p.}}{\longrightarrow}$ means convergence in probability uniformly over all finite time intervals):

(i) For all semimartingales X it holds that $\frac{1}{k_n} V(Z,g,p,0)_t^n \stackrel{\mathbb{P}}{\longrightarrow} \overline{g}(p) \sum_{s \leq t} |\Delta X_s|^p$ for $p > 2$ and $\frac{1}{k_n} V(Z,g,2,0)_t^n - \frac{1}{2k_n} V(Z,g,0,1)_t^n \stackrel{\mathbb{P}}{\longrightarrow} \overline{g}(2)[X,X]_t$, where the $\overline{g}(p)$'s are known constants (which depend on g) and $[X,X]$ is the quadratic variation of X.

(ii) When X is a continuous Itô semimartingale it holds that $\Delta_n^{1-p/4} V(Z,g,p,0)_t^n \stackrel{\text{u.c.p.}}{\longrightarrow} m_p \int_0^t \left|\theta \overline{g}(2)\sigma_s^2 + \frac{\overline{g}'(2)}{\theta} \alpha_s^2\right|^{p/2} ds$, where m_p, θ are certain constants, (σ_s^2) is the volatility process and (α_s^2) is the local conditional variance of the noise process χ. Furthermore, a proper linear combination of $V(Z,g,p,r)_t^n$ for integers p, r with $p+2r = l$ converges in probability to $\int_0^t |\sigma_s|^l ds$, when l is an even integer.

For each of the afore-mentioned cases we prove a joint stable central limit theorem for a given family of weight functions $(g_i)_{1 \leq i \leq d}$ (for the first functional in (i) we additionally have to assume that $p > 3$). The corresponding convergence rate is $\Delta_n^{1/4}$.

We end this introduction by emphasizing that only the 1-dimensional case for X is studied here. The extension to multi-dimensional semimartingales is possible, and even mathematically rather straightforward, but extremely cumbersome, and this paper is already quite complicated as it is.

This paper is organized as follows: in Section 2 we introduce the setting and the assumptions. Sections 3 and 4 are devoted to stating the results, first the various convergences in probability, and second the associated central limit theorems. The proof are gathered in Section 5.

2 The setting

We have a 1-dimensional underlying process $X = (X_t)_{t\geq 0}$, and observation times $i\Delta_n$ for all $i = 0, 1, \cdots, k, \cdots$, with $\Delta_n \to 0$. We suppose that X is a semimartingale, which can thus be written as

$$X = X_0 + B + X^c + (x1_{\{|x|\leq 1\}}) \star (\mu - \nu) + (x1_{\{|x|>1\}}) \star \mu. \tag{2.1}$$

Here μ is the jump measure of X and ν is its predictable compensator, and X^c is the continuous (local) martingale part of X, and B is the drift. All these are defined on some filtered probability space $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)})_{t\geq 0}, \mathbb{P}^{(0)})$. We use here the traditional notation of stochastic calculus, and for any unexplained (but standard) notation we refer to [9]; for example $\psi \star (\mu - \nu)_t = \int_0^t \int_{\mathbb{R}} \psi(s, x)(\mu - \nu)(ds, dx)$ is the stochastic integral of the predictable function $\psi(\omega, t, x)$ with respect to the martingale measure $\mu - \nu$, when it exists.

The process X is observed with an error: that is, at stage n and instead of the values $X_i^n = X_{i\Delta_n}$ for $i \geq 0$, we observe $X_i^n + \chi_i^n$, where the χ_i^n's are "errors" which are, conditionally on the process X, centered and independent (this allows for errors which are depending on X and thus may be unconditionally dependent). It is convenient to define the noise χ_t for any time t, although at stage n only the values $\chi_{i\Delta_n}$ are really used.

Mathematically speaking, this can be formalized as follows: for each $t \geq 0$, we have a transition probability $Q_t(\omega^{(0)}, dz)$ from $(\Omega^{(0)}, \mathcal{F}_t^{(0)})$ into $\mathbb{R}$. We endow the space $\Omega^{(1)} = \mathbb{R}^{[0,\infty)}$ with the product Borel σ-field $\mathcal{F}^{(1)}$ and the "canonical process" $(\chi_t : t \geq 0)$ and with the probability $\mathbb{Q}(\omega^{(0)}, d\omega^{(1)})$ which is the product $\otimes_{t\geq 0} Q_t(\omega^{(0)}, .)$. We introduce the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the filtration $(\mathcal{G}_t)$ as follows:

$$\left.\begin{array}{l} \Omega \;=\; \Omega^{(0)} \times \Omega^{(1)}, \qquad \mathcal{F} \;=\; \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}, \\ \mathcal{F}_t \;=\; \mathcal{F}_t^{(0)} \otimes \sigma(\chi_s : s \in [0, t)), \qquad \mathcal{G}_t \;=\; \mathcal{F}^{(0)} \otimes \sigma(\chi_s : s \in [0, t)), \\ \mathbb{P}(d\omega^{(0)}, d\omega^{(1)}) = \mathbb{P}^{(0)}(d\omega^{(0)})\; \mathbb{Q}(\omega^{(0)}, d\omega^{(1)}). \end{array}\right\} \tag{2.2}$$

Any variable or process which is defined on either $\Omega^{(0)}$ or $\Omega^{(1)}$ can be considered in the usual way as a variable or a process on Ω. Note that X is still a semimartingale, with the same decomposition (2.1), on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, despite the fact that the filtration $(\mathcal{F}_t)$ is not right-continuous. On the other hand, the "process" χ typically has no measurability property in time, since under $\mathbb{Q}(\omega^{(0)}, .)$ it is constituted of independent variables; as mentioned before, only the values of χ at the observation times are relevant, and the extension as a process indexed by $\mathbb{R}_+$ is for notational convenience only.

At time t, instead of X_t we observe the variable

$$Z_t \;=\; X_t + \chi_t \tag{2.3}$$

We make the following crucial assumption on the noise, for some $q \geq 2$:

Hypothesis (N-q): There is a sequence of $(\mathcal{F}_t^{(0)})$-stopping times (T_n) increasing to ∞, such that $\int Q_t(\omega^{(0)}, dz)\,|z|^q \leq n$ whenever $t < T_n(\omega^{(0)})$. We write for any integer $r \leq q$:

$$\beta(r)_t(\omega^{(0)}) \;=\; \int Q_t(\omega^{(0)}, dz)\, z^r, \qquad \alpha_t \;=\; \sqrt{\beta(2)_t}, \tag{2.4}$$

and we also assume that

$$\beta(1) \;\equiv\; 0. \tag{2.5}$$

□

In most applications, the local boundedness of the qth moment of the noise, even for all $q > 0$, is not a genuine restriction. The condition (2.5), on the other hand, is quite a serious restriction, and for instance it rules out the case where Z_t is a pure rounding of X_t: see [11] for a discussion of the implications of this assumption, and some examples.

We choose a sequence of integers k_n satisfying for some $\theta > 0$:

$$k_n\sqrt{\Delta_n} \;=\; \theta + \mathrm{o}(\Delta_n^{1/4}); \qquad \text{we write } \; u_n \;=\; k_n\Delta_n. \tag{2.6}$$

We will also consider weight functions g on $[0,1]$, satisfying

$$\left.\begin{array}{l} g \text{ is continuous, piecewise } C^1 \text{ with a piecewise Lipschitz derivative } g', \\ g(0) = g(1) = 0, \qquad \int_0^1 g(s)^2 ds \;>\; 0. \end{array}\right\} \tag{2.7}$$

It is convenient to extend such a g to the whole of $\mathbb{R}$ by setting $g(s) = 0$ if $s \notin [0,1]$. We associate with g the following numbers (where $p \in (0,\infty)$ and $i \in \mathbb{Z}$):

$$\left.\begin{array}{ll} g_i^n \;=\; g(i/k_n), & g_i'^n \;=\; g_i^n - g_{i-1}^n, \\ \overline{g}(p)_n = \sum_{i=1}^{k_n} |g_i^n|^p, & \overline{g}'(p)_n = \sum_{i=1}^{k_n} |g_i'^n|^p. \end{array}\right\} \tag{2.8}$$

If g, h are bounded functions with support in $[0,1]$ and $p > 0$ and $t \in \mathbb{R}$ we set

$$\overline{g}(p) \;=\; \int |g(s)|^p\, ds, \qquad \overline{(gh)}(t) \;=\; \int g(s)h(s-t)\, ds. \tag{2.9}$$

For example $\overline{g'}(p)$ is associated with g' by the first definition above, and $\overline{g}(2) = \overline{(gg)}(0)$. Note that, as $n \to \infty$,

$$\overline{g}(p)_n \;=\; k_n\, \overline{g}(p) + \mathrm{O}(1), \qquad \overline{g}'(p)_n \;=\; k_n^{1-p}\, \overline{g'}(p) + \mathrm{O}(k_n^{-p}). \tag{2.10}$$

With any process $Y = (Y_t)_{t\geq 0}$ we associate the following random variables

$$\left.\begin{array}{l} Y_i^n \;=\; Y_{i\Delta_n}, \qquad \Delta_i^n Y \;=\; Y_{i\Delta_n} - Y_{(i-1)\Delta_n}, \\ \overline{Y}(g)_i^n \;=\; \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n Y \;=\; -\sum_{j=1}^{k_n} g_j'^n Y_{i+j-1}^n, \\ \widehat{Y}(g)_i^n \;=\; \sum_{j=1}^{k_n} (g_j'^n \Delta_{i+j}^n Y)^2, \end{array}\right\} \tag{2.11}$$

and we define the σ-fields $\mathcal{F}_i^n = \mathcal{F}_{i\Delta_n}$ and $\mathcal{G}_i^n = \mathcal{G}_{i\Delta_n}$.

Now we can define the processes of interest for this paper. Below, p and r are nonnegative reals, and typically the process Y will be X or Z:

$$V(Y,g,p,r)_t^n = \sum_{i=0}^{[t/\Delta_n]-k_n} |\overline{Y}(g)_i^n|^p \, |\widehat{Y}(g)_i^n|^r. \tag{2.12}$$

We end this section by stating a number of assumptions on X, which are needed for some of the results below.

One of these assumptions is that X is an *Itô semimartingale.* This means that its characteristics are absolutely continuous with respect to Lebesgue measure, or equivalently that it can be written as

$$X_t \;=\; X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + (\delta 1_{\{|\delta|\leq 1\}}) \star (\underline{\mu} - \underline{\nu})_t + (\delta 1_{\{|\delta|>1\}}) \star \underline{\mu}_t, \tag{2.13}$$

where W is a Brownian motion and $\underline{\mu}$ and $\underline{\nu}$ are a Poisson random measure on $\mathbb{R}_+ \times E$ and its compensator $\nu(dt,dz) = dt \otimes \lambda(dz)$ (where $(E,\mathcal{E})$ is an auxiliary space and λ a σ-finite measure). The required regularity and boundedness conditions on the coefficients b, σ, δ are gathered in the following:

Hypothesis (H) : The process X has the form (2.13) (on $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)}), \mathbb{P}^{(0)})$), and further:

a) the process (b_t) is optional and locally bounded;

b) the processes (σ_t) is càdlàg (= right-continuous with left limits) and adapted;

c) the function δ is predictable, and there is a bounded function γ in $\mathbb{L}^2(E,\mathcal{E},\lambda)$ such that the process $\sup_{z\in E}(|\delta(\omega^{(0)},t,z)| \wedge 1)/\gamma(z)$ is locally bounded. □

In particular, a *continuous Itô semimartingale* is of the form

$$X_t \;=\; X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s. \tag{2.14}$$

where the processes b and σ are optional (relative to $(\mathcal{F}_t^{(0)})$) and such that the integrals above make sense. When this is the case, we sometimes need the process σ itself to be an Itô semimartingale: it can then be written as in (2.13), but another way of expressing this property is as follows (we are again on the space $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)}), \mathbb{P}^{(0)})$):

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + M_t + \sum_{s\leq t} \Delta\sigma_s \, 1_{\{|\Delta\sigma_s|>v\}}, \tag{2.15}$$

where M is a local martingale orthogonal to W and with bounded jumps and $\langle M,M\rangle_t = \int_0^t a_s ds$, and the compensator of $\sum_{s\leq t} 1_{\{|\Delta\sigma_s|>v\}}$ is $\int_0^t a'_s ds$, and where $\widetilde{b}_t$, a_t, a'_t and $\widetilde{\sigma}_t$ are optional processes, the first three ones being locally integrable and the fourth one being locally square-integrable. Then we set:

Hypothesis (K) : We have (2.14) and (2.15), and the processes $\widetilde{b}_t$, a_t, a'_t are locally bounded, whereas the processes b_t and $\widetilde{\sigma}_t$ are left-continuous with right limits. □

Remark 2.1 The intuition behind the quantities $\overline{Z}(g)_i^n$ and $\widehat{Z}(g)_i^n$ can be explained as follows. Assume for simplicity that X is a continuous Itô semimartingale of the form (2.14) and the noise process χ is independent of X. Now, conditionally on $\mathcal{F}_i^n$, it holds that

$$\Delta_n^{-1/4}\overline{Z}(g)_i^n \overset{asy}{\sim} \mathcal{N}\left(0, \theta\overline{g}(2)\sigma_{i\Delta_n}^2 + \frac{\overline{g}'(2)}{\theta}\ \alpha_{i\Delta_n}^2\right)$$

when the processes α and σ are continuous on the interval $(i\Delta_n, (i+k_n)\Delta_n]$. Thus, $\Delta_n^{-1/4}\overline{Z}(g)_i^n$ contains a "biased information" about $\sigma_{i\Delta_n}^2$ (which is usually the main object of interest). On the other hand, we have that

$$\widehat{Z}(g)_i^n \approx \frac{2\overline{g}'(2)}{k_n}\ \alpha_{i\Delta_n}^2$$

when the process α is continuous on the interval $(i\Delta_n, (i+k_n)\Delta_n]$ (this approximation holds even for all semimartingales X). It is now intuitively clear that a certain combination of the quantities $\overline{Z}(g)_i^n$ and $\widehat{Z}(g)_i^n$ can be used to recover some functions of $\sigma_{i\Delta_n}$. In particular, a proper linear combination of $V(Y, g, p-2l, l)_t^n$, $l = 0, \ldots, p/2$, for an even number p, converges in probability to $\int_0^t |\sigma_s|^p\, ds$. This intuition is formalized in Theorem 3.3 and 3.4.

3 Results: the Laws of Large Numbers

3.1 LLN for all semimartingales.

We consider here an LLN which holds for *all* semimartingales, and we start with the version without noise, that is $Z = X$. For the sake of comparison, we recall the following classical result:

$$\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p \ \stackrel{\mathbb{P}}{\longrightarrow}\ \begin{cases} \sum_{s\leq t} |\Delta X_s|^p & \text{if } p > 2 \\ [X,X]_t & \text{if } p = 2. \end{cases} \tag{3.1}$$

Below, and throughout the paper, g always denotes a weight function satisfying (2.7).

Theorem 3.1 *For any $t \geq 0$ which is not a fixed time of discontinuity of X we have*

$$\frac{1}{k_n}\ V(X, g, p, 0)_t^n \ \stackrel{\mathbb{P}}{\longrightarrow}\ \begin{cases} \overline{g}(p)\ \sum_{s\leq t} |\Delta X_s|^p & \textit{if } p > 2 \\ \overline{g}(2)\ [X,X]_t & \textit{if } p = 2. \end{cases} \tag{3.2}$$

This convergence also holds for any t such that t/Δ_n is an integer for all n, if this happens, but it *never* holds in the Skorokhod sense, except of course when X is continuous. Taking in (2.12) test functions which are $f(x) = |x|^p$ is essential. For this we do not need the full force of (2.6), but only that $u_n \to 0$ and $k_n \to \infty$.

Next we have the version with noise, again for an arbitrary semimartingale X. The reader will have noticed in the previous theorem that nothing is said about $V(X,g,p,r)_t^n$ when $r \geq 1$, and in fact those functionals are of little interest. However, when noise is present, we need those processes to remove an intrinsic bias, as in (b) below, and so we provide their behavior, or at least some (rough) estimates on them.

Theorem 3.2 *a) For any $t \geq 0$ which is not a fixed time of discontinuity of X we have*

$$p > 2 \text{ and (N-}p\text{) holds} \quad \Rightarrow \quad \frac{1}{k_n} V(Z,g,p,0)_t^n \stackrel{\mathbb{P}}{\longrightarrow} \overline{g}(p) \sum_{s \leq t} |\Delta X_s|^p. \tag{3.3}$$

Moreover if $r > 0$ and $p + 2r > 2$ and if (N-$(p+2r)$) holds, then

$$\text{the sequence } \left(k_n^{r - \frac{p+4r}{p+2r}} V(Z,g,p,r)_t^n \right) \text{ is tight.} \tag{3.4}$$

b) Under (N-2) we have for all t as above:

$$\frac{1}{k_n} V(Z,g,2,0)_t^n - \frac{1}{2k_n} V(Z,g,0,1)_t^n \stackrel{\mathbb{P}}{\longrightarrow} \overline{g}(2)\ [X,X]_t. \tag{3.5}$$

It is worth emphasizing that the behaviors of $V(Z,g,p,0)^n$ and of $V(X,g,p,0)^n$ are basically the same when $p > 2$, at least for the convergence in probability. That is, by using the pre-averaging procedure we wipe out completely the noise in this case. On the opposite, when $p = 2$ the two processes $V(Z,g,2,0)^n$ and $V(X,g,2,0)^n$ behave differently, even at the level of convergence in probability.

3.2 LLN for continuous Itô semimartingales - 1.

When X is continuous, Theorem 3.2 gives a vanishing limit when $p > 2$, so it is natural in this case to look for a normalization which provides a non-trivial limit. This is possible only when X is a continuous Itô semimartingale, of the form (2.14).

Theorem 3.3 *Assume (N-q) for some $q > 2$ and that X is given by (2.14). Assume also that b is locally bounded and that σ and α are càdlàg. Then if $0 < p \leq q/2$ we have*

$$\Delta_n^{1-p/4}\ V(Z,g,p,0)_t^n \stackrel{u.c.p.}{\longrightarrow} m_p \int_0^t \left| \theta \overline{g}(2) \sigma_s^2 + \frac{\overline{g}'(2)}{\theta}\ \alpha_s^2 \right|^{p/2} ds\,, \tag{3.6}$$

where m_p denotes the pth absolute moment of $\mathcal{N}(0,1)$.

(The assumption $p \leq q/2$ could be replaced by $p < q$, with some more work.) This result should be compared to the well known result which states that, under the same assumptions on X, the processes $\Delta_n^{1-p/2} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p$ converge to the limiting process $m_p \int_0^t |\sigma_s|^p\, ds$.

This theorem is not really satisfactory, since contrary to what happens in Theorem 3.2-(a) the limit depends on the noise process, through α_s, and further we do not know how to prove a CLT associated to it, because of an intrinsic bias due again to the noise, see Remark 2.1. However, at least when p is an even integer, we can prove a useful substitute. That is, by an application of the binomial formula and the estimation of the terms that involve the process α_s, we obtain (up to a constant factor) the process $\int_0^t |\sigma_s|^p\,ds$ in the limit. This result, which we explain below, is much more useful for practical applications.

For any even integer $p \geq 2$ we introduce the numbers $\rho_{p,l}$ for $l = 0, \cdots, p/2$ which are the solutions of the following triangular system of linear equations ($C_q^p = \frac{q!}{p!(q-p)!}$ denote the binomial coefficients):

$$\left.\begin{array}{l} \rho_{p,0} \;=\; 1, \\ \sum_{l=0}^{j} 2^l \; m_{2j-2l} \; C_{p-2l}^{2j-2l} \; \rho_{p,l} \;=\; 0, \qquad j = 1, 2, \cdots, p/2. \end{array}\right\} \tag{3.7}$$

These could of course be explicitly computed, and for example we have

$$\rho_{p,1} \;=\; -\,\frac{1}{2}\,C_p^2, \quad \rho_{p,2} \;=\; \frac{3}{4}\,C_p^4, \quad \rho_{p,3} \;=\; -\,\frac{15}{8}\,C_p^6. \tag{3.8}$$

Then for any process Y and for $p \geq 2$ an even integer we set

$$\overline{V}(Y, g, p)_t^n \;=\; \sum_{l=0}^{p/2} \rho_{p,l}\; V(Y, g, p-2l, l)_t^n. \tag{3.9}$$

Theorem 3.4 *a) Let X be an arbitrary semimartingale, and assume (N-p) for some even integer $p \geq 2$. Then for all $t \geq 0$ we have*

$$\frac{1}{k_n}\,\overline{V}(Z, g, p)_t^n \;\stackrel{\mathbb{P}}{\longrightarrow}\; \left\{\begin{array}{ll} \overline{g}(p) \sum_{s\leq t} |\Delta X_s|^p & \textit{if } \; p \geq 4 \\ \overline{g}(2)\,[X, X]_t & \textit{if } \; p = 2. \end{array}\right. \tag{3.10}$$

b) Let X satisfy (2.14), and assume (N-2p) for some even integer $p \geq 2$. Assume also that b is locally bounded and that σ and α are càdlàg. Then we have

$$\Delta_n^{1-p/4}\;\overline{V}(Z, g, p)_t^n \;\stackrel{u.c.p.}{\longrightarrow}\; m_p\;(\theta\overline{g}(2))^{p/2} \int_0^t |\sigma_s|^p\,ds\,. \tag{3.11}$$

The first part of (3.10) is an obvious consequence of (a) of Theorem 3.2, whereas the second part of (3.10) is nothing else than (3.5), because $\rho_{2,1} = -1/2$.

3.3 LLN for continuous Itô semimartingales - 2.

For statistical applications we need to have "estimates" for the conditional variance which will appear in the CLTs associated with some of the previous LLNs. In other words, we need to provide some other laws of large numbers, which *a priori* seem artificial but are motivated by potential applications.

To this end we need a few, somehow complicated, notation. Some of it will be of use for the CLTs below. First, we consider two independent Brownian motions W^1 and W^2, given on another auxiliary filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$. With any function g satisfying (2.7), and extended as before on $\mathbb{R}$ by setting it to be 0 outside $[0,1]$, we define the following Wiener integral processes

$$L(g)_t \;=\; \int g(s-t)dW^1_s, \qquad L'(g)_t \;=\; \int g'(s-t)dW^2_s. \tag{3.12}$$

If h is another function satisfying (2.7), we define $L(h)$ and $L'(h)$ likewise, *with the same* W^1 *and* W^2. The four dimensional process $U := (L(g), L'(g), L(h), L'(h))$ is continuous in time, centered, Gaussian and stationary. Clearly $(L(g), L(h))$ is independent of $(L'(g), L'(h))$, and the variables U_t and U_{t+s} are independent if $s \geq 1$.

We set

$$\left.\begin{array}{lcl} m_p(g;\eta,\zeta) &=& \mathbb{E}'((\eta L(g)_0 + \zeta L'(g)_0)^p) \\ m_{p,q}(g,h;\eta,\zeta) &=& \int_0^2 \mathbb{E}'\Big((\eta L(g)_1 + \zeta L'(g)_1)^p \; (\eta L(h)_t + \zeta L'(h)_t)^q\Big)\, dt. \end{array}\right\} \tag{3.13}$$

These could of course be expressed by the mean of expectations with respect to the joint law of U above and, considered as functions of (η,ζ), they are C^∞. In particular, since $L(g)_0$ and $L'(g)_0$ are independent centered Gaussian variables with respective variances $\overline{g}(2)$ and $\overline{g}'(2)$, when p in an integer we have

$$m_p(g;\eta,\zeta) \;=\; \begin{cases} \sum_{v=0}^{p/2} C_p^{2v}\, (\eta^2\overline{g}(2))^v\, (\zeta^2\overline{g}'(2))^{p/2-v}\, m_{2v}\, m_{p-2v} & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd.} \end{cases} \tag{3.14}$$

Next, recalling (3.7), we set for $p \geq 2$ an even integer:

$$\left.\begin{array}{l} \mu_p(g;\eta,\zeta) \;=\; \sum_{r=0}^{p/2} \rho_{p,r}\, (2\zeta^2\overline{g}'(2))^r\, m_{p-2r}(g;\eta,\zeta) \\ \mu_{2p}(g,h;\eta,\zeta) = \sum_{r,r'=0}^{p/2} \rho_{p,r}\, \rho_{p,r'}\, (2\zeta^2\overline{g}'(2))^r (2\zeta^2\overline{h}'(2))^{r'}\, m_{p-2r,p-2r'}(g,h;\eta,\zeta) \\ \overline{\mu}_{2p}(g,h;\eta,\zeta) \;=\; \mu_{2p}(g,h;\eta,\zeta) - 2\mu_p(g;\eta,\zeta)\, \mu_p(h;\eta,\zeta). \end{array}\right\} \tag{3.15}$$

The following lemma will be useful in the sequel:

Lemma 3.5 *We have*

$$\mu_p(g;\eta,\zeta) \;=\; m_p\, \eta^p\, \overline{g}(2)^{p/2}. \tag{3.16}$$

Moreover if g_i is a finite family of functions satisfying (2.7), for any (η,ζ) the matrix with entries $\overline{\mu}_{2p}(g_i,g_j;\eta,\zeta)$ is symmetric nonnegative.

We need a final notation, associated with any process Y and any even integer p:

$$\begin{aligned} M(Y,g,h;p)^n_t \;=\; \sum_{r,r'=0}^{p/2} \rho_{p,r}\rho_{p,r'} \sum_{i=0}^{[t/\Delta_n]-3k_n} (\widehat{Y}(g)^n_i)^r\, (\widehat{Y}(h)^n_i)^{r'} \Big(|\overline{Y}(g)^n_{i+k_n}|^{p-2r} \cdot \\ \frac{1}{k_n}\sum_{j=1}^{2k_n} |\overline{Y}(h)^n_{i+j}|^{p-2r'} - 2|\overline{Y}(g)^n_i|^{p-2r} |\overline{Y}(h)^n_{i+k_n}|^{p-2r'}\Big). \end{aligned} \tag{3.17}$$

Then our last LLN goes as follows:

Theorem 3.6 *Let X satisfy (2.14), and let $p \geq 2$ be an even integer. Assume (N-2p), that b is locally bounded and that σ and α are càdlàg. Then if $p \leq q/2$ and if g and h are two functions satisfying (2.7), we have*

$$\Delta_n^{1-p/2}\ M(Z,g,h;p)_t^n \ \stackrel{u.c.p.}{\longrightarrow}\ \theta^{-p/2}\int_0^t \overline{\mu}_{2p}(g,h;\theta\sigma_s,\alpha_s)\,ds\,. \tag{3.18}$$

The reader will observe that the limit in (3.18) is symmetrical in g and h, although $M(Y,g,h;p)_t^n$ is not.

4 Results: the Central Limit Theorems

4.1 CLT for continuous Itô semimartingales

As mentioned before, we do not know whether a CLT associated with the convergence (3.6) exists. But there is one associated with (3.11) when p is an even integer. Below we give a joint CLT for several weight functions g at the same time. We use the notation

$$\widetilde{V}(g,p)_t^n \ = \ \frac{1}{\Delta_n^{1/4}}\ \Big(\Delta_n^{1-p/4}\overline{V}(Z,g,p)_t^n - m_p\,(\theta\,\overline{g}(2))^{p/2}\int_0^t |\sigma_s|^p\,ds\Big). \tag{4.1}$$

In view of Lemma 3.5, the square-root matrix ψ referred to below exists, and by a standard selection theorem one can find a measurable version for it. For the stable convergence in law used below, we refer for example to [9].

Theorem 4.1 *Assume (K) and (N-4p), where p is an even integer, and also that the processes α and $\beta(3)$ are càdlàg. If $(g_i)_{1\leq i\leq d}$ is a family of functions satisfying (2.7), for each $t\geq 0$ the variables $(\widetilde{V}(g_i,p)_t^n)_{1\leq i\leq d}$ converge stably in law to a d-dimensional variable of the form*

$$\Big(\theta^{1/2-p/2}\sum_{j=1}^d\int_0^t \psi_{ij}(\theta\sigma_s,\alpha_s)\ dB_s^j\Big)_{1\leq i\leq d}, \tag{4.2}$$

where B is a d-dimensional Brownian motion independent of $\mathcal{F}$ (and defined on an extension of the space), and ψ is a measurable $d\times d$ matrix-valued function such that $(\psi\psi^\star)(\eta,\zeta)$ is the matrix with entries $\overline{\mu}_{2p}(g_i,g_j;\eta,\zeta)$, as defined by (3.15).

Observe that, up to the multiplicative constant $\theta^{1-p/2}$, the covariance of the jth and kth components of the limit above, conditionally on the σ-field $\mathcal{F}$, is exactly the right side of (3.18) for $g=g_j$ and $h=g_k$.

Remark 4.2 An application of Theorem 3.6 and the properties of stable convergence gives now a a *feasible* version of Theorem 4.1. We obtain, for example, that the quantity

$$\frac{\widetilde{V}(g,p)_t^n}{\sqrt{\theta^{1-p/2}\Delta_n^{1-p/2}\ M(Z,g,g;p)_t^n}}$$

converges stably in law (for any fixed t) to a variable $U \sim \mathcal{N}(0,1)$ that is independent of $\mathcal{F}$. The latter can be used to construct confidence regions for the quantity $\int_0^t |\sigma_s|^p\,ds$ for even p's.

Remark 4.3 We only have above the stable convergence in law for a given (arbitrary) time t. Obviously this can be extended to the convergence along any finite family of times, but we do not know whether a *functional* stable convergence in law holds, although it is quite likely.

4.2 CLT for discontinuous Itô semimartingales

Now we turn to the case when X jumps. There is a CLT for Theorem 3.2, at least when $p=2$ and $p>3$, exactly as in [10] for the processes of type (3.1). The CLT for Theorem 3.4, when p is an even integer, takes exactly the same form. In this subsection we are interested in the case $p>3$, whereas the case $p=2$ is dealt with in the next subsection.

In view of statistical applications, and as in the previous subsection, we need to consider a family $(g_i)_{1\leq i\leq d}$ of weight functions. We use the notation

$$\widetilde{V}^{\star}(g,p)^n_t \;=\; \frac{1}{\Delta_n^{1/4}}\,\Big(\frac{1}{k_n}\,V(Z,g,p,0)^n_t-\overline{g}(p)\sum_{s\leq t}|\Delta X_s|^p\Big) \tag{4.3}$$

and, when further $p\geq 4$ is an even integer,

$$\overline{V}^{\star}(g,p)^n_t \;=\; \frac{1}{\Delta_n^{1/4}}\,\Big(\frac{1}{k_n}\,\overline{V}(Z,g,p)^n_t-\overline{g}(p)\sum_{s\leq t}|\Delta X_s|^p\Big). \tag{4.4}$$

These are the processes whose asymptotic behavior is studied, but to describe the limit we need some rather cumbersome notation, which involves the d weight functions g_j satisfying (2.7), in which we are interested. For any real x and any $p>0$ we write $\{x\}^p=|x|^p\,\mathrm{sign}(x)$. Then we introduce four $d\times d$ symmetric matrices Ψ_{p-}, Ψ_{p+}, $\overline{\Psi}_{p-}$ and $\overline{\Psi}_{p+}$ with entries:

$$\left.\begin{array}{l}
\Psi^{ij}_{p-}=\int_0^1\Big(\int_t^1\{g_i(s)\}^{p-1}g_i(s-t)ds\Big)\Big(\int_t^1\{g_j(s)\}^{p-1}g_j(s-t)ds\Big)\,dt\\
\Psi^{ij}_{p+}=\int_0^1\Big(\int_0^{1-t}\{g_i(s)\}^{p-1}g_i(s+t)ds\Big)\Big(\int_0^{1-t}\{g_j(s)\}^{p-1}g_j(s+t)ds\Big)\,dt\\
\overline{\Psi}^{ij}_{p-}=\int_0^1\Big(\int_t^1\{g_i(s)\}^{p-1}g_i'(s-t)ds\Big)\Big(\int_t^1\{g_j(s)\}^{p-1}g_j'(s-t)ds\Big)\,dt\\
\overline{\Psi}^{ij}_{p+}=\int_0^1\Big(\int_0^{1-t}\{g_i(s)\}^{p-1}g_i'(s+t)ds\Big)\Big(\int_0^{1-t}\{g_j(s)\}^{p-1}g_j'(s+t)ds\Big)\,dt
\end{array}\right\} \tag{4.5}$$

These matrices are semi-definite positive, and we can thus consider four independent sequences of i.i.d. d-dimensional variables $(U_{m-})_{m\geq1}$, $(U_{m+})_{m\geq1}$, $(\overline{U}_{m-})_{m\geq1}$ and $(\overline{U}_{m+})_{m\geq1}$, defined on an extension of the space, independent of $\mathcal{F}$, and such that for each m the d-dimensional variables U_{m-}, U_{m+}, $\overline{U}_{m-}$ and $\overline{U}_{m+}$ are centered Gaussian vectors with respective covariances Ψ_{p-}, Ψ_{p+}, $\overline{\Psi}_{p-}$ and $\overline{\Psi}_{p+}$. Note that these variables also depend on p and on the family (g_j), although it does not show in the notation.

Now let $(T_m)_{m\geq 1}$ be a sequence of stopping times with pairwise disjoint graphs, such that $\Delta X_t \neq 0$ implies that $t = T_m$ for some m. As is well known (see [10]), the following d-dimensional processes are well-defined when $p > 3$ and α is càdlàg, and are $\mathcal{F}$-conditional martingales:

$$\begin{array}{rcl} U(p)_t &=& p\sum_{m\geq 1} \left\{\Delta X_{T_m}\right\}^{p-1} \Big(\sqrt{\theta}\ \sigma_{T_m-} U_{m-} + \frac{\alpha_{T_m-}}{\sqrt{\theta}}\ \overline{U}_{m-} \\ && \qquad +\sqrt{\theta}\ \sigma_{T_m} U_{m+} + \frac{\alpha_{T_m}}{\sqrt{\theta}}\ \overline{U}_{m+}\Big)\ 1_{\{T_m\leq t\}}. \end{array} \tag{4.6}$$

Moreover, although these processes obviously depend on the choice of the times T_m, their $\mathcal{F}$-conditional laws do not; so if the stable convergence in law below holds for a particular "version" of $U(p)_t$, it also holds for all other versions.

Theorem 4.4 *Assume (H) and let $p > 3$. Assume also (N-2p) and that the process α is càdlàg. If $(g_i)_{1\leq i\leq d}$ is a family of functions satisfying (2.7), for each $t \geq 0$ the variables $(\widetilde{V}^*(g_i,p)_t^n)_{1\leq i\leq d}$ converge stably in law to the d-dimensional variable $U(p)_t$.*

The same holds for the sequence $(\overline{V}^*(g_i,p)_t^n)_{1\leq i\leq d}$ if further p is an even integer.

4.3 CLT for the quadratic variation

Finally we give a CLT for the quadratic variation, associated with (3.5) when $p = 2$, or equivalently with (3.10) which is exactly the same in this case. In contrast to the preceding results the function g is kept fixed, thus we will only show a one-dimensional result. So the processes of interest are simply

$$\overline{V}_t^n \;=\; \frac{1}{\Delta_n^{1/4}}\Big(\frac{1}{k_n}\ \overline{V}(Z,g,2)_t^n - \overline{g}(2)\,[X,X]_t\Big). \tag{4.7}$$

In order to describe the limit, we introduce an extension of the space on which are defined a Brownian motion B and variables $U_{m-}, \overline{U}_{m-}, U_{m+}, \overline{U}_{m+}$ indexed by $m \geq 1$, all these being independent one from the others and independent of $\mathcal{F}$, and such that the variables U_{m-}, U_{m+}, $\overline{U}_{m-}$, $\overline{U}_{m+}$ are centered Gaussian variables with respective variances Ψ_{2-}^{11}, Ψ_{2+}^{11}, $\overline{\Psi}_{2-}^{11}$ and $\overline{\Psi}_{2+}^{11}$, as defined in (4.5).

As in the previous section, $(T_m)_{m\geq 1}$ is a sequence of stopping times with pairwise disjoint graphs, such that $\Delta X_t \neq 0$ implies that $t = T_m$ for some m. Then we associate with these data the process $U(2)$, as defined by (4.6). The result goes as follows:

Theorem 4.5 *Assume (H). Assume also (N-4) and that the process α is càdlàg. Then for each t the variables $\overline{V}_t^n$ converges stably in law to the variable*

$$\overline{U}_t \;=\; \theta^{-1/2}\int_0^t \sqrt{\overline{\mu}_4(g,g;\theta\sigma_s,\alpha_s)}\ dB_s + U(2)_t, \tag{4.8}$$

where $\overline{\mu}_4(g,g;\eta,\zeta)$ is defined by (3.15), which here takes the form

$$\overline{\mu}_4(g,g;\eta,\zeta) \;=\; 4\int_0^1 \Big(\eta^2\int_s^1 g(u)g(u-s)du + \zeta^2\int_s^1 g'(u)g'(u-s)du\Big)^2\, ds. \tag{4.9}$$

When further X is continuous, the processes $\overline{V}^n$ converge stably (in the functional sense) to the process (4.8), with $U(2)=0$ in this case.

When X is continuous, we exactly recover Theorem 4.1 when $d=1$ and $g_1=g$, for $p=2$. Note that we do not need Hypothesis (K) here, because of the special feature of the case $p=2$. When X has jumps, though, the functional convergence does not hold.

5 The proofs

In the whole proof, we denote by K a constant which may change from line to line. This constant may depend on the characteristics of the process X and the law of the noise χ, on θ and the two sequences $(k_n)_{n\geq 1}$ and $(\Delta_n)_{n\geq 1}$ in (2.6), but it does not depend on n itself, nor on the index i of the increments $\Delta_i^n X$ or $\Delta_i^n Z$ under consideration. If it depends on an additional parameter q, we write it K_q.

For the proof of all the results we can use a localization procedure, described in details in [10] for instance, and which allows to systematically replace the hypotheses (N-q), (H) or (K), according to the case, by the following strengthened versions:

Hypothesis (SN-q): We have (N-q), and further $\int Q_t(\omega^{(0)},dz)\,|z|^q \leq K$. □

Hypothesis (SH): We have (H), and the processes b_t, σ_t, $\sup_{z\in E}|\delta(t,z)|/\gamma(z)$ and X itself are bounded. □

Hypothesis (SK): We have (K), and the processes b_t, σ_t, $\widetilde{b}_t$, a_t, a'_t, $\widetilde{\sigma}_t$ and X itself are bounded. □

Observe that under (SK), and upon taking v large enough in (2.15) (changing v changes the coefficients $\widetilde{b}_t$ and a_t without altering their boundedness), we can also suppose that the last term in (2.15) vanishes identically, that is

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + M_t. \tag{5.1}$$

Recall that $|g_j'^n| \leq K/k_n$. Then the fact that conditionally on $\mathcal{F}^{(0)}$ the χ_t's are independent and centered, plus Hölder inequality, give us that under (SN-q) we have (the σ-fields $\mathcal{F}_i^n$ and $\mathcal{G}_i^n$ have been defined after (2.11)):

$$\left.\begin{array}{lll} p\leq q & \Rightarrow & \mathbb{E}(|\overline{\chi}(g)_i^n|^p \mid \mathcal{G}_i^n) \;\leq\; K_p k_n^{-p/2} \\ 2r\leq q & \Rightarrow & \mathbb{E}(|\widehat{\chi}(g)_i^n|^r \mid \mathcal{G}_i^n) \;\leq\; K_r k_n^{-r}. \end{array}\right\} \tag{5.2}$$

We will also often use the following property, valid for all semimartingales Y:

$$\overline{Y}(g)_i^n = \int_{i\Delta_n}^{i\Delta_n+u_n} g_n(s-i\Delta_n)dY_s, \quad \text{where} \quad g_n(s) = \sum_{j=1}^{k_n-1} g_j^n \, 1_{((j-1)\Delta_n, j\Delta_n]}(s). \tag{5.3}$$

5.1 Proof of Theorem 3.1.

We start with an arbitrary semimartingale X, written as (2.1). The proof follows several steps.

Step 1) Denote by B' the variation process of B, and let $C = \langle X^c, X^c\rangle$. The process $B' + C + (x^2 \wedge 1) \star \nu$ is predictable increasing finite-valued, hence locally bounded. Then by an obvious localization procedure it is enough to prove the result under the assumption that

$$B'_\infty + C_\infty + (x^2 \wedge 1) \star \nu_\infty \ \leq\ K \tag{5.4}$$

for some constant K.

For each $\varepsilon \in (0,1]$ we set:

$$\left.\begin{array}{llllll} X(\varepsilon) & = & (x1_{\{|x|>\varepsilon\}}) \star \mu, & M(\varepsilon) & = & (x1_{\{|x|\leq\varepsilon\}}) \star (\mu-\nu) \\ A(\varepsilon) & = & \langle M(\varepsilon), M(\varepsilon)\rangle, & B(\varepsilon) & = & B - (x1_{\{\varepsilon<|x|\leq 1\}}) \star \nu \\ A'(\varepsilon) & = & (x^2 1_{\{|x|\leq\varepsilon\}}) \star \nu, & B'(\varepsilon) & = & \text{variation process of } B(\varepsilon), \end{array}\right\} \tag{5.5}$$

so that we have

$$X \ = \ X_0 + B(\varepsilon) + X^c + M(\varepsilon) + X(\varepsilon). \tag{5.6}$$

We also denote by $T_n(\varepsilon)$ the successive jump times of $X(\varepsilon)$, with the convention $T_0(\varepsilon) = 0$ (which of course is not a jump time). If $0 < \varepsilon < \eta \leq 1$ we have

$$\left.\begin{array}{l} A(\varepsilon) \leq A'(\varepsilon), \qquad \Delta B'(\varepsilon) \leq \varepsilon, \qquad |\Delta M(\varepsilon)| \leq 2\varepsilon \\ B'(\varepsilon) \leq B' + \frac{1}{\varepsilon}\, A'(\eta) + \frac{1}{\eta}\, (x^2 \wedge 1) \star \nu. \end{array}\right\} \tag{5.7}$$

Finally, we write $V(Y,p)^n = V(Y,g,p,0)^n$ and $\overline{Y}^n_i = \overline{Y}(g)^n_i$ in this proof. We also set $\theta(Y,u,t) = \sup_{s\leq r\leq s+u, r\leq t} |Y_r - Y_s|$. Observe that $\overline{Y}^n_i = -\sum_{j=1}^{k_n} (g((j+1)/k_n) - g(j/k_n))(Y_{(i+j)\Delta_n} - Y_{i\Delta_n})$. Hence, since the derivative g' is bounded, we obtain

$$i \leq [t/\Delta_n] - k_n + 1 \quad \Rightarrow \quad |\overline{Y}^n_i| \ \leq\ K\theta(Y, u_n, t). \tag{5.8}$$

Step 2) Here we study $B(\varepsilon)$. By (5.8) and $\theta(B(\varepsilon),u,t) \leq \theta(B'(\varepsilon),u,t)$ we obtain for $p > 1$:

$$V(B(\varepsilon), p)^n_t \ \leq\ K k_n\, B'(\varepsilon)_t\, \theta(B'(\varepsilon), u_n, t)^{p-1}.$$

Since $\Delta B'(\varepsilon) \leq \varepsilon$ we have $\limsup_{n\to\infty} \theta(B'(\varepsilon), u_n, t) \leq \varepsilon$, so by (5.4) and (5.7) we have $\limsup_n \frac{1}{k_n} V(B'(\varepsilon), p)^n_t \leq K\varepsilon^{p-1}\Big(\frac{1}{\eta} + \frac{1}{\varepsilon}\, A'(\eta)_t\Big)$ for all $0 < \varepsilon < \eta \leq 1$. Since $A'(\eta)_t \to 0$ as $\eta \to 0$, we deduce (choose first η small, then ε smaller) that for $p \geq 2$:

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \frac{1}{k_n}\, V(B(\varepsilon), p)^n_t \ = \ 0. \tag{5.9}$$

Step 3) In this step, we consider a square-integrable martingale Y with $D = \langle Y, Y\rangle$ bounded. In view of (5.3)),

$$\mathbb{E}((\overline{Y}^n_i)^2) \ = \ \mathbb{E}\Big(\int_{i\Delta_n}^{i\Delta_n+u_n} g_n(s - i\Delta_n)^2 dD_s\Big) \ \leq\ K\mathbb{E}(D_{i\Delta_n+u_n} - D_{i\Delta_n}).$$

On the other hand, $\mathbb{E}(\overline{Y}^n_i\overline{Y}^n_{i+j})=0$ whenever $j\geq k_n$. Therefore

$$\mathbb{E}\Big((V(Y,2)^n_t)^2\Big) \;\leq\; k_n\sum_{i=0}^{[t/\Delta_n]-k_n}\mathbb{E}((\overline{Y}^n_i)^2) \;\leq\; Kk_n^2\mathbb{E}(D_t). \tag{5.10}$$

We first apply this with $Y=M(\varepsilon)$, hence $D=A(\varepsilon)$. In view of (5.10) and since $A'(\varepsilon)_t\to 0$ as $\varepsilon\to 0$ and $A'(\varepsilon)_t\leq K$, we deduce

$$\lim_{\varepsilon\to 0}\;\sup_n\;\mathbb{E}\Big(\Big(\frac{1}{k_n}\,V(M(\varepsilon),2)^n_t\Big)^2\Big) \;=\; 0.$$

Since by (5.8) we have $V(M(\varepsilon),p)^n_t\leq KV(M(\varepsilon),2)^n_t\;\theta(M(\varepsilon),u_n,t)^{p-2}$ when $p>2$, and since $\limsup_n\theta(M(\varepsilon),u_n,t)\leq 2\varepsilon$, we get for $p\geq 2$:

$$p\geq 2,\quad \eta>0\quad\Rightarrow\quad \lim_{\varepsilon\to 0}\;\limsup_{n\to\infty}\;\mathbb{P}\Big(\frac{1}{k_n}\,V(M(\varepsilon),p)^n_t>\eta\Big)\;=\;0. \tag{5.11}$$

Next, (5.10) with $Y=X^c$ yields that the sequence $\frac{1}{k_n}V(X^c,2)^n_t$ is bounded in $\mathbb{L}^2$. Exactly the same argument as above, where now $\theta(X^c,u_n,t)\to 0$, yields

$$p>2,\quad \eta>0\quad\Rightarrow\quad \lim_{\varepsilon\to 0}\;\limsup_{n\to\infty}\;\mathbb{P}\Big(\frac{1}{k_n}\,V(X^c,p)^n_t>\eta\Big)\;=\;0. \tag{5.12}$$

Step 4) In this step we study $V(X(\varepsilon),p)^n_t$. We fix $t>0$ such that $\mathbb{P}(\Delta X_t\neq 0)=0$. For any $m\geq 1$ we set

$$I(m,n,\varepsilon)\;=\;\inf(i:\;i\Delta_n\geq T_m(\varepsilon)).$$

We consider the set $\Omega_n(t,\varepsilon)$ on which all intervals between two successive jumps of $X(\varepsilon)$ in $[0,t]$ are of length bigger than u_n, and also $[0,u_n)$ and $[t-u_n,t]$ contain no jump. Then $u_n\to 0$ and $\mathbb{P}(\Delta X_t\neq 0)=0$ yield $\Omega_n(t,\varepsilon)\to\Omega$ a.s. as $n\to\infty$. On the set $\Omega_n(t,\varepsilon)$ we have for $i\leq[t/\Delta_n]-k_n+1$:

$$\overline{X(\varepsilon)}^n_i=\begin{cases} g^n_{I(m,n,\varepsilon)-i}\,\Delta X_{T_m(\varepsilon)} & \text{if }\; I(m,n,\varepsilon)-k_n+1\leq i\leq I(m,n,\varepsilon)-1 \\ & \qquad\text{for some } m \\ 0 & \text{otherwise.}\end{cases} \tag{5.13}$$

Therefore on the set $\Omega_n(t,\varepsilon)$ we have

$$V(X(\varepsilon),p)^n_t\;=\;\overline{g}(p)_n\sum_{s\leq t}|\Delta X_s|^p 1_{\{|\Delta X_s|>\varepsilon\}},$$

and (2.10) yields

$$\frac{1}{k_n}\,V(X(\varepsilon),p)^n_t\;\to\;\overline{g}(p)\sum_{s\leq t}|\Delta X_s|^p 1_{\{|\Delta X_s|>\varepsilon\}}. \tag{5.14}$$

Step 5) In this step we study $V(X^c,2)^n_t$. For easier notation we write $Y = X^c$ and $Y(n,i)_s = \int_{i\Delta_n}^s g_n(r-i\Delta_n)dY_r$ when $s > i\Delta_n$. Using (5.3) and Itô's formula, we get $(\overline{Y}^n_i)^2 = \zeta^n_i + \zeta'^n_i$, where

$$\zeta^n_i = \int_{i\Delta_n}^{i\Delta_n+u_n} g_n(s-i\Delta_n)^2 dC_s, \qquad \zeta'^n_i = 2\int_{i\Delta_n}^{i\Delta_n+u_n} Y(n,i)_s dY_s.$$

On the one hand, $\sum_{i=0}^{[t/\Delta_n]-k_n} \zeta^n_i$ is equal to $\overline{g}(2)_n\, C_t$ plus a term smaller in absolute value than KC_{u_n} and another term smaller than $K(C_t - C_{t-u_n})$. Then obviously

$$\frac{1}{k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} \zeta^n_i \;\to\; \overline{g}(2)\, C_t. \tag{5.15}$$

On the other hand, we have $\mathbb{E}(\zeta'^n_i \zeta'^n_{i+j}) = 0$ when $j \geq k_n$, and

$$\mathbb{E}((\zeta'^n_i)^2) \;\leq\; 4\mathbb{E}\Big((C_{i\Delta_n+u_n} - C_{i\Delta_n}) \sup_{s\in[i\Delta_n, i\Delta_n+u_n]} Y(n,i)^2_s\Big).$$

Now, by Doob's inequality $\mathbb{E}\Big(\sup_{s\in[i\Delta_n,i\Delta_n+u_n]} Y(n,i)^4_s\Big) \leq K\mathbb{E}((C_{i\Delta_n+u_n} - C_{i\Delta_n})^2)$, hence Cauchy-Schwarz inequality yields

$$\mathbb{E}((\zeta'^n_i)^2) \;\leq\; K\mathbb{E}((C_{i\Delta_n+u_n} - C_{i\Delta_n})^2) \;\leq\; K\mathbb{E}\Big((C_{i\Delta_n+u_n} - C_{i\Delta_n})\theta(C,u_n,t)\Big)$$

whenever $i \leq [t/\Delta_n] - k_n + 1$. At this point, the same argument as for (5.10) gives

$$\mathbb{E}\Big(\Big(\sum_{i=0}^{[t/\Delta_n]-k_n} \zeta'^n_i\Big)^2\Big) \;\leq\; Kk_n^2\mathbb{E}(C_t\theta(C,u_n,t)) \;\leq\; Kk_n^2\mathbb{E}(\theta(C,u_n,t)).$$

But $\theta(C,u_n,t)$ tends to 0 and is smaller uniformly in n than a square-integrable variable. We then deduce that $\frac{1}{k_n}\sum_{i=0}^{[t/\Delta_n]-k_n} \zeta'^n_i \stackrel{\mathbb{P}}{\longrightarrow} 0$, and this combined with (5.15) yields

$$\frac{1}{k_n} V(X^c,2)^n_t \;\stackrel{\mathbb{P}}{\longrightarrow}\; \overline{g}(2)\, C_t. \tag{5.16}$$

Step 6) It remains to put all the previous partial results together. For this we use the following obvious property: for any $p \geq 2$ and $\eta > 0$ there is a constant $K_{p,\eta}$ such that

$$x,y \in \mathbb{R} \quad\Rightarrow\quad \Big||x+y|^p - |x|^p\Big| \;\leq\; K_{p,\eta}|y|^p + \eta|x|^p. \tag{5.17}$$

Suppose first that $p > 2$. Applying (5.17) and (5.6), we get

$$\begin{aligned}
&\Big|V(X,p)^n_t - V(X(\varepsilon),p)^n_t\Big| \\
&\quad\leq\; \eta V(X(\varepsilon),p)^n_t + K_{p,\eta}\Big(V(B(\varepsilon),p)^n_t + V(X^c,p)^n_t + V(M(\varepsilon),p)^n_t\Big).
\end{aligned}$$

Then by (5.9), (5.11), (5.12) and (5.14), plus $\sum_{s\leq t}|\Delta X_s|^p 1_{\{|\Delta X_s|>\varepsilon\}} \to \sum_{s\leq t}|\Delta X_s|^p$ as $\varepsilon\to 0$, and by taking η arbitrarily small in the above, we obtain the first part of (3.2).

Next suppose that $p=2$. The same argument shows that it is enough to prove that

$$\frac{1}{k_n}\,V(X^c+X(\varepsilon),2)^n_t \;\stackrel{\mathbb{P}}{\longrightarrow}\; \overline{g}(2)\Big(C_t+\sum_{s\leq t}|\Delta X_s|^2 1_{\{|\Delta X_s|>\varepsilon\}}\Big). \tag{5.18}$$

On the set $\Omega_n(t,\varepsilon)$, one easily sees that

$$V(X^c+X(\varepsilon),2)^n_t \;=\; V(X^c,2)^n_t+V(X(\varepsilon),2)^n_t+\sum_{m\geq 1:\;T_m(\varepsilon)\leq t}\zeta^n_m,$$

where $\zeta^n_m=\sum_{i=I(m,n,\varepsilon)-k_n+1}^{I(m,n,\varepsilon)-1}\zeta(m,n,i)$ and (with again $Y=X^c$)

$$\zeta(m,n,i)\;=\;\Big|g^n_{I(m,n,\varepsilon)-i}\Delta X_{T_m(\varepsilon)}+\overline{Y}^n_i\Big|^2-\Big|g^n_{I(m,n,\varepsilon)-i}\Delta X_{T_m(\varepsilon)}\Big|^2-|\overline{Y}^n_i|^2.$$

In view of (5.8), we deduce from (5.17) that for all $\eta>0$,

$$|\zeta(m,n,i)|\;\leq\; K_\eta\theta(X^c,u_n,t)^2+K\eta|\Delta X_{T_m(\varepsilon)}|^2$$

if $I(m,n,\varepsilon)-k_n<i<I(m,n,\varepsilon)$ and $T_m(\varepsilon)\leq t$. Then obviously (since η is arbitrarily small) we have $\zeta^n_m/k_n\to 0$ for all m with $T_m(\varepsilon)\leq t$. Hence (5.18) follows from (5.16) and (5.14), and we are finished.

5.2 Proof of Theorem 3.2.

Now we turn to the case where noise is present. X is still an arbitrary semimartingale, and as in the previous theorem we can assume by localization that (5.4) holds.

We first prove (a), and we assume (N-q) with $q=p$ for proving (3.3) and $q=p+2r$ for proving (3.4). Another localization allows to assume (SN-q), in which case (5.2) implies

$$\mathbb{E}(V(\chi,g,q,0)^n_t)+\mathbb{E}(V(\chi,g,0,q/2)^n_t)\;\leq\;\frac{Kt}{\Delta_n k_n^{q/2}}\;\leq\; Ktk_n^{2-q/2}. \tag{5.19}$$

We deduce from (5.17) that, for all $\eta>0$,

$$\Big|V(Z,g,q,0)^n_t-V(X,g,q,0)^n_t\Big|\;\leq\;\eta V(X,g,q,0)^n_t+K_{q,\eta}V(\chi,g,q,0)^n_t, \tag{5.20}$$

and thus (3.3) follows from (3.2) and (5.19).

Next, Hölder's inequality yields when $p,r>0$ with $p+2r=q>2$:

$$V(Z,g,p,r)^n_t\;\leq\;(V(Z,g,q,0)^n_t)^{p/q}\;(V(Z,g,0,q/2)^n_t)^{2r/q}.$$

By (3.3) applied with q instead of p we see that the sequence $k_n^{-1}V(Z,g,q,0)^n_t$ is tight, so for (3.4) it is enough to show that the sequence $k_n^{q/2-2}V(Z,g,0,q/2)^n_t$ is also tight.

To see this we first deduce from $|g_j'^n| \leq K/k_n$ that

$$\widehat{X}(g)_i^n \;\leq\; \frac{K}{k_n^2} \sum_{j=i}^{i+k_n-1} (\Delta_j^n X)^2, \tag{5.21}$$

implying by Hölder inequality (recall $q>2$) that $(\widehat{X}(g)_i^n)^{q/2} \leq \frac{K}{k_n^{1+q/2}} \sum_{j=i}^{i+k_n-1} |\Delta_j^n X|^q$, hence by (3.1) the sequence $k_n^{q/2} V(X,g,0,q/2)_t^n$ is tight. Second, (5.19) yields that the sequence $k_n^{q/2-2} V(\chi,g,0,q/2)_t^n$ is tight, and (3.4) follows because $V(Z,g,0,q/2)_t^n \leq K_q(V(X,g,0,q/2)_t^n + V(\chi,g,0,q/2)_t^n)$.

Now we turn to (b), and by localization we can assume (SN-2). The left side of (3.5) can be written as

$$\frac{1}{k_n}\, V(X,g,2,0)_t^n + \frac{1}{k_n} \sum_{l=1}^{4} U(l)_t^n,$$

where

$$U(l)_t^n \;=\; \begin{cases} 2\sum_{i=0}^{[t/\Delta_n]-k_n} \overline{X}(g)_i^n \;\overline{\chi}(g)_i^n & \text{if } \; l=1 \\ -\sum_{i=0}^{[t/\Delta_n]-k_n} \;\; \sum_{j=1}^{k_n} (g_j'^n)^2 \Delta_{i+j}^n X \;\Delta_{i+j}^n \chi & \text{if } \; l=2 \\ -\frac{1}{2}\, V(X,g,0,1)_t^n & \text{if } \; l=3 \\ V(\chi,g,2,0)_t^n - \frac{1}{2}\, V(\chi,g,0,1)_t^n & \text{if } \; l=4 \end{cases}$$

and by (3.2) it is enough to prove that for $l=1,2,3,4$,

$$\frac{1}{k_n}\, U(l)_t^n \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0. \tag{5.22}$$

First, (5.21) yields $|U(3)_t^n| \leq \frac{K}{k_n} \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^2$, so (5.22) for $l=3$ follows from (3.1). Next, (2.5) implies $\mathbb{E}(U(l)_t^n \mid \mathcal{F}^{(0)}) = 0$ for $l=1,2$, hence (5.22) for $l=1,2$ will be implied by

$$\mathbb{E}\Big(\Big(\frac{1}{k_n}\; U(l)_t^n\Big)^2 \mid \mathcal{F}^{(0)}\Big) \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0. \tag{5.23}$$

By (2.5) and (2.11) and (5.2), the variables $|\mathbb{E}(\overline{\chi}(g)_i^n \overline{\chi}(g)_j^n \mid \mathcal{F}^{(0)})|$ vanish if $j \geq k_n$ and are smaller than K/k_n otherwise, whereas the variable $|\mathbb{E}(\Delta_i^n \chi \;\Delta_{i+j}^n \chi \mid \mathcal{F}^{(0)})|$ are bounded, and vanish if $j \geq 2$. Then we get

$$\mathbb{E}((U(1)_t^n)^2 \mid \mathcal{F}^{(0)}) \;\leq\; \frac{K}{k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} \;\; \sum_{j=1}^{k_n} \overline{X}(g)_i^n \overline{X}(g)_{i+j}^n \;\leq\; K V(X,g,2,0)_t^n,$$

$$\begin{aligned} \mathbb{E}((U(2)_t^n)^2 \mid \mathcal{F}^{(0)}) \;&\leq\; \frac{K}{k_n^4} \sum_{i,i''=0}^{[t/\Delta_n]-k_n} \;\; \sum_{j,j'=0}^{k_n-1} |\Delta_{i+j}^n X \Delta_{i'+j'}^n X| 1_{\{|i'+j'-i-j| \leq 2\}} \\ &\leq\; \frac{K}{k_n^2} \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^2, \end{aligned}$$

and (5.23) follows from (3.2) when $l = 1$ and from (3.1) when $l = 2$.

Finally, an easy calculation shows that $U(4)_t^n = U(5)_t^n + U(6)_t^n$, where

$$U(5)_t^n \;=\; \sum_{i=0}^{[t/\Delta_n]} \chi_i^n \sum_{j=1}^{k_n} \alpha_{ij}^n \chi_{i+j}^n, \quad U(6)_t^n = \sum_{i=0}^{k_n} \Big(\alpha_i'^n (\chi_i^n)^2 + \alpha_i''^n (\chi_{i+[t/\Delta_n]-k_n}^n)^2 \Big)$$

for some coefficients $\alpha_{ij}^n, \alpha_i'^n, \alpha_i''^n$, all smaller than K/k_n. Then obviously $\mathbb{E}(|U(6)_t^n|) \leq K$ and $E(U(5)_t^n) = 0$ and, since $\mathbb{E}(\chi_i^n \chi_{i+j}^n \chi_{i'}^n \chi_{i'+j'}^n)$ vanishes unless $i = i'$ and $j = j'$ when $j, j' \geq 1$, we also have $\mathbb{E}((U(5)_t^n)^2) \leq Kt/k_n\Delta_n \leq Ktk_n$. Then (5.22) and (5.23) hold for $l = 6$ and $l = 5$ respectively, and thus (5.22) finally holds for $l = 4$. $\square$

5.3 A key lemma.

In this section we prove a key result, useful for deriving the other LNNs, when the process X is continuous, and for all CLTs. Before that, we prove Lemma 3.5.

Proof of Lemma 3.5. By virtue of (3.14) we have

$$\mu_p(g; \eta, \zeta) \;=\; \sum_{v=0}^{p/2} m_{2v} (\eta^2 \overline{g}(2))^v \, (\zeta^2 \overline{g}'(2))^{p/2-v} \sum_{r=0}^{p/2-v} C_{p-2r}^{2v} \, \rho_{p,r} \, 2^r \, m_{p-2r-2v}.$$

By (3.7) the last sum above vanishes if $v < p/2$ and equals 1 when $v = p/2$, hence (3.16). Next, we put $a_i = \mu_p(g_i; \eta, \zeta)$ and $U_t^i = \eta L(g_i)_t + \zeta L'(g_i)_t$ and, for $T \geq 2$,

$$V_T^i \;=\; \sum_{r=0}^{p/2} \rho_{p,r} (2\zeta^2 \overline{g}_i'(2))^r \int_0^T |U_t^i|^{p-2r} dt.$$

The process $(L(g_i), L'(g_i))$ is stationary, hence $\mathbb{E}'(V_T^i) = Ta_i$ for some constant a_i. Moreover if

$$f_{ij}(s,t) \;=\; \sum_{r,r'=0}^{p/2} \rho_{p,r} \, \rho_{p,r'} \, (2\zeta^2 \overline{g_i}'(2))^r (2\zeta^2 \overline{g_j}'(2))^{r'} \, \mathbb{E}'(|U_s^i|^{p-2r} |U_t^j|^{p-2r'}) - a_i a_j,$$

then f_{ij} satisfies $f_{ij}(s,t) = f_{ij}(s+u, t+u)$ and $f_{ij}(s,t) = 0$ if $|s-t| > 1$. Thus if $T > 2$,

$$\begin{aligned} \mathrm{Cov}(V_T^i, V_T^j) \;&=\; \int_{[0,T]^2} f_{ij}(s,t) ds dt \\ &= \int_0^1 ds \int_0^{s+1} f_{ij}(s,t) dt + \int_{T-1}^T ds \int_{s-1}^T f_{ij}(s,t) dt + \int_1^{T-1} ds \int_{s-1}^{s+1} f_{ij}(s,t) dt \end{aligned}$$

Therefore $\frac{1}{T}$ $\mathrm{Cov}(V_T^i, V_T^j)$ converges to $\int_0^2 f_{ij}(1,u) du$ as $T \to \infty$, and this limit equals $\overline{\mu}_{2p}(g_i, g_j; \eta, \zeta)$. Since the limit of a sequence of covariance matrices is symmetric nonnegative, we have the result. $\square$

Now, we fix a sequence i_n of integers, and we associate the following processes, with g an arbitrary function satisfying (2.7):

$$\overline{L}(g)^n_t = \sqrt{k_n}\ \overline{W}(g)^n_{i_n+[k_nt]}, \quad \overline{L}'(g)^n_t = \sqrt{k_n}\ \overline{\chi}(g)^n_{i_n+[k_nt]}, \quad \widehat{L}'(g)^n_t = k_n\ \widehat{\chi}(g)^n_{i_n+[k_nt]}. \tag{5.24}$$

We do not mention the sequence i_n in this notation, but those processes obviously depend on it.

Below, we fix a family $(g_l)_{1\leq l\leq d}$ of weight functions satisfying (2.7). We denote by $\overline{L}^n_t$ and $\overline{L}'^n_t$ and $\widehat{L}'^n_t$ the d-dimensional processes with respective components $\overline{L}(g_l)^n_t$ and $\overline{L}'(g_l)^n_t$ and $\widehat{L}'(g_l)^n_t$. These processes can be considered as variables with values in the Skorokhod space $\mathbb{D}^d$ of all càdlàg functions from $\mathbb{R}_+$ into $\mathbb{R}^d$. The processes L_t and L'_t with components $L(g_l)_t$ and $L'(g_l)_t$, defined by (3.12) *with the same Wiener processes* W^1 *and* W^2 for all components, are also $\mathbb{D}^d$-valued variables, and the probability on $\mathbb{D}^{2d} = \mathbb{D}^d \times \mathbb{D}^d$ which is the law of the pair (L, L') is denoted by $R = R_{(g_v)} = R(dx, dy)$.

We also have a sequence (f_n) of functions on $\mathbb{D}^{3d}$, which all depend on $w \in \mathbb{D}^{3d}$ only through its restriction to $[0, m+1]$ for some $m \geq 0$, and which satisfy the following property for some $q' \geq 2$ (below, $x, y, z \in \mathbb{D}^d$, so $v = (x, y) \in \mathbb{D}^{2d}$ and $(x, y, z) = (v, z) \in \mathbb{D}^{3d}$, and the same for x', y', z', and v'; moreover for any multidimensional function u on $\mathbb{R}_+$ we put $u^\star_m = \sup_{s\in[0,m+1]} \|u(s)\|$):

$$\left.\begin{array}{rcl} |f_n(v,z)| & \leq & K\ (1 + (v^\star_m)^{q'} + (z^\star_m)^{q'/2}) \\ |f_n(v,z) - f_n(v',z')| & \leq & K((v-v')^\star_m + (z-z')^*_m)\ (1 + (v^\star_m)^{q'-1} + (v'^\star_m)^{q'-1} \\ & & \qquad +(z^\star_m)^{q'/2-1} + (z'^\star_m)^{q'/2-1}). \end{array}\right\} \tag{5.25}$$

We can now state the main result of this subsection:

Lemma 5.1 *Assume (SN-q) for some $q > 4$ and that the process σ is bounded. Let Γ be the set of all times $s \geq 0$ such that both σ and α are almost surely continuous at time s. Take any sequence (i_n) of integers such that $s_n = i_n\Delta_n$ converges to some $s \in \Gamma$. If the sequence (f_n) satisfies (5.25) for some $q' < q$ and converges pointwise to a limit f, we have the almost sure convergence:*

$$\mathbb{E}\Big(f_n(\sigma_{s_n}\overline{L}^n, \overline{L}'^n, \widehat{L}'^n) \mid \mathcal{F}_{s_n}\Big) \ \to\ \int f(\theta\sigma_s x, \alpha_s y, 2(\alpha_s)^2 z_0)\ R(dx, dy), \tag{5.26}$$

where z_0 is the constant function with components $(\overline{g}'_l(2))_{1\leq l\leq d}$.

Proof. 1) We first prove an auxiliary result. Let $\Omega^{(0)}_s$ be the set of all $\omega^{(0)}$ such that both $\sigma(\omega^{(0)})$ and $\alpha(\omega^{(0)})$ are continuous at time s. We have $\mathbb{P}^{(0)}(\Omega^{(0)}_s) = 1$ because $s \in \Gamma$, and we fix $\omega^{(0)} \in \Omega^{(0)}_s$. We consider the probability space $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{Q})$, where $\mathbb{Q} = \mathbb{Q}(\omega^{(0)}, .)$, and our aim is to show that under $\mathbb{Q}$,

$$\overline{L}'^n \ \stackrel{\mathcal{L}}{\longrightarrow}\ \alpha_s(\omega^{(0)})L' \tag{5.27}$$

(functional convergence in law in $\mathbb{D}^d$), with $L' = (L'_t)$ the process introduced after (5.24).

We first prove the finite-dimensional convergence. Let $0 < t_1 < \cdots < t_r$. By (5.24) and (2.11) the rd-dimensional variable $Z_n = (\overline{L}'^{n,l}_{t_i} : 1 \le l \le d, 1 \le i \le r)$ is

$$\left.\begin{array}{lcl} Z_n & = & \sum_{j=1}^\infty z_j^n, \quad \text{where } z_j^n \;=\; \zeta_j^n\, a_j^n, \quad \zeta_j^n = \frac{1}{\sqrt{k_n}}\,\chi^n_{i_n+j-1} \text{ and} \\ a_j^{n,l,i} & = & \begin{cases} -k_n (g_l)'^n_{j-[k_nt_i]} & \text{if } 1+[k_nt_i] \le j \le k_n + [k_nt_i] \\ 0 & \text{otherwise.} \end{cases} \end{array}\right\} \tag{5.28}$$

Under $\mathbb{Q}$ the variables ζ_j^n are independent and centered with $\mathbb{E}_Q(|\zeta_j^n|^4) \le K k_n^{-2}$ by (SN-q), recall $q > 4$. The numbers $a_j^{n,l,i}$ being uniformly bounded and equal to 0 when $j > k_n + [k_n tr]$, we deduce that under $\mathbb{Q}$ again the variables z_j^n are independent, with

$$\mathbb{E}_Q(z_j^n) \;=\; 0, \quad \mathbb{E}_Q(\|z_j^n\|^4) \le K k_n^{-2}, \quad \sum_{j=1}^\infty \mathbb{E}_Q(\|z_j^n\|^4) \;\to\; 0. \tag{5.29}$$

Next,

$$\sum_{j=1}^\infty \mathbb{E}_Q(z_j^{n,l,i} z_j^{n,l',i'}) \;=\; \frac{1}{k_n} \sum_{j=1}^\infty \alpha_{(i_n+j-1)\Delta_n}(\omega^{(0)})^2\, a_j^{n,l,i}\, a_j^{n,l',i'}.$$

On the one hand $\alpha_{(i_n+j-1)\Delta_n}(\omega^{(0)})^2$ converges uniformly in $j \le k_n + [t_r k_n]$ to $\alpha_s(\omega^{(0)})^2$ because $s \mapsto \alpha_s(\omega^{(0)})$ is continuous at s. On the other hand (recall $g_l = 0$ outside $[0,1]$),

$$\frac{1}{k_n} \sum_{j=1}^\infty a_j^{n,l,i}\, a_j^{n,l',i'} \;=\; k_n \sum_{j=1}^\infty \int_{(j-1)/k_n}^{j/k_n} g_l'(u - \frac{[k_nt_i]}{k_n})du \int_{(j-1)/k_n}^{j/k_n} g_{l'}'(u - \frac{[k_nt_{i'}]}{k_n})du$$

which clearly converges to $c^{l,i,l',i'} = \int g_l'(v-t_i) g_{l'}'(v-t_{i'})dv$ by the mean value theorem, the piecewise continuity of each g_l', and Riemann approximation. Hence

$$\sum_{j=1}^\infty \mathbb{E}_Q(z_j^{n,l,i} z_j^{n,l',i'}) \;\to\; c^{l,i,l',i'}\, \alpha_s(\omega^{(0)})^2. \tag{5.30}$$

Then a standard limit theorem on rowwise independent triangular arrays of infinitesimal variables yield that Z_n converges in law under $\mathbb{Q}$ to a centered Gaussian variable with covariance matrix $(c^{l,i,l',i'})$, see e.g. Theorem VII-2-36 of [9]. Now, in view of (3.12), this matrix is the covariance of the centered Gaussian vector $(L'^{,l}_{t_i} : 1 \le l \le d, 1 \le i \le q)$, and the finite-dimensional convergence in (5.27) is proved.

To obtain the functional convergence in (5.27) it remains to prove that for each component the processes $\overline{L}'(g_l)^n$ are C-tight. For this we use a criterion given in [7] for example. Namely, since $q > 2$, the tightness of the sequence $\overline{L}'(g_l)^n$ is implied by

$$0 < v \le 1 \quad \Rightarrow \quad \mathbb{E}_{\mathbb{Q}}(|\overline{L}'(g_l)^n_{t+v} - \overline{L}'(g_l)^n_t|^q) \;\le\; K v^{q/2}. \tag{5.31}$$

A simple computation shows $\overline{L}'(g_l)^n_{t+v} - \overline{L}'(g_l)^n_t = \sum_j \delta_j^n \chi_j^n$ for suitable coefficients δ_j^n, such that at most $2[k_n v]$ are smaller that $K_1/\sqrt{k_n}$, and at most k_n of them are smaller than $K_2 v/\sqrt{k_n}$, and all others vanish. Then Burkholder-Davis-Gundy inequality yields

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(|\overline{L}'(g_l)^n_{t+v} - \overline{L}'(g_l)^n_t|^q) &\leq K\mathbb{E}_{\mathbb{Q}}\Big(\Big(\sum_j (\delta^n_j \chi^n_j)^2\Big)^{q/2}\Big) \\ &\leq K\beta(q)(\omega^{(0)})\Big(K_1^q (2v)^{q/2} + K_2^q v^q\Big),\end{aligned}$$

and (5.31) follows. Note also that the same argument implies

$$\mathbb{E}_{\mathbb{Q}}\Big(\sup_{v\leq t}(|\overline{L}'(g_l)^n_v|^q)\Big) \leq K_t. \tag{5.32}$$

2) In exactly the same setting than in the previous step, we want to prove here that

$$\left.\begin{array}{l} \widehat{L}'(g_l)^n_t \stackrel{\text{u.c.p.}}{\longrightarrow} 2(\alpha_s(\omega^{(0)})^2\,\overline{g}'_l(2) \\ \mathbb{E}_{\mathbb{Q}}\Big(\sup_{v\leq t} |\widehat{L}'(g_l)^n_v|^{q/2}\Big) \leq K_t \end{array}\right\} \tag{5.33}$$

(under $\mathbb{Q}$ again). Under $\mathbb{Q}$ the variable $\zeta^n_{t,j} = k_n(g'_l(j/k_n)\,\Delta^n_{i_n+[k_nt]+j}\chi)^2$ satisfies

$$\begin{aligned}&a^n_{t,j} := \mathbb{E}_{\mathbb{Q}}(\zeta^n_{t,j}) = k_n(g'_l(j/k_n))^2\Big((\alpha(\omega^{(0)})_{(i_n+[k_nt]+j)\Delta_n})^2 + (\alpha(\omega^{(0)})_{(i_n+[k_nt]+j-1)\Delta_n})^2\Big) \\ &\mathbb{E}_{\mathbb{Q}}(|\zeta^n_{t,j}|^{q/2}) \leq K/k_n^{q/2}.\end{aligned}$$

In view of the continuity of $\alpha(\omega^{(0)})$ at time s and of (2.10), and since $\widehat{L}'(g_l)^n_t = \sum_{j=1}^{k_n} \zeta^n_{t,j}$, we see that $B^n_t = \mathbb{E}_{\mathbb{Q}}(\widehat{L}'(g_l)^n_t) = \sum_{j=1}^{k_n} a^n_{t,j}$ converges locally uniformly to the "constant" $2(\alpha_s(\omega^{(0)})^2\,\overline{g}'_l(2)$, and also $B^n_t \leq K$. Hence it is enough to prove that $V^n_t = \widehat{L}'(g_l)^n_t - B^n_t \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ and that the second part of (5.33) holds when $\widehat{L}'(g_l)^n_t$ is substituted with V^n_t.

Now, V^n_t is the sum of the k_n centered variables $\zeta^n_{t,j} - a^n_{t,j}$, with $(q/2)$th absolute moment smaller than $K/k_n^{q/2}$, and $\zeta^n_{t,j}$ is independent of $(\zeta^n_{t,l} : |l-j| \geq 2)$. Then obviously $\mathbb{E}_Q((V^n_t)^2) \leq K/k_n \to 0$. Moreover if $v \in (0,1]$, $\widehat{L}'(g_l)^n_{t+v} - \widehat{L}'(g_l)^n_t = \sum_i \delta^n_i(\Delta^n_i\chi)^2$ for suitable coefficients δ^n_j, such that at most $2[k_nv]$ are smaller that K_1/k_n, and at most k_n of them are smaller than K_2v/k_n, and all others vanish. Then by Burkholder-Davis-Gundy inequality (applied separately for the sum of even indices and the sum of odd indices, to ensure the independence of the summands), we have

$$\mathbb{E}_{\mathbb{Q}}(|V^n_{t+v} - V^n_t|^{q/2}) \leq K\mathbb{E}_{\mathbb{Q}}\Big(\Big(\sum_j (\delta^n_j \chi^n_j)^2\Big)^{q/4}\Big) \leq Kv^{q/4}.$$

The second part of (5.33) for V^n_t follows and, together with the property $q > 4$ and the fact that $V^n_t \stackrel{\mathbb{P}}{\longrightarrow} 0$ for all t, it also implies $V^n_t \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Therefore we have (5.33).

3) Now we draw some consequences of the previous facts. We set for $y, z \in \mathbb{D}^d$, and with z_0 the constant function with components $\overline{g}'_l(2)$:

$$f^n_{\omega^{(0)}}(y,z) = f_n(\sigma_{s_n}(\omega^{(0)})L^n(\omega^{(0)}), y, z),$$

$$A_j^n(\omega^{(0)}) = \left\{ \begin{array}{ll} \int \mathbb{Q}(\omega^{(0)}, d\omega^{(1)})\ f^n_{\omega^{(0)}}(\overline{L}'^n(\omega^{(1)}), \widehat{L}'^n(\omega^{(1)})), & j=1 \\ \int f^n_{\omega^{(0)}}(\alpha_{s_n}(\omega^{(0)})y, 2\alpha_s^2\, z_0)\ R(dx,dy), & j=2. \end{array} \right.$$

The $\mathcal{F}^{(0)}$-measurable variables

$$\Phi_n \ = \ 1 + \sup_{v\in[0,(m+1)u_n]} \sqrt{k_n}\ |W_{s_n+v} - W_{s_n}|$$

satisfy $\mathbb{E}(\Phi_n^u) \leq K_u$ for any $u > 0$, by scaling of the Brownian motion W, whereas $|\overline{L}(g_l)^n_t| \leq K\Phi_n$ if $t \leq m$. Then we deduce from (5.25) and from the boundedness of σ and α that if y, y', z, z' are in $\mathbb{D}^d$ and $u = (y,z)$ and $u' = (y', z')$:

$$\left. \begin{array}{ll} |f^n_{\omega^{(0)}}(u)| \ \leq \ K\Phi_n(\omega^{(0)})^{q'}\ (1 + (y^\star_m)^{q'} + (z^\star_m)^{q'/2}) \\ |f^n_{\omega^{(0)}}(u) - f^n_{\omega^{(0)}}(u')| \ \leq \ K\Phi_n(\omega^{(0)})^{q'}\ (u-u')^\star_m\ (1 + (y^\star_m)^{q'-1} + (y'^\star_m)^{q'-1} \\ \qquad\qquad + (z^\star_m)^{q'/2-1} + (z'^\star_m)^{q'/2-1}). \end{array} \right\}$$

Moreover $\alpha_{s_n}(\omega^{(0)}) \to \alpha_s(\omega^{(0)})$, so by the Skorokhod representation theorem according to which, in case of convergence in law, one can replace the original variables by variables having the same laws and converging pointwise, one deduces from (5.27) and (5.32) and (5.33) (these imply that the variables $f^n_{\omega^{(0)}}(\overline{L}'^n, \widehat{L}'^n)$ are uniformly integrable, since $q' < q$), that

$$\left. \begin{array}{l} \omega^{(0)} \in \Omega_s^{(0)} \quad \Rightarrow \quad A_1^n(\omega^{(0)}) - A_2^n(\omega^{(0)}) \ \to \ 0, \\ \mathbb{E}\Big(|A_j^n|^{q/q'}\Big) \ \leq \ K. \end{array} \right\} \tag{5.34}$$

Next, we make the following observation: due to the $\mathcal{F}^{(0)}$-conditional independence of the χ_t's, a version of the conditional expectation in (5.26) is $\mathbb{E}(A_1^n \mid \mathcal{F}_{s_n})$. Therefore in view of (5.34) (which ensures the uniform integrability and the a.s. convergence to 0 of the sequence $A_1^n - A_2^n$), (5.26) is implied by

$$\mathbb{E}(A_2^n \mid \mathcal{F}_{s_n}) \ \to \ F(\sigma_s, \alpha_s) \quad \text{a.s.}, \tag{5.35}$$

where

$$F(\eta, \zeta) \ = \ \int f(\theta\eta x, \zeta y, 2(\zeta)^2 z_0)\ R(dx,dy).$$

4) For proving (5.35) we start again with an auxiliary result, namely

$$\overline{L}^n \ \stackrel{\mathcal{L}}{\longrightarrow} \ \theta L. \tag{5.36}$$

For this, we see that $Z_n = (\overline{L}^{n,l}_{t_i} : 1 \leq l \leq d, 1 \leq i \leq r)$ is given by (5.28), except that

$$\zeta_j^n \ = \ \sqrt{k_n}\ \Delta^n_{i_n+j} W, \quad a_j^{n,l,i} \ = \ \left\{ \begin{array}{ll} (g_l)^n_{j-[k_n t_i]} & \text{if } \ 1 + [k_n t_i] \leq j \leq k_n + [k_n t_i] \\ 0 & \text{otherwise.} \end{array} \right.$$

Then the proof of (5.36), both for the finite-dimensional convergence and the C-tightness, is exactly the same as for (5.27) (note that the right side of (5.30) is now $\theta^2 \int g_l(v -$

$t_i)g_{l'}(v-t_{i'})dv$, which is the covariance matrix of $(\theta L^l_{t_i} : 1 \le l \le d, 1 \le i \le r))$. Further, an elementary calculation yields

$$\mathbb{E}\Big(\sup_{v\le t}(|\overline{L}(g_l)^n_v|^q\Big) \le K_t. \tag{5.37}$$

5) Now we introduce some functions on $\mathbb{R}^2$:

$$F_n(\eta,\zeta) = \int \mathbb{E}\Big(f_n(\eta\overline{L}^n, \zeta y, 2(\zeta)^2 z_0)\Big)\ R(dx,dy).$$
$$F'_n(\eta,\zeta) = \int \mathbb{E}\Big(f_n(\theta\eta L, \zeta y, , 2(\zeta)^2 z_0)\Big)\ R(dx,dy).$$

Under R the canonical process is locally in time bounded in each $\mathbb{L}^r$. Then in view of (5.25) we deduce from (5.36) and (5.37), and exactly as for (5.34), that $F_n - F'_n \to 0$ locally uniformly in $\mathbb{R}^2$. We also deduce from (5.25) that $F'_n(\eta_n,\zeta_n) - F'_n(\eta,\zeta) \to 0$ whenever $(\eta_n,\zeta_n) \to (\eta,\zeta)$, and also that $F'_n \to F$ pointwise because $f_n \to f$ pointwise, hence we have $F_n(\eta_n,\zeta_n) \to F(\eta,\zeta)$.

At this point it remains to observe that, because $(W_{s_n+t} - W_{s_n})_{t\ge 0}$ is independent of $\mathcal{F}_{s_n}$, we have $\mathbb{E}(A^n_2 \mid \mathcal{F}_{s_n}) = F_n(\sigma_{s_n}, \alpha_{s_n})$. Since $(\sigma_{s_n}, \alpha_{s_n}) \to (\sigma_s, \alpha_s)$ a.s., we readily deduce (5.26), and we are done. □

Remark 5.2 In the previous lemma, suppose that all f_n (hence f as well) only depend on (x,y) and not on z; that is, the processes $\widehat{L}'^n$ do not enter the picture. Then it is easily seen from the previous proof that we do not need $q > 4$, but only $q > 2$. □

5.4 Asymptotically negligible arrays.

An array (δ^n_i) of *nonnegative* variables is called AN (for "asymptotically negligible") if

$$\sqrt{\Delta_n}\ \sup_{0\le j\le k_n}\ \mathbb{E}\Big(\sum_{i=0}^{[t/u_n]} \delta^n_{ik_n+j}\Big)\ \to\ 0, \qquad |\delta^n_i|\ \le\ K \tag{5.38}$$

for all $t > 0$. With all processes γ and reals $p > 0$ and integers m we associate the variables

$$\left.\begin{array}{lll} \Gamma(\gamma,m)^n_i &=& \sup_{t\in[i\Delta_n, i\Delta_n+(m+1)u_n]} |\gamma_t - \gamma_{i\Delta_n}|, \\ \Gamma'(\gamma,m)^n_i &=& \mathbb{E}(\Gamma(\gamma,m)^n_i \mid \mathcal{F}^n_i). \end{array}\right\} \tag{5.39}$$

Lemma 5.3 *a) If (δ^n_i) is an AN array, we have*

$$\Delta_n \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} \delta^n_i\Big)\ \to\ 0 \tag{5.40}$$

for all $t > 0$, and the array $((\delta^n_i)^q)$ is also AN for each $q > 0$.

b) If γ is a càdlàg bounded process, then for all $m \ge 1$ the two arrays $(\Gamma(\gamma,m)^n_i)$ and $(\Gamma'(\gamma,m)^n_i)$ are AN.

Proof. a) The left side of (5.40) is smaller than a constant times the left side of (5.38), hence the first claim. The second claim follows from Hölder inequality if $q<1$, and from $\sum_{i\in I}(\delta_i^n)^q\le K\sum_{i\in I}\delta_i^n$ if $q>1$ (recall that $|\delta_i^n|\le K$).

b) Let $\delta_i^n=\Gamma(\gamma,m)_i^n$. If $\varepsilon>0$, denote by $N(\varepsilon)_t$ the number of jumps of γ with size bigger than ε on the interval $[0,t]$, and by $v(\varepsilon,t,\eta)$ the supremum of $|\gamma_s-\gamma_r|$ over all pairs (r,s) with $s\le r\le s+\eta$ and $s\le t$ and such that $N(\varepsilon)_s-N(\varepsilon)_r=0$. Since γ is bounded,

$$u_n\sup_{0\le j\le k_n}\mathbb{E}\Big(\sum_{i=0}^{[t/u_n]}\delta^n_{ik_n+j}\Big)\ \le\ \mathbb{E}\Big(t\,v(\varepsilon,t+1,(m+1)u_n)+(Kt)\wedge(K\,u_n\,N(\varepsilon)_{t+1})\Big)$$

as soon as $(m+2)u_n\le 1$. Since $\limsup_{n\to\infty}\ v(\varepsilon,t+1,(m+1)u_n)\le\varepsilon$, Fatou's lemma implies that the $\limsup$ of the left side above is smaller than $Kt\varepsilon$, so we have (5.38) because ε is arbitrarily small. Since $\mathbb{E}(\Gamma'(\gamma,m)_i^n)=\mathbb{E}(\Gamma(\gamma,m)_i^n)$, the second claim follows. $\square$

5.5 Some estimates.

In this subsection we provide a (somewhat tedious) list of estimates, under the following assumption for some $q>2$:

- we have (2.14) and (SN-q) and b and σ are bounded, and σ and α are càdlàg. (5.41)

We first introduce some notation, where i and j are integers, Y is an arbitrary process, and $\rho_{p,l}$ is given by (3.7), and $i+j\ge1$ in the first line below, and p an even integer in (5.43):

$$\left.\begin{array}{l}\kappa^n_{i,j}\ =\ \sigma^n_i\Delta^n_{i+j}W+\Delta^n_{i+j}\chi,\quad \lambda^n_{i,j}\ =\ \Delta^n_{i+j}Z-\kappa^n_{i,j}\ =\ \Delta^n_{i+j}X-\sigma^n_i\Delta^n_{i+j}W\\ \overline{\kappa}(g)^n_{i,j}=\sum_{l=1}^{k_n-1}g^n_l\kappa^n_{i,j+l},\quad \overline{\lambda}(g)^n_{i,j}=\sum_{l=1}^{k_n-1}g^n_l\lambda^n_{i,j+l},\quad \widehat{\lambda}(g)^n_{i,j}=\sum_{l=1}^{k_n}(g'^n_l\lambda^n_{i,j+l})^2\end{array}\right\}\quad(5.42)$$

$$\left.\begin{array}{rcl}\phi(Y,g,p)^n_i&=&\sum_{l=0}^{p/2}\rho_{p,l}\ (\overline{Y}(g)^n_i)^{p-2l}\ (\widehat{Y}(g)^n_i)^l,\\ \phi(g,p)^n_{i,j}&=&\sum_{l=0}^{p/2}\rho_{p,l}\ (\overline{\kappa}(g)^n_{i,j})^{p-2l}\ (\widehat{\chi}(g)^n_{i,j})^l.\end{array}\right\}\quad(5.43)$$

Note that, by (3.9) ,

$$\overline{V}(Y,g,p)^n_t\ =\ \sum_{i=0}^{[t/\Delta_n]-k_n}\phi(Y,g,p)^n_i.\quad(5.44)$$

In the forthcoming inequalities, we have $0\le j\le mk_n$, where m is a fixed integer. First, if we use (5.3) and the boundedness of g, and also (5.2), we obtain for $u>0$:

$$\left.\begin{array}{l}\mathbb{E}(|\overline{X}(g)^n_i|^u+|\overline{W}(g)^n_i|^u\mid\mathcal{F}^n_i)\le K_u\Delta_n^{u/4},\\ u\le q\ \Rightarrow\ \mathbb{E}(|\overline{Z}(g)^n_i|^u+|\overline{\kappa}(g)^n_i|^u\mid\mathcal{F}^n_i)\le K_u\Delta_n^{u/4}\end{array}\right\}\quad(5.45)$$

Next,

$$\left.\begin{array}{rcl}\lambda^n_{i,j}&=&\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}\Big(b_sds+(\sigma_s-\sigma^n_i)dW_s\Big)\\ \overline{\lambda}(g)^n_{i,j}&=&\int_{(i+j)\Delta_n}^{(i+j+k_n)\Delta_n}g_n(s-(i+j)\Delta_n)\Big(b_sds+(\sigma_s-\sigma^n_i)dW_s\Big).\end{array}\right\}\quad(5.46)$$

Hence we obtain for $u \geq 1$, and recalling that $\Gamma(\sigma,m)_i^n \leq K$:

$$\left.\begin{array}{l} \mathbb{E}(|\lambda_{i,j}^n|^u \mid \mathcal{F}_i^n\Big) \leq K_u \Delta_n^{u/2}\Big(\Delta_n^{u/2} + \Gamma'(\sigma,m)_i^n)\Big), \\ \mathbb{E}(|\overline{\lambda}(g)_{i,j}^n|^u \mid \mathcal{F}_i^n\Big) \leq K_u \Delta_n^{u/4}\Big(\Delta_n^{u/4} + \Gamma'(\sigma,m)_i^n)\Big). \end{array}\right\} \quad (5.47)$$

If u is an *odd integer*, (5.45), (5.47) and an expansion of $(\sigma_i^n \overline{W}(g)_i^n + \overline{\lambda}(g)_i^n)^u$ yield

$$\mathbb{E}((\overline{W}(g)_i^n)^u \mid \mathcal{F}_i^n) = 0, \quad \mathbb{E}((\overline{X}(g)_i^n)^u \mid \mathcal{F}_i^n\Big) \leq K_u \Delta_n^{u/4}\Big(\Delta_n^{1/4} + \sqrt{\Gamma'(\sigma,m)_i^n}\Big). \quad (5.48)$$

Next, using $|g_i'^n| \leq K/k_n$ and (5.3) and the first part of (5.47), plus Hölder inequality and the definition of $\widehat{Y}(g)_i^n$, plus the obvious fact that $\mathbb{E}(|\kappa_{i,j}^n|^u \mid \mathcal{F}_i^n) \leq K_u$ if $u \leq q$, and after some calculations, we get for $u \geq 1$:

$$\left.\begin{array}{l} \mathbb{E}(|\widehat{X}(g)_i^n|^u + |\widehat{W}(g)_i^n|^u \mid \mathcal{F}_i^n) \leq K_u \Delta_n^{3u/2}, \\ u \leq q/2 \;\Rightarrow\; \mathbb{E}(|\widehat{Z}(g)_{i+j}^n|^u + |\widehat{\chi}(g)_{i+j}^n|^u \mid \mathcal{F}_i^n) \leq K_u \Delta_n^{u/2} \\ u \leq q \;\Rightarrow\; \mathbb{E}(|\widehat{Z}(g)_{i+j}^n - \widehat{\chi}(g)_{i+j}^n|^u \mid \mathcal{F}_i^n) \leq K_u \Delta_n^u. \end{array}\right\} \quad (5.49)$$

Then, if we combine (5.45), (5.47) and (5.49), and use again Hölder inequality, we obtain for all reals $l,u \geq 1$ and $r \geq 0$:

$$\left.\begin{array}{l} (l+2r)u \leq q \;\Rightarrow\; \mathbb{E}\Big(\Big|(\overline{Z}(g)_{i+j}^n)^l(\widehat{Z}(g)_{i+j}^n)^r - (\overline{\kappa}(g)_{i,j}^n)^l(\widehat{\chi}(g)_{i,j}^n)^r\Big|^u \mid \mathcal{F}_i^n\Big) \\ \qquad\qquad \leq\; K_{u,l,r}\Delta_n^{ul/4+ur/2}\Big(\Delta_n^{u/4} + (\Gamma'(\sigma,m)_i^n)^{1-u(l+2r-1)/q}\Big), \\ 2ru \leq q \;\Rightarrow\; \mathbb{E}\Big(\Big|(\widehat{Z}(g)_{i+j}^n)^r - (\widehat{\chi}(g)_{i,j}^n)^r\Big|^u \mid \mathcal{F}_i^n\Big) \;\leq\; K_{u,r}\Delta_n^{ru/2+u/2}. \end{array}\right\} \quad (5.50)$$

Finally, by (5.43), this readily gives for $p \geq 2$ an even integer and $u \geq 1$ a real, such that $pu \leq q$:

$$\begin{array}{l} \mathbb{E}(|\phi(Z,g,p)_{i+j}^n|^u + |\phi(g,p)_{i,j}^n|^u \mid \mathcal{F}_i^n) \leq K_{u,p}\Delta_n^{pu/4} \\ \mathbb{E}(|\phi(Z,g,p)_{i+j}^n - \phi(g,p)_{i,j}^n|^u \mid \mathcal{F}_i^n) \leq K_{u,p}\Delta_n^{pu/4}\Big(\Delta_n^{u/4} + (\Gamma'(\sigma,m)_i^n)^{1-u(p-1)/q}\Big). \end{array} \quad (5.51)$$

5.6 Proof of Theorem 3.3.

By localization we can and will assume (5.41). We set

$$\mu_i^n = \Delta_n^{-p/4}\;|\overline{Z}(g)_i^n|^p, \quad \zeta_i^n = \Delta_n^{-p/4}|\overline{\kappa}(g)_{i,0}^n|^p, \quad \gamma_t = m_p\Big|\theta\overline{g}(2)\sigma_t^2 + \frac{\overline{g}'(2)}{\theta}\;\alpha_t^2\Big|^{p/2}.$$

We deduce from (5.50) with $r=0$ and Lemma 5.3 that $\Delta_n\sum_{i=0}^{[t/\Delta_n]-k_n}|\mu_i^n - \zeta_i^n| \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Then it remains to prove

$$\Delta_n\sum_{i=0}^{[t/\Delta_n]-k_n}\zeta_i^n \;\stackrel{\text{u.c.p.}}{\longrightarrow}\; \int_0^t \gamma_s\,ds. \quad (5.52)$$

Set $\zeta_i'^n = \mathbb{E}(\zeta_i^n \mid \mathcal{F}_i^n)$. By (5.45), $\mathbb{E}((\zeta_i^n)^2 \mid \mathcal{F}_i^n) \leq K$, and in particular $\zeta_i'^n \leq K$. Moreover ζ_i^n is $\mathcal{F}_{i+k_n}^n$-measurable, hence $\mathbb{E}((\zeta_i^n - \zeta_i'^n)(\zeta_j^n - \zeta_j'^n)) = 0$ if $|j-i| \geq k_n$, and

$$\mathbb{E}\Big(\Big|\Delta_n \sum_{i=0}^{[t/\Delta_n]-k_n} (\zeta_i^n - \zeta_i'^n)\Big|^2\Big) \;=\; \Delta_n^2 \sum_{i,j=1}^{[t/\Delta_n]-k_n} \mathbb{E}\Big((\zeta_i^n - \zeta_i'^n)(\zeta_j^n - \zeta_j'^n)\Big) \;\leq\; K\Delta_n k_n \;\to\; 0.$$

Thus it is enough to prove (5.52) with ζ_i^n substituted with $\zeta_i'^n$. Since $\gamma_t + \zeta_i'^n \leq K$,

$$\Big|\Delta_n \sum_{i=0}^{[t/\Delta_n]-k_n} \zeta_i'^n - \int_0^t \gamma_s\,ds\Big| \;\leq\; \int_{\Delta_n}^{([t/\Delta_n]-k_n)\Delta_n} |\gamma_s^n - \gamma_s| ds + K k_n \Delta_n,$$

where $\gamma_s^n = \zeta_i'^n$ when $(i-1)\Delta_n \leq s < i\Delta_n$. Therefore, since $|\gamma_s^n - \gamma_s| \leq K$, in order to obtain (5.52) it is enough to prove that for Lebesgue-almost all s we have $\gamma_s^n \to \gamma_s$ a.s. In particular it is enough to prove that, for all $s \in \Gamma$ (cf. Lemma 5.1), we have

$$\zeta_{[s/\Delta_n]+1}'^n \;\to\; \gamma_s \quad \text{a.s.} \tag{5.53}$$

With the notation of Lemma 5.1, we take $d = 1$ and the weight function $g_1 = g$, and the functions $f_n = f$ on $\mathbb{D}^3$ as $f(x,y,z) = |x(0) + y(0)|^p$, so (5.25) is satisfied with $q' = p < q$. Moreover we fix $s \in \Gamma$ and set $i_n = [s/\Delta_n] + 1$, so $s_n = i_n\Delta_n \to s$. The left side of (5.26) is $\Delta_n^{p/4} k_n^{p/2} \zeta_{i_n}'^n$, whereas its right side is $\mathbb{E}'\Big(|\theta\eta L(g)_0 + \eta' L'(g)_0|^p\Big)$ (recall (3.12)) evaluated at $\eta = \sigma_s$ and $\eta' = \alpha_s$. Since $L(g)_0$ and $L'(g)_0$ are independent centered normal with respective variances $\overline{g}(2)$ and $\overline{g}'(2)$, this right side is $m_p\Big(\theta^2\overline{g}(2)\sigma_s^2 + \overline{g}'(2)\alpha_s^2\Big)^{p/2} = \theta^{p/2}\gamma_s$. Since $\Delta_n^{p/4} k_n^{p/2} \to \theta^{p/2}$, we get (5.53). □

5.7 Proof of Theorem 3.4.

As said already, (a) is a particular case of (3.3) when $p \geq 4$, and of (3.5) when $p = 2$. For (b), we can again assume (5.41). We set

$$\mu_i^n = \Delta_n^{-p/4} \phi(Z, g, p)_i^n, \quad \zeta_i^n = \Delta_n^{-p/4} \phi(g, p)_{i,0}^n, \quad \gamma_t = m_p\,(\theta\,\overline{g}(2))^{p/2}\,|\sigma_t|^p,$$

and $\zeta_i'^n = \mathbb{E}(\zeta_i^n \mid \mathcal{F}_i^n)$. We deduce from (5.51) and Lemma 5.3 that $\Delta_n \sum_{i=0}^{[t/\Delta_n]-k_n} |\mu_i^n - \zeta_i^n| \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Then it is enough to prove (5.52).

By (5.51) we have $\mathbb{E}((\zeta_i^n)^2 \mid \mathcal{F}_i^n) \leq K$, hence $\zeta_i'^n \leq K$. Then, exactly as in the previous proof, it remains to show (5.53) when $s \in \Gamma$. For this, we use Lemma 5.1 with $d = 1$ and $g_1 = g$ and the functions $f_n = f$ given by

$$f(x,y,z) \;=\; \sum_{l=0}^{p/2} \rho_{p,l} |x(0) + y(0)|^{p-2l}\,|z(0)|^l.$$

The left side of (5.26) is again $\Delta_n^{p/4} k_n^{p/2} \zeta_{i_n}'^n$. Its right side is $\mu_p(g; \theta\sigma_s, \alpha_s)$, as given by (3.15), and by (3.16) this is also $\theta^{p/2}\gamma_s$. Then (5.9) holds. □

5.8 Proof of Theorem 3.6.

Once more, the proof is the basically same as in the previous subsection. We can assume (5.41). We have

$$\begin{aligned}\overline{\mu}_{2p}(g,h;\eta,\zeta) &= \sum_{r,r'=0}^{p/2} \rho_{p,r}\rho_{p,r'}(2\zeta^2\overline{g}'(2))^r(2\zeta^2\overline{h}'(2))^{r'}\Big(m_{p-2r,p-2r'}(g,h;\eta,\zeta)\\ &\qquad -2m_{p-2r}(g;\eta,\zeta)m_{p-2r'}(h;\eta,\zeta)\Big).\end{aligned} \tag{5.54}$$

Therefore is is enough to prove that for r,r' between 0 and $p/2$, and with the notation

$$\begin{aligned}\mu_i^n &= \Delta_n^{-p/2}(\widehat{Z}(g)_i^n)^r\,(\widehat{Z}(h)_i^n)^{r'}\Big(|\overline{Z}(g)_{i+k_n}^n|^{p-2r}\cdot\\ &\qquad \frac{1}{k_n}\sum_{j=1}^{2k_n}|\overline{Z}(h)_{i+j}^n|^{p-2r'} - 2|\overline{Z}(g)_i^n|^{p-2r}|\overline{Z}(h)_{i+k_n}^n|^{p-2r'}\Big),\\ \gamma_t &= \theta^{-p/2}(2\alpha_t^2\overline{g}'(2))^{r+r'}\Big(m_{p-2r,p-2r'}(g,h;\theta\sigma_t,\alpha_t)\\ &\qquad -2m_{p-2r}(g;\theta\sigma_t,\alpha_t)m_{p-2r'}(h;\theta\sigma_t,\alpha_t)\Big),\end{aligned}$$

we have

$$\Delta_n\sum_{i=0}^{[t/\Delta_n]-3k_n}\mu_i^n \stackrel{\text{u.c.p.}}{\longrightarrow} \int_0^t \gamma_s ds.$$

By (5.49) and (5.50) we have $\Delta_n\sum_{i=0}^{[t/\Delta_n]-3k_n}|\mu_i^n-\zeta_i^n| \stackrel{\text{u.c.p.}}{\longrightarrow} 0$, where

$$\begin{aligned}\zeta_i^n &= \Delta_n^{-p/2}(\widehat{\chi}(g)_i^n)^r\,(\widehat{\chi}(h)_i^n)^{r'}\,\Big(|\overline{\kappa}(g)_{i,k_n}^n|^{p-2r}\cdot\\ &\qquad \frac{1}{k_n}\sum_{j=1}^{2k_n}|\overline{\kappa}(h)_{i,j}^n|^{p-2r'} - 2|\overline{\kappa}(g)_{i,0}^n|^{p-2r}\,|\overline{\kappa}(h)_{i,k_n}^n|^{p-2r'}\Big),\end{aligned}$$

and thus it is enough to prove

$$\Delta_n\sum_{i=0}^{[t/\Delta_n]-3k_n}\zeta_i^n \stackrel{\text{u.c.p.}}{\longrightarrow} \int_0^t \gamma_s ds.$$

We set $\zeta_i'^n = \mathbb{E}(\zeta_i^n \mid \mathcal{F}_i^n)$, so as in the proof of Theorem 3.3 it is enough to prove (5.53) when $s\in\Gamma$. We apply Lemma 5.1 with $d=2$ and $g_1=g$ and $g_2=h$ and the functions f_n and f on $\mathbb{D}^6$ defined by

$$\begin{aligned}f_n((x,x'),(y,y'),(z,z')) &= z(0)^r\,z'(0)^{r'}\Big(|x(1)+y(1)|^{p-2r}\frac{1}{k_n}\sum_{j=1}^{2k_n}|x'(\frac{j}{k_n})+y'(\frac{j}{k_n})|^{p-2r'}\\ &\qquad -2|x(0)+y(0)|^{p-2r}|x'(1)+y'(1)|^{p-2r'}\Big),\\ f((x,x'),(y,y'),(z,z')) &= z(0)^r\,z'(0)^{r'}\Big(|x(1)+y'(1)|^{p-2r}\int_0^2|x'(t)+y'(t)|^{p-2r'}dt\\ &\qquad -2|x(0)+y(0)|^{p-2r}\,|x'(1)+y'(1)|^{p-2r'}\Big),\end{aligned}$$

and again $i_n = [s\Delta_n] + 1$. Then (5.25) is satisfied with $q' = 2p < q$, and $f_n \to f$ pointwise. The left side of (5.26) is $\Delta_n^{p/2} k_n^p \zeta'^n_{i_n}$, whereas its right side is $\theta^{p/2}\gamma_s$ (recall that $(L(g)_0, L'(g)_0)$ and $(L(h)_1, L'(h)_1)$ are independent). Since $\Delta_n^{p/2} k_n^p \to \theta^{p/2}$, we get (5.53) by the lemma, and the proof is finished. □

5.9 Auxiliary results on the noise process.

At this stage we start the proof of our CLTs, and this is done through a large number of steps. In the first step, we derive some results on the noise process χ. Recall that $\mathcal{G}_i^n$ denotes the σ-field generated by $\mathcal{F}^{(0)}$ and $\mathcal{F}_i^n$. We set

$$A(g)_i^n \;=\; \sum_{j=1}^{k_n} (g_j'^n)^2 (\alpha_{i+j-1}^n)^2. \tag{5.55}$$

For random variables U_γ and V_γ indexed by a parameter γ (for example $\gamma = (n,i)$ just below), with $V_\gamma > 0$, we write $U_\gamma = \mathrm{O}_u(V_\gamma)$ if the family U_γ/V_γ is bounded in probability.

Lemma 5.4 *Assume (SN-q) for some $q \geq 2$, and let v and r be integers such that $2 \leq v + 2r \leq q$. Let also $m \geq 0$ and j be arbitrary in $\{0,1,\cdots,mk_n\}$.*

a) When v is even we have

$$\mathbb{E}\Big((\overline{\chi}(g)_{i+j}^n)^v (\widehat{\chi}(g)_{i+j}^n)^r \mid \mathcal{G}_i^n\Big) \;=\; m_v\; 2^r\; (A(g)_{i+j}^n)^{r+v/2} + \mathrm{O}_u(\Delta_n^{r/2+v/4+1/2}) \tag{5.56}$$

$$= m_v\; 2^r \frac{\overline{g}'(2)^{r+v/2}}{k_n^{r+v/2}}\; (\alpha_i^n)^{2r+v} + \mathrm{O}_u\Big(\Delta_n^{r/2+v/4}\Big(\Delta_n^{1/2} + \Gamma(\alpha,m)_i^n\Big)\Big), \tag{5.57}$$

b) When v is odd we have

$$\mathbb{E}\Big((\overline{\chi}(g)_{i+j}^n)^v (\widehat{\chi}(g)_{i+j}^n)^r \mid \mathcal{G}_i^n\Big) \;=\; \mathrm{O}_u(\Delta_n^{r/2+v/4+1/4}), \tag{5.58}$$

and also, for some suitable numbers $\gamma_{v,r}$, depending on g,

$$\mathbb{E}\Big((\overline{\chi}(g)_{i+j}^n)^v (\widehat{\chi}(g)_{i+j}^n)^r \mid \mathcal{G}_i^n\Big) \;=\; \frac{\gamma_{v,r}}{k_n^{r+v/2+1/2}}\; (\alpha_i^n)^{2r+v-3}\; \beta(3)_i^n$$
$$+\mathrm{O}_u\Big(\Delta_n^{r/2+v/4+1/4}\Big(\Delta_n^{1/4} + \Gamma(\alpha,m)_i^n + \Gamma(\beta(3),m)_i^n\Big)\Big). \tag{5.59}$$

Proof. (5.57) and (5.58) are simple consequences of (5.56) and (5.59) respectively, upon observing that $A(g)_{i+j}^n = \overline{g}'(2)(\alpha_i^n)^2/k_n + \mathrm{O}_u(\Delta_n^{1/2}(\Delta_n^{1/2} + \Gamma(\alpha,m)_i^n))$. As for (5.56) and (5.59), and up to taking a further conditional expectation, it is enough to prove them when $j = 0$, so in the rest of the proof we take $j = 0$, and thus $m = 0$ as well. The product $(\overline{\chi}(g)_i^n)^v (\widehat{\chi}(g)_i^n)^r$ is the sum of all the terms of the form

$$\left.\begin{array}{l} \Phi(J,n) = (-1)^v \prod_{l=1}^v g_{j_l}'^n \chi_{i+j_l-1}^n \prod_{l=1}^s (g_{j_l'}'^n \chi_{i+j_l'+\overline{j}_l'-1}^n)^2 \prod_{l=1}^{r-s} \Big(-2(g_{j_l''}'^n)^2 \chi_{i+j_l''}^n \chi_{i+j_l''-1}^n\Big), \\ \quad J = \{s, j_1,\cdots,j_v, j_1',\cdots,j_s', \overline{j}_1',\cdots,\overline{j}_s', j_1'',\cdots,j_{r-s}''\}, \\ \quad \text{where} \quad s \in \{0,\cdots,r\}, \quad j_l, j_l', j_l'' \in \{1,\cdots,k_n\}, \quad \overline{j}_l' \in \{0,1\}. \end{array}\right\} \tag{5.60}$$

We denote by $I(J)$ the family of all indices of the variables χ_j^n occurring in (5.60), the index j appearing l times if χ_j^n is taken at the power l, so that $I(J)$ contains $v+2r$ indices. We also denote by $D(u)^n$ the class of all J's such that among the $v+2r$ indices in $I(J)$, there are exactly u different indices, each one appearing at least twice. Note that $D(u)^n$ is the disjoint union over $s'=0,\cdots,r$ of the set $D(u,s')^n$ of all $J\in D(u)^n$ such that $s=s'$. Note also that $D(u)^n=\emptyset$ if $u>v/2+r$.

By (2.5) and the $\mathcal{F}^{(0)}$-conditional independence of the χ_t's, the conditional expectation $\mathbb{E}(\Phi(J,n)\mid\mathcal{G}_i^n)$ is always smaller than K/k_n^{v+2r}, and it vanishes if J is outside $\cup_{u\geq 1}D(u)^n$, that is

$$\begin{array}{rcl}\mathbb{E}\Big((\overline{\chi}(g)_i^n)^v(\widehat{\chi}(g)_i^n)^r\mid\mathcal{G}_i^n\Big) & = & \sum_{u=1}^{[v/2]+r}\overline{\Phi}_u^n,\qquad\text{where}\\ \overline{\Phi}_u^n & = & \sum_{s=0}^r\overline{\Phi}(u,s)^n,\qquad \overline{\Phi}(u,s)^n=\sum_{J\in D(u,s)^n}\mathbb{E}\Big(\Phi(J,n)\mid\mathcal{G}_i^n\Big).\end{array}$$

Now $\#D(u,s)^n\leq Kk_n^u$, so $|\overline{\Phi}(u,s)^n|\leq Kk_n^{u-v-2r}$, hence $\overline{\Phi}_u^n=\mathrm{O}_u(\Delta_n^{r/2+v/4+1/4})$ as soon as $u\leq r-1/2+v/2$. We deduce that for proving (5.56), so v is even, it is enough to show that $\overline{\Phi}_u^n$ equals the right side of (5.56) for $u=r+v/2$. In the same way, for proving (5.59), so v is odd, it is enough to show that $\overline{\Phi}_u^n$ equals the right side of (5.59) for $u=r+(v-1)/2$.

a) Suppose that v is even and $u=r+v/2$. The definition of $D(u)^n$ and the property $u=r+v/2$ yield that, if $J\in D(u)^n$, there is a nonnegative integer $w\leq\frac{v}{2}\wedge\frac{r-s}{2}$ such that $\Phi(J,n)$ is the product of $\frac{v+s+r-w}{2}$ terms, of three types, all for different indices for χ^n:

(1) $s-w+\frac{v}{2}$ terms of the form $(g_j'^n\chi_{i+j-1}^n)^2$ or $(g_j'^n\chi_{i+j}^n)^2$,

(2) w terms of the form $-2(g_j'^n)^3g_{j+1}'^n(\chi_{i+j-1}^n\chi_{i+j}^n)^2$,

(3) $\frac{r-s-w}{2}$ terms of the form $4(g_j'^n)^4(\chi_{i+j-1}^n\chi_{i+j}^n)^2$.

Hence $\#D(u,s)^n\leq Kk_n^{\frac{v+s+r}{2}}$, because the number of terms for a particular J is smaller than $\frac{v+s+r}{2}$ and the indices range from 1 to k_n. Moreover, since α is bounded and $|g_j'^n|\leq K/k_n$, we have $\mathbb{E}(|\Phi(J,n)||\mid\mathcal{G}_i^n)\leq K/k_n^{v+2r}$. We then deduce that

$$|\overline{\Phi}(u,s)^n|\;\leq\;Kk_n^{\frac{v+s+r}{2}-v-2r}\;\leq\;K\Delta_n^{r/2+v/4+\frac{r-s}{2}}. \tag{5.61}$$

In particular, $\overline{\Phi}(u,s)^n=\mathrm{O}(\Delta_n^{r/2+v/4+1/2})$ when $s<r$, and it thus remains to prove that $\overline{\Phi}(u,r)^n$ is equal to the right side of (5.56). If $J\in D(u,r)^n$ then $\Phi(J,n)$ contains only terms of type (1). In fact $D(u,r)^n$ contains exactly the families J for which $s=r$, and among $j_1,\ldots,j_v$ there are $v/2$ distinct indices, each one appearing twice (we then denote by J_1 the set of the $v/2$ distinct indices), and the sets $J_2=\{j_l'+\overline{j}_l':1\leq l\leq r,\overline{j}_l'=0\}$ and $J_3=\{j_l'+\overline{j}_l':1\leq l\leq r,\overline{j}_l'=1\}$ have distinct indices, and J_1, J_2 and J_3 are pairwise disjoint. With this notation, we have (with u terms all together in the products):

$$\mathbb{E}(\Phi(J,n)\mid\mathcal{G}_i^n)\;=\;\prod_{j\in J_1\cup J_2}(g_j'^n\alpha_{i+j-1}^n)^2\prod_{j\in J_3}(g_{j-1}'^n\alpha_{i+j-1}^n)^2. \tag{5.62}$$

The assumption (2.7) on g yields that $|g_j'^n-g_{j-1}'^n|\leq K/k_n^2$, except for j belonging to set Q_n of indices for which g' fails to exist or to be Lipschitz on $[(j-1)/k_n,jk_n]$, so $\#Q_n\leq K$.

Since $\alpha_i^n \leq K$, we thus have

$$\mathbb{E}(\Phi(J,n) \mid \mathcal{G}_i^n) = \begin{cases} \prod_{j\in J_1\cup J_2\cup J_3} (g_j'^n \alpha_{i+j-1}^n)^2 + \mathrm{O}_u(k_n^{-2u-1}) & \text{if } Q_n \cap (J_1\cup J_2\cup J_3) = \emptyset \\ \mathrm{O}_u(k_n^{-2u}) & \text{otherwise.} \end{cases}$$

Consider now $L = \{l_1, \cdots, l_u\}$ in the set $\mathcal{L}_n$ of all families of indices with $1 \leq l_1 < \cdots < l_u \leq k_n$, and let $w_n(L)$ be the number of $J \in D(u,r)^n$ such that the associated sets J_1, J_2, J_3 satisfy $J_1 \cup J_2 \cup J_3 = L$. Then since $\#D(u,r)^n \leq K k_n^u$ and $\sup_n \#Q_n < \infty$, we deduce from the above that

$$\overline{\Phi}(u,r)^n \;=\; \sum_{L\in\mathcal{L}_n} w_n(L) \prod_{j\in L} (g_j'^n \alpha_{i+j-1}^n)^2 + \mathrm{O}_u(\Delta_n^{u/2+1/2}). \tag{5.63}$$

Now we have to evaluate $w_n(L)$. There are C_u^r many ways of choosing the two complementary subsets J_1 and $J_2 \cup J_3$ of L. Next, with J_1 given, there are $(v/2)!\,(v-1)(v-3)\cdots 3\cdot 1$ ways of choosing the indices j_l so that $j_1, \cdots, j_v$ has $v/2$ paired distinct indices which are the indices in J_1, and we recall that $(v-1)(v-3)\cdots 3\cdot 1 = m_v$ (if $v=0$ then J_1 is empty and there is $m_0 = 1$ ways again of choosing J_1). Finally with $J_2 \cup J_3$ fixed, there are $2^r\, r!$ ways of choosing the indices $j_l' + \overline{j}_l'$, all different, when the smallest index in $J_2 \cup J_3$ is bigger than 1, and $2^{r-1}\, r!$ ways if this smallest index is 1. Summarizing, we get

$$w_n(L) \;=\; \begin{cases} m_v\, 2^r\, u! & \text{if } \; 1 \notin L \\ m_v\, 2^{r-1}\, u! & \text{if } \; 1 \in L. \end{cases} \tag{5.64}$$

On the other hand, we have by (5.55):

$$(A(g)_i^n)^u \;=\; u! \sum_{L\in\mathcal{L}_n} \prod_{j\in L} (g_j'^n \alpha_{i+j-1}^n)^2 + \mathrm{O}_u(k_n^{-1-u}).$$

Therefore, by (5.63) and (5.64), we deduce that

$$m_v\, 2^r\, (A(g)_i^n)^u - \overline{\Phi}(u,r)^n \;=\; m_v\, 2^{r-1} \sum_{L\in\mathcal{L}_n:\; 1\in L} \;\prod_{j\in L} (g_j'^n \alpha_{i+j-1}^n)^2 + \mathrm{O}_u(\Delta_n^{u/2+1/2}).$$

Since $|g_j'^n| \leq K/k_n$ and since the number of $L \in \mathcal{L}_n$ such that $1 \in L$ is smaller than k_n^{u-1}, the right side above is smaller than $K\Delta_n^{u/2+1/2}$, and we deduce that $\overline{\Phi}(u,r)^n$ is equal to the right side of (5.56). In view of (5.61), this finishes the proof of (5.56).

b) Suppose that v is odd and $u = r + v/2 - 1/2$, and recall that we need to prove that $\overline{\Phi}_u^n$ equals the right side of (5.59). Again, the definition of $D(u)^n$ and the property $u = r + v/2 - 1/2$ yield that, if $J \in D(u)^n$, there is a number a in $\{0,1\}$ and a nonnegative integer $w \leq \frac{v-1}{2} \wedge \frac{r-s-2a}{2}$ such that $\Phi(J,n)$ is the product of $\frac{v+s+r-w-1}{2}$ terms, all for different indices for χ^n, with $s - w + a + \frac{v-3}{2}$ terms of type 1, w terms of type 2, $\frac{r-s-w-2a}{2}$ terms of type 3, and $1-a$ and a term respectively of the types (4) and (5) described below:

(4) terms of the form $(g_j'^n \chi_{i+j-1}^n)^3$ or $(g_j'^n)^2 g_{j+1}'^n (\chi_{i+j}^n)^3$,

(5) terms of the form $-2(g_j'^n)^4 g_{j+1}'^n (\chi_{i+j-1}^n)^3 (\chi_{i+j}^n)^2$ or $-2(g_j'^n)^3 (g_{j+1}'^n)^2 (\chi_{i+j-1}^n)^2 (\chi_{i+j}^n)^3$,

the whole product being multiplied by -1. It follows that $\#D(u,s)^n \leq K k_n^{\frac{v+s+r-1}{2}}$, by the same argument as in (a), whereas $\mathbb{E}(|\Phi(J,n)|| \mid \mathcal{G}_i^n) \leq K/k_n^{v+2r}$ still holds. Hence, instead of (5.61) we get $|\overline{\Phi}(u,s)^n| \leq K\Delta_n^{r/2+v/4+1/4+\frac{r-s}{2}}$. In particular, $\overline{\Phi}(u,s)^n = \mathrm{O}(\Delta_n^{r/2+v/4+1/2})$ when $s<r$, and it thus remains to prove that $\overline{\Phi}(u,r)^n$ is equal to the right side of (5.59).

If $J\in D(u,r)^n$ then $\Phi(J,n)$ has $u-1$ terms of type (1) and one of type (4), and there is exactly one common index among $j_1,\cdots,j_v$ and $j_1'+\overline{j}_1',\cdots,j_s'+\overline{j}_s'$. In other words, we can associate with J three sets J_1,J_2,J_3 pairwise disjoint (with the same description than when v is even, except that $\#J_1=\frac{v-1}{2}$ and $\#(J_2\cup J_3)=r-1$), plus an index l outside $J_1\cup J_2\cup J_3$ and an integer $\overline{l}$ equal to 0 or 1, such that instead of (5.62) we have

$$\mathbb{E}(\Phi(J,n)\mid \mathcal{G}_i^n) \;=\; -\,(g_l'^n)^2 g_{l+\overline{l}}'^n\,\beta(3)_{i+l+\overline{l}-1}^n \prod_{j\in J_1\cup J_2}(g_j'^n\alpha_{i+j-1}^n)^2 \prod_{j\in J_3}(g_{j-1}'^n\alpha_{i+j-1}^n)^2.$$

This is equal to

$$-\beta(3)_i^n\,(\alpha_i^n)^{2u-2}\,(g_l'^n)^3 \prod_{j\in J_1\cup J_2\cup J_3}(g_j'^n)^2,$$

up to $\mathrm{O}_u(k_n^{-2u}(k_n^{-1}+\Gamma(\alpha,m)_i^n+\Gamma(\beta(3),m)_i^n))$ when $Q_n\cap(\{k\}\cup J_1\cup J_2\cup J_3)=\emptyset$ and to $\mathrm{O}_u(k_n^{-2u})$ otherwise. Therefore, since $\#D(u,r)^n\leq Kk_n^u$, we deduce that

$$\overline{\Phi}(u,r)^n \;=\; -\,\beta(3)_i^n\,(\alpha_i^n)^{2u-2} \sum_{l,J_1,J_2,J_3}(g_l'^n)^3 \prod_{j\in J_1\cup J_2\cup J_3}(g_j'^n)^2 + R_n,$$

where the remainder term R_n is like the last term in (5.59), and the sum is extended over all l,J_1,J_2,J_3 such that $\{l\},J_1,J_2,J_3$ are pairwise disjoint in the set $\{1,\cdots,k_n\}$. Then, with R_n' as R_n above, we have

$$\overline{\Phi}(u,r)^n \;=\; -\,\beta(3)_i^n\,(\alpha_i^n)^{2u-2}\Big(\sum_{j=1}^{k_n}(g_j'^n)^3\Big)\Big(\sum_{j=1}^{k_n}(g_j'^n)^2\Big)^{u-1}+R_n'.$$

Then by an estimate similar to (2.10) (without the absolute value), we deduce (5.59), with $\gamma_{v,r}=-\overline{g}(2)^{r+v/2-1/2}\int_0^2(g'(s))^3ds$. □

Lemma 5.5 *Assume (SN-q) for some $q\geq 2$, and let p be an even integer. The variables*

$$\Psi(g,p)_{i,j}^n \;=\; \mathbb{E}\Big(\phi(g,p)_{i,j}^n \mid \mathcal{G}_i^n\Big)-(\sigma_i^n\overline{W}(g)_{i+j}^n)^p$$

satisfy, for all $u\leq q/p$ and $m\geq 0$ and $0\leq j\leq mk_n$,

$$|\mathbb{E}(\Psi(g,p)_{i,j}^n\mid\mathcal{F}_i^n)| \;\leq\; K\Delta_n^{p/4+1/4}\Big(\Delta_n^{1/4}+\Gamma'(\alpha,m)_i^n+\Gamma'(\beta(3),m)_i^n\Big), \tag{5.65}$$

$$\mathbb{E}(|\Psi(g,p)_{i,j}^n|^u\mid\mathcal{F}_i^n) \;\leq\; K\Delta_n^{up/4+u/4}. \tag{5.66}$$

Proof. In view of (5.43), and recalling that $\sigma_i^n \overline{W}(g)_{i+j}^n$ is $\mathcal{G}_i^n$-measurable, we see that

$$\mathbb{E}(\phi(g,p)_{i,j}^n \mid \mathcal{G}_i^n) = \sum_{r=0}^{p/2} \sum_{w=0}^{p-2r} C_{p-2r}^w \rho_{p,r} \, (\sigma_i^n \overline{W}(g)_{i+j}^n)^w \, \mathbb{E}\Big((\overline{\chi}(g)_{i+j}^n)^{p-2r-w} \, (\widehat{\chi}(g)_{i+j}^n)^r \mid \mathcal{G}_i^n \Big).$$

By (3.7) and a change of the order of summation, we easily get

$$\sum_{r=0}^{p/2} \sum_{v=0}^{p/2-r} C_{p-2r}^{2v} \, \rho_{p,r} \, 2^r \, m_{p-2r-2v} \, (\sigma_i^n \overline{W}(g)_{i+j}^n)^{2v} \, (A(g)_{i+j}^n)^{p/2-v} \;=\; (\sigma_i^n \overline{W}(g)_{i+j}^n)^p,$$

hence

$$\begin{aligned}\Psi(g,p)_{i,j}^n \;=\; & \sum_{r=0}^{p/2} \sum_{v=0}^{p/2-r} C_{p-2r}^{2v} \, \rho_{p,r} \, (\sigma_i^n \overline{W}(g)_{i+j}^n)^{2v} \, \Big(\mathbb{E}\Big((\overline{\chi}(g)_{i+j}^n)^{p-2r-2v} \, (\widehat{\chi}(g)_{i+j}^n)^r \mid \mathcal{G}_i^n \Big) \\ & \qquad -2^r \, m_{p-2r-2v} \, (A(g)_{i+j}^n)^{p/2-v} \Big) \\ & + \sum_{r=0}^{p/2} \sum_{v=0}^{p/2-r-1} C_{p-2r}^{2v+1} \, \rho_{p,r} \, (\sigma_i^n \overline{W}(g)_{i+j}^n)^{2v+1} \, \mathbb{E}\Big((\overline{\chi}(g)_{i+j}^n)^{p-2r-2v-1} \, (\widehat{\chi}(g)_{i+j}^n)^r \mid \mathcal{G}_i^n \Big).\end{aligned}$$

Now, (5.65) is a simple consequence of (5.45) and (5.56) applied to the terms in the first sum above and of (5.48) and (5.59) for those in the second sum. Finally, (5.66) follows from (5.45), (5.56) and (5.58), plus Hölder's inequality. □

5.10 Block splitting.

In this subsection we are going to split the sum over i which defines $\overline{V}(Z,g,p)_t^n$ into blocks of size mk_n, separated by blocks of size k_n, in order to ensure some "conditional independence" of the successive summands, and it remains a residual sum for the summands occurring just before time t.

More specifically, we fix an integer $m \geq 2$ (which will eventually go to infinity). Recalling (5.43), the ith block of size mk_n contains $\phi(Z,g,p)_j^n$ for all j between $I(m,n,i) = (i-1)(m+1)k_n$ and $I(m,n,i) + mk_n - 1$. In a similar way, the ith block of size k_n corresponds to indices j between $\overline{I}(m,n,i) = i(m+1)k_n - k_n$ and $\overline{I}(m,n,i) + k_n - 1$. The number of pairs of blocks which can be accommodated without using data after time t is then $i_n(m,t) = \left[\frac{t-(k_n-1)\Delta_n}{(m+1)k_n\Delta_n}\right]$. The "real" times corresponding to the beginnings of the ith big and small blocks are then $t(m,n,i) = I(m,n,i)\Delta_n$ and $\overline{t}(m,n,i) = \overline{I}(m,n,i)\Delta_n$.

At this stage, we need some more notation. We consider the partial sums (we drop the mention of p, but we keep the function g):

$$\zeta(g,m)_i^n = \sum_{j=0}^{mk_n-1} \phi(Z,g,p)_{I(m,n,i)+j}^n, \qquad \overline{\zeta}(g,m)_i^n = \sum_{j=0}^{k_n-1} \phi(Z,g,p)_{\overline{I}(m,n,i)+j}^n, \tag{5.67}$$

$$U(g,m)^n_t \;=\; \sum_{i=i_n(m,t)(m+1)k_n}^{[t/\Delta_n]-k_n} \phi(Z,g,p)^n_i. \tag{5.68}$$

Consider the discrete time filtrations $\mathcal{F}(m)^n_i = \mathcal{F}^n_{I(m,n,i+1)}$ and $\overline{\mathcal{F}}(m)^n_i = \mathcal{F}^n_{\overline{I}(m,n,i+1)}$. Observe that $\zeta(g,m)^n_i$ is $\mathcal{F}(m)^n_i$-measurable and $\overline{\zeta}(g,m)^n_i$ is $\overline{\mathcal{F}}(m)^n_i$-measurable, and set

$$\eta(g,m)^n_i \;=\; \mathbb{E}(\zeta(g,m)^n_i \mid \mathcal{F}(m)^n_{i-1}), \quad \overline{\eta}(g,m)^n_i \;=\; \mathbb{E}(\overline{\zeta}(g,m)^n_i \mid \overline{\mathcal{F}}(m)^n_{i-1}), \tag{5.69}$$

$$\left.\begin{array}{l} B(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} \eta(g,m)^n_i, \quad M(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} (\zeta(g,m)^n_i - \eta(g,m)^n_i) \\ \overline{B}(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} \overline{\eta}(g,m)^n_i, \quad \overline{M}(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} (\overline{\zeta}(g,m)^n_i - \overline{\eta}(g,m)^n_i). \end{array}\right\} \tag{5.70}$$

The key point is the following obvious relation, for any $m \geq 1$:

$$\overline{V}(Z,g,p)_t = M(g,m)^n_t + B(g,m)^n_t + \overline{M}(g,m)^n_t + \overline{B}(g,m)^n_t + U(g,m)^n_t. \tag{5.71}$$

Lemma 5.6 *Under (SN-p) we have* $\Delta_n^{3/4-p/4}\, U(g,m)^n_t \stackrel{\mathbb{P}}{\longrightarrow} 0$ *as* $n \to \infty$.

Proof. The variable $U(g,m)^n_t$ is the sum of at most $(m+1)k_n$ terms $\phi(Z,g,p)^n_i$, all of them satisfying (5.51). Then the expectation of the absolute value of $\Delta_n^{3/4-p/4}\, U(g,m)^n_t$ is less than $K_{m,p} k_n \Delta_n^{3/4}$, which clearly goes to 0. □

Next, we show that, in (5.67), we can replace $\phi(Z,g,p)^n_j$ by $\phi(g,p)^n_{i,j}$, see (5.43). This leads us to introduce some additional notation, similar to the previous ones:

$$\left.\begin{array}{l} \delta(g,m)^n_i = \sum_{j=0}^{mk_n-1} \phi(g,p)^n_{I(m,n,i),j}, \quad \overline{\delta}(g,m)^n_i = \sum_{j=0}^{k_n-1} \phi(g,p)^n_{\overline{I}(m,n,i),j}, \\ \gamma(g,m)^n_i \;=\; \mathbb{E}(\delta(g,m)^n_i \mid \mathcal{F}(m)^n_{i-1}), \quad \overline{\gamma}(g,m)^n_i \;=\; \mathbb{E}(\overline{\delta}(g,m)^n_i \mid \overline{\mathcal{F}}(m)^n_{i-1})), \end{array}\right\} \tag{5.72}$$

$$\left.\begin{array}{l} D(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} \gamma(g,m)^n_i, \quad N(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} (\delta(g,m)^n_i - \gamma(g,m)^n_i) \\ \overline{D}(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} \overline{\gamma}(g,m)^n_i, \quad \overline{N}(g,m)^n_t = \sum_{i=1}^{i_n(m,t)} (\overline{\delta}(g,m)^n_i - \overline{\gamma}(g,m)^n_i). \end{array}\right\} \tag{5.73}$$

Lemma 5.7 *Under (SN-2p) we have, as* $n \to \infty$:

$$\left.\begin{array}{l} \Delta_n^{3/4-p/4}(M(g,m)^n - N(g,m)^n) \stackrel{u.c.p.}{\longrightarrow} 0, \\ \Delta_n^{3/4-p/4}(\overline{M}(g,m)^n - \overline{N}(g,m)^n) \stackrel{u.c.p.}{\longrightarrow} 0. \end{array}\right\} \tag{5.74}$$

Proof. The proof of the two assertions is the same, and we prove for example the first one. By a convergence theorem for martingale arrays, it is enough to prove that

$$\Delta_n^{3/2-p/2} \sum_{i=1}^{i_n(m,t)} \mathbb{E}\Big(\Big|\zeta(g,m)^n_i - \delta(g,m)^n_i\Big|^2\Big) \;\to\; 0,$$

By Cauchy-Schwarz inequality, the left side above is smaller than

$$\Delta_n^{3/2-p/2}\, mk_n \sum_{i=1}^{i_n(m,t)} \sum_{j=0}^{mk_n-1} \mathbb{E}(|\phi(Z,g,p)^n_{i+j} - \phi(g,p)^n_{i,j}|^2),$$

which by (5.51) and Lemma 5.3 goes to 0. □

Lemma 5.8 *Under (SN-p) we have, as $n \to \infty$:*

$$\left.\begin{array}{lll} \Delta_n^{3/4-p/4}(B(g,m)^n - D(g,m)^n) & \stackrel{u.c.p.}{\longrightarrow} & 0, \\ \Delta_n^{3/4-p/4}(\overline{B}(g,m)^n - \overline{D}(g,m)^n) & \stackrel{u.c.p.}{\longrightarrow} & 0. \end{array}\right\} \tag{5.75}$$

Proof. 1) Again we prove only the first assertion, and we drop g from the notation. In view of (5.43), it is enough to show that, for all integers l between 0 and $p/2$, we can find an AN array (δ_i^n) (depending on l) such that for all $0 \le j \le mk_n$ we have

$$\Big|\mathbb{E}\Big((\overline{Z}^n_{i+j})^{p-2l}(\widehat{Z}^n_{i+j})^l - (\overline{\kappa}^n_{i,j})^{p-2l}(\widehat{\chi}^n_{i+j})^l \mid \mathcal{F}^n_i\Big)\Big| \;\le\; K\Delta_n^{p/4+1/4}\,\delta^n_{i+j}.$$

When $l = p/2$ the second estimate (5.50) with $u = 1$ gives the result, but otherwise the first estimate (5.50) with $u = 1$ is not quite enough. Below we fix l between 0 and $p/2-1$, and the result will be true if we have the following:

$$\begin{array}{ll} |\mathbb{E}(F^n_{i,j} \mid \mathcal{F}^n_i)| \;\le\; K\Delta_n^{p/4+1/4}\,\delta^n_{i+j}, \quad \text{where} & (5.76) \\ F^n_{i,j} \;=\; \left\{\begin{array}{ll} (\overline{\kappa}^n_{i,j})^{p-2l}((\widehat{Z}^n_{i+j})^l - (\widehat{\chi}^n_{i+j})^l) & \text{(called Case A)} \\ (\widehat{\chi}^n_{i+j})^l((\overline{Z}^n_{i+j})^{p-2l} - (\overline{\kappa}^n_{i,j})^{p-2l}) & \text{(called Case B)} \\ ((\overline{Z}^n_{i+j})^{p-2l} - (\overline{\kappa}^n_{i,j})^{p-2l})((\widehat{Z}^n_{i+j})^l - (\widehat{\chi}^n_{i,j})^l) & \text{(called Case C),} \end{array}\right. & \end{array}$$

and where again (δ^n_i) is an AN array (perhaps different for each case).

The proof of (5.76) is simple in Cases A and C. Indeed, in those two cases we have $F^n_{i,j} = 0$ when $l = 0$, and when $l \ge 1$ we have $|\mathbb{E}(F^n_{i,j} \mid \mathcal{F}^n_i)| \le K\Delta_n^{p/4+1/2}$. To see this, we apply (5.45) and the second part of (5.50), plus the fact that $\Gamma'(\sigma, m+1)^n_i \le K$, and Cauchy-Schwarz inequality.

2) Now we consider Case B. Recall that $\overline{Z}^n_{i+j} = \overline{\kappa}^n_{i,j} + \overline{\lambda}^n_{i,j}$, hence

$$F^n_{i,j} \;=\; \sum_{u=1}^{p-2l} C^u_{p-2l}\,G^{u,n}_{i,j}, \qquad G^{u,n}_{i,j} \;=\; (\widehat{\chi}^n_{i+j})^l\,(\overline{\kappa}^n_{i,j})^{p-2l-u}\,(\overline{\lambda}^n_{i,j})^u, \tag{5.77}$$

and we will prove the estimate (5.76) separately for each $G^{u,n}_{i,j}$. For this, we begin with a decomposition of $\overline{\lambda}^n_{i,j}$. Recall (5.1) and the boundedness of the coefficients. By (5.46) we have $\overline{\lambda}^n_{i,j} = \xi^n_{i,j} + \xi'^n_{i,j}$, where, with the simplifying notation $S = i\Delta_n$ and $T = (i+j)\Delta_n$,

$$\xi^n_{i,j} = \int_T^{T+u_n} g_n(s-(i+j)\Delta_n)\left((b_s - b^n_i)ds + \Big(\int_S^s \Big(\widetilde{b}_r\,dr + (\widetilde{\sigma}_r - \widetilde{\sigma}^n_i)dW_r\Big)\Big)\,dW_s\right),$$

$$\xi'^n_{i,j} \;=\; \int_T^{T+u_n} g_n(s-(i+j)\Delta_n)\Big(b^n_i ds + \widetilde{\sigma}^n_i(W_s - W_S)dW_s + (M_s - M_S)dW_s\Big).$$

Then for $v \geq 1$ and $j \leq mk_n$, we have (recall that $\mathbb{E}(|M_s - M_S|^v \mid \mathcal{F}_S) \leq K_{m,v} u_n^{1\wedge(v/2)}$ if $S \leq s \leq S + mu_n$):

$$\left.\begin{array}{rcl}\mathbb{E}(|\xi_{i,j}^n|^v \mid \mathcal{F}_i^n) & \leq & K_{m,v}\Delta_n^{v/2}\Big(\Delta_n^{v/2} + \Gamma'(b,m+1)_i^n + \Gamma'(\widetilde{\sigma},m+1)_i^n\Big) \\ \mathbb{E}(|\xi_{i,j}'^n|^v \mid \mathcal{F}_i^n) & \leq & K_{m,v}\Delta_n^{v/4+((1/2)\wedge(v/4))} \\ \mathbb{E}(|\overline{\lambda}_{i,j}^n|^v \mid \mathcal{F}_i^n) & \leq & K_{m,v}\Delta_n^{v/4+((1/2)\wedge(v/4))}.\end{array}\right\} \qquad (5.78)$$

3) Next we prove that, for u an odd integer,

$$\mathbb{E}\Big((\overline{W}_{i+j}^n)^u \ \xi_{i,j}'^n \mid \mathcal{F}_i^n\Big) \ = \ 0. \qquad (5.79)$$

We prove this separately for each of the three terms constituting $\xi_{i,j}'^n$. Since $x \mapsto x^u$ is an odd function, this is obvious for the (first) term involving b_i^n, and also for the (second) term involving $\widetilde{\sigma}_i^n$. For the (third) term involving M, we observe first that $(\overline{W}_{i+j}^n)^u = Y + \int_S^{T+u_n} \rho_s dW_s$ for some $\mathcal{F}_S = \mathcal{F}_i^n$-measurable variable Y and process ρ adapted to the filtration ($\mathcal{F}_t^W$) generated by the Brownian motion). Since the third term is a martingale increment it is enough to prove that $\mathbb{E}(U_{T+u_n} \mid \mathcal{F}_S) = 0$, where

$$U_t \ = \ \Big(\int_S^t \rho_s dW_s\Big)\Big(\int_T^{T+u_n} g_n(s-(i+j)\Delta_n)(M_s - M_S)dW_s\Big).$$

Itô's formula yields that $U_t = M_t' + \int_T^t g_n(s-(i+j)\Delta_n)\rho_s(M_s - M_S)ds$ for $t \geq T$, where M' is a martingale with $M_S' = 0$, so it is enough to prove that

$$\mathbb{E}(\rho_t(M_t - M_S) \mid \mathcal{F}_S) = 0. \qquad (5.80)$$

But for any fixed $t \geq T$ we again have $\rho_t = Y_t' + \int_S^t \rho_s' dW_s$ where Y_t' is $\mathcal{F}_S$-measurable. Hence (5.80) follows from the orthogonality of W and M, and we have (5.79).

4) Now, we use (5.45), (5.49) and (5.78), and the form of $G_{i,j}^{u,n}$ as a product of three terms at the respective powers l, $v = p - 2l - u$ and u. Then Hölder inequality with the respective exponents $l' = 2p/l$ and $v' = 4p/(p-2l-u)$ (so $2ll' = vv' = 4p$ and (5.45) and (5.49) apply) and $u' = 4p/(3p+u)$ yields $\mathbb{E}(|G_{i,j}^{u,n}| \mid \mathcal{F}_i^n) \leq K\Delta_n^{p/4+((u/4)\wedge(1/2u'))}$. Observing that $(u/4)\wedge(1/2u') > 1/4$ when $u \geq 2$, we deduce that (5.76) holds for $G_{i,j}^{u,n}$ when $u \geq 2$. It remains to study $G_{i,j}^{1,n}$, which is the sum $G_{i,j}^n + G_{i,j}'^n$, where

$$G_{i,j}^n \ = \ (\widehat{\chi}_{i+j}^n)^l \ (\overline{\kappa}_{i,j}^n)^{p-2l-1} \ \xi_{i,j}^n, \qquad G_{i,j}'^n \ = \ (\widehat{\chi}_{i+j}^n)^l \ (\overline{\kappa}_{i,j}^n)^{p-2l-1} \ \xi_{i,j}'^n.$$

By (5.45), (5.49) and (5.78),and Hölder inequality as above, we get

$$\mathbb{E}(|G_{i,j}^n| \mid \mathcal{F}_i^n) \ \leq \ K\Delta_n^{p/4+1/4}\Big(\sqrt{\Delta_n} + \sqrt{\Gamma'(b,m+1)_i^n + \Gamma'(\widetilde{\sigma},m+1)_i^n}\ \Big).$$

Then by Lemma 5.3 we deduce that $G_{i,j}^n$ satisfies (5.76).

5) We are left to study $G'^n_{i,j}$, which can be written as $G'^n_{i,j} = \sum_{w=0}^{p-2l-1} C^w_{p-2l-1} a(n,w,i,j)$, where $a(n,w,i,j) = (\sigma^n_i \overline{W}^n_{i+j})^{p-2l-1-w}\, \xi'^n_{i,j}\, (\widehat{\chi}^n_{i+j})^l\, (\overline{\chi}^n_{i+j})^w$. On the one hand, by successive conditioning we deduce from (5.58) and (5.78) that $\mathbb{E}(|a(n,w,i,j)| \mid \mathcal{F}^n_i)) \leq K\Delta_n^{p/4+1/2}$ when w is odd. On the other hand, when w is even, the same argument with (5.56), plus (5.79) and the fact that $p-2l-1-w$ is then odd yield

$$|\mathbb{E}(a(n,w,i,j) \mid \mathcal{F}^n_i))| \;=\; \mathrm{O}_u\Big(\Delta_n^{p/4+1/4}\Big(\Delta_n^{1/2} + \Gamma(\alpha,m)^n_i\Big)\Big),$$

and by Lemma 5.3 this finishes the proof. □

Lemma 5.9 *Under (SN-p) we have, as $n \to \infty$:*

$$\left.\begin{array}{l} \frac{1}{\Delta_n^{1/4}}\, \Big(\Delta_n^{1-p/4} D(g,m)^n_t - \frac{m}{m+1}\; m_p\, (\theta\, \overline{g}(2))^{p/2} \int_0^t |\sigma_s|^p\, ds\Big) \;\stackrel{u.c.p.}{\longrightarrow}\; 0 \\ \frac{1}{\Delta_n^{1/4}}\, \Big(\Delta_n^{1-p/4} \overline{D}(g,m)^n_t - \frac{1}{m+1}\; m_p\, (\theta\, \overline{g}(2))^{p/2} \int_0^t |\sigma_s|^p\, ds\Big) \;\stackrel{u.c.p.}{\longrightarrow}\; 0. \end{array}\right\} \tag{5.81}$$

Proof. By (2.11), $\overline{W}(g)^n_{i+j}$ is independent of $\mathcal{F}^n_i$, and $\mathcal{N}(0, \overline{g}(2)_n \Delta_n)$. So by virtue of (2.6) and (2.10) we have $\mathbb{E}((\overline{W}(g)^n_{i+j})^p \mid \mathcal{F}^n_i) = m_p(\theta\, \overline{g}(2))^{p/2} \Delta_n^{p/4} + \mathrm{O}_u(\Delta_n^{p/4+1/2})$. Therefore by (5.66), the left side of the first expression in (5.81) is smaller in absolute value than

$$\begin{array}{l} \dfrac{K}{\Delta_n^{1/4}}\, \Big|(m+1) k_n \Delta_n \displaystyle\sum_{i=1}^{i_n(m,t)} |\sigma_{t(m,n,i)}|^p - \int_0^t |\sigma_s|^p\, ds\Big| \\ \qquad + Kt(m+1)\Delta_n^{1/4} + Kt(m+1)\sqrt{\Delta_n} \displaystyle\sum_{i=1}^{i_n(m,t)} \Big(\Gamma'(\alpha,m)^n_{I(m,n,i)} + \Gamma'(\beta(3),m)^n_{I(m,n,i)}\Big). \end{array}$$

The second term above goes to 0, as the last term (locally uniformly in t, in probability) by Lemma 5.3. As to the first term, it goes to 0 locally uniformly in t in probability as well, because of our assumption (K): see for example [10]. Therefore the first assertion in (5.81) holds, and the second one is proved in the same way. □

Lemma 5.10 *Under (SN-2p) we have for all $m \geq 2$ and $t > 0$:*

$$\mathbb{E}\Big(\sup_{s \leq t}\; \Big(\Delta_n^{3/4-p/4} \overline{N}(g,m)^n_s\Big)^2\Big) \;\leq\; \frac{Kt}{m}. \tag{5.82}$$

Proof. By Doob's inequality, the left side above is smaller than

$$4\Delta_n^{3/2-p/2} \sum_{i=1}^{i_n(m,t)} \mathbb{E}\Big((\overline{\delta}(g,m)^n_i)^2\Big),$$

and (5.72) and (5.51) yield $\mathbb{E}\Big((\overline{\delta}(g,m)^n_i)^2\Big) \leq K\Delta_n^{p/2-1}$. Since $i_n(m,t) \leq Kt/m\sqrt{\Delta_n}$, we readily deduce the result. □

5.11 An auxiliary CLT.

Here we prove a CLT for the vector $(N(g_i,m)^n)_{1\leq i\leq d}$, when $m\geq 2$ is fixed and $(g_i)_{1\leq i\leq d}$ is a family of functions satisfying (2.7). We first complement the notation (3.13). For $\zeta,\eta\in\mathbb{R}$ and $p>0$ and $m\geq 1$ we set

$$\left.\begin{array}{rcl}\mu^m_{2p}(g,h;\eta,\zeta) &=& \sum_{r,r'=0}^{p/2}\rho_{p,r}\rho_{p,r'}(2\zeta^2\overline{g}'(2))^r(2\zeta^2\overline{h}'(2))^{r'} \\ & & \int_0^m\int_0^m \mathbb{E}'\Big((\eta L(g)_s+\zeta L'(g)_s)^{p-2r}(\eta L(h)_t+\zeta L'(h)_t))^{p-2r'}\,ds\,dt, \\ \overline{\mu}^m_{2p}(g,h;\eta,\zeta) &=& \frac{1}{m+1}\Big(\mu^m_{2p}(g,h;\eta,\zeta)-m^2\,\mu_p(g;\eta,\zeta)\,\mu_p(h;\eta,\zeta)\Big).\end{array}\right\} \tag{5.83}$$

Exactly as in Lemma 3.5, for any (η,ζ) the matrix with entries $\overline{\mu}^m_{2p}(g_i,g_j;\eta,\zeta)$ is symmetric nonnegative.

Proposition 5.11 *Assume (SN-4p), and let $m\geq 2$. The sequence of d-dimensional processes with components $\Delta_n^{3/4-p/4}N(g_i,m)$ converges stably in law to a process of the following form*

$$\left(\theta^{1/2-p/2}\sum_{j=1}^d\int_0^t\psi^m_{ij}(\theta\sigma_s,\alpha_s)\,dB^j_s\right)_{1\leq i\leq d}, \tag{5.84}$$

where B is as in Theorem 4.1 and ψ^m is a measurable $d\times d$ matrix-valued function such that $(\psi^m\psi^{m\star})(\eta,\zeta)$ is the matrix with entries $\overline{\mu}^m_{2p}(g_i,g_j;\eta,\zeta)$, as defined by (5.83).

We begin with a lemma, for which we use the notation Γ of Lemma 5.1.

Lemma 5.12 *Let $m\geq 2$ and $s\in\Gamma$ and $i_n=\min(i: I(m,n,i)\Delta_n\geq s)$. Then under (SN-4p) we have the following almost sure convergences:*

$$\Delta_n^{1/2-p/4}\gamma(g,m)^n_{i_n}\ \to\ m\,m_p\,\theta^{1+p/2}\,\overline{g}(2)^{p/2}\ |\sigma_s|^p\ =\ m\theta^{1-p/2}\mu_p(g;\theta\sigma_s,\alpha_s), \tag{5.85}$$

$$\Delta_n^{1-p/2}\mathbb{E}\Big(\delta(g,m)^n_{i_n}\,\delta(h,m)^n_{i_n}\mid\mathcal{F}(m)^n_{i_n-1}\Big)\ \to\ \theta^{2-p}\,\mu^m_{2p}(g,h;\theta\sigma_s,\alpha_s). \tag{5.86}$$

Proof. We set $i'_n=I(m,n,i_n)$ and $s_n=i'_n\Delta_n$, which converges to s. Both results are consequences of Lemma 5.1.

By (5.66) (with $u=1$), we see that (5.85) follows from

$$\Delta_n^{1/2-p/4}\,\mathbb{E}\Big(\sum_{j=0}^{mk_n-1}|\sigma_{s_n}\overline{W}(g)^n_{i'_n+j}|^p\mid\mathcal{F}_{s_n}\Big)\ \to\ m\,m_p\,\theta^{1+p/2}\,\overline{g}(2)^{p/2}\ |\sigma_s|^p. \tag{5.87}$$

Then we apply Lemma 5.1 with $d=1$ and $g_1=g$ and with the functions

$$f_n(x,y,z)=\frac{1}{k_n}\sum_{j=0}^{mk_n-1}|x(j/k_n)|^p,\qquad f(x,y,z)=\int_0^m|x(s)|^p\,ds,$$

which satisfy (5.25) and $f_n \to f$ pointwise. The left (resp. right) side of (5.87) is equal to $\Delta_n^{1/2-p/4}/k_n^{p/2-1}$ times (resp. $\theta^{1-p/2}$ times) the left (resp. right) side of (5.26), hence (5.85) holds.

For (5.86) we apply Lemma 5.1 with $d=2$ and $g_1=g$ and $g_2=h$ and the functions

$$\begin{array}{rcl} f_n((x,x'),(y,y'),(z,z')) & = & \sum_{r,r'=0}^{p/2} \rho_{p,r}\rho_{p,r'} \frac{1}{k_n^2} \sum_{j,j'=0}^{mk_n-1} \Big(x\Big(\frac{j}{k_n}\Big)+y\Big(\frac{j}{k_n}\Big)\Big)^{p-2r} \\ & & \Big(x'\Big(\frac{j'}{k_n}\Big)+y'\Big(\frac{j'}{k_n}\Big)\Big)^{p-2r'} z\Big(\frac{j}{k_n}\Big)^r \; z'\Big(\frac{j'}{k_n}\Big)^{r'} \\ f((x,x'),(y,y'),(z,z')) & = & \sum_{r,r'=0}^{p/2} \rho_{p,r}\rho_{p,r'} \int_0^m \int_0^m (x(s)+y(s))^{p-2r} \\ & & (x'(t)+y'(t))^{p-2r'} \; z(s)^r \; z'(t)^{r'} \, ds \, dt, \end{array}$$

which satisfy (5.25) and $f_n \to f$ pointwise. The left (resp. right) side of (5.86) is equal to $\Delta_n^{1-p/2}/k_n^{p-2}$ times (resp. θ^{2-p} times) the left (resp. right) side of (5.26), hence (5.86) holds. □

Proof of Proposition 5.11. 1) As is well known, and with the d-dimensional variables with components $\xi_i^{n,k} = \Delta_n^{1/2-p/4}\ (\delta(g_k,m)_i^n - \gamma(g_k,m)_i^n)$ (which are martingale differences), it suffices to prove the next three convergences, for all $t>0$ and all bounded martingales N:

$$\sqrt{\Delta_n} \sum_{i=1}^{i_n(m,t)} \mathbb{E}(\xi_i^{n,k}\, \xi_i^{n,l} \mid \mathcal{F}(m)_{i-1}^n) \ \stackrel{\mathbb{P}}{\longrightarrow}\ \theta^{1-p} \int_0^t \overline{\mu}_{2p}^m(g_k,g_l;\theta\sigma_s,\alpha_s)\, ds, \tag{5.88}$$

$$\Delta_n \sum_{i=1}^{i_n(m,t)} \mathbb{E}(\|\xi_i^n\|^4 \mid \mathcal{F}(m)_{i-1}^n) \ \stackrel{\mathbb{P}}{\longrightarrow}\ 0, \tag{5.89}$$

$$\Delta_n^{1/4} \sum_{i=1}^{i_n(m,t)} \mathbb{E}(\xi_i^n\ (N_{i(m+1)u_n} - N_{(i-1)(m+1)u_n}) \mid \mathcal{F}(m)_{i-1}^n) \ \stackrel{\mathbb{P}}{\longrightarrow}\ 0 \tag{5.90}$$

(we use Theorem IX.7.28 of [9], with Z being a bounded martingale of the form $Z_t = \int_0^t u_s dW_s$ for some predictable process u with values in $(0,1]$).

Observe that (5.51) and Hölder's inequality imply that $\mathbb{E}(|\delta(g_k,m)_i^n|^4 \mid \mathcal{F}(m)_{i-1}^n) \le K_m \Delta_n^{p-2}$. Then the expected value of the left side of (5.89) is smaller than $K_m\sqrt{\Delta_n}$: hence (5.89) holds. The proof of the other two properties is a bit more involved.

2) The proof of (5.88) is similar to the proof of Theorem 3.3. We denote by ζ_i^n the variable $\mathbb{E}\Big(\xi_i^{n,k}\xi_i^{n,l} \mid \mathcal{F}(m)_{i-1}^n\Big)$, and $\gamma_s = \overline{\mu}_{2p}^m(g_k,g_l;\theta\sigma_s,\alpha_s)$. Then, since $k_n\sqrt{\Delta_n} \to \theta$, we need to show that

$$(m+1)k_n\Delta_n \sum_{i=1}^{i_n(m,t)} \zeta_i^n \ \stackrel{\mathbb{P}}{\longrightarrow}\ (m+1)\theta^{2-p} \int_0^t \gamma_s\, ds. \tag{5.91}$$

Note that $|\zeta_i^n| \leq K_m$. Then, exactly as in the proof of Theorem 3.3, the above property will follow from the fact that for any $s \in \Gamma$, and with the notation i_n of Lemma 5.12, we have (similar to (5.53)):

$$\zeta_{i_n}^n \;\to\; \theta^{2-p}\gamma_s \quad \text{a.s.} \tag{5.92}$$

Now, observe that

$$\zeta_{i_n}^n \;=\; \Delta_n^{1-p/2}\Big(\mathbb{E}\Big(\delta(g_k,m)_{i_n}^n\,\delta(g_l,m)_{i_n}^n \mid \mathcal{F}(m)_{i_n-1}^n\Big) - \gamma(g_k,m)_{i_n}^n\,\gamma(g_l,m)_{i_n}^n\Big).$$

Then (5.92) readily follows from Lemma 5.12 and (5.83).

3) Now we turn to (5.90), which we prove for the first component, say with $g = g_1$. For simplicity we write $D_i^n(Y) = Y_{i(m+2)u_n} - Y_{(i-1)(m+2)u_n}$ for any process Y. In view of the definition of ξ_i^n, and since N is a martingale, it is enough to prove that

$$\Delta_n^{3/4-p/4}\sum_{i=1}^{i_n(m,t)}\mathbb{E}(\delta(g,m)_i^n D_i^n(N) \mid \mathcal{F}(m)_{i-1}^n) \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0. \tag{5.93}$$

By (5.66) we see that $\delta(g,m)_i^n = \delta_i'^n + \Psi_i'^n$, where

$$\delta_i'^n \;=\; \sum_{j=0}^{mk_n-1}(\sigma_{I(m,n,i)}^n\overline{W}(g)_{I(m,n,i)+j}^n)^p, \qquad \mathbb{E}(|\Psi_i'^n|^2 \mid \mathcal{F}(m)_{i-1}^n) \;\leq\; K\Delta_n^{p/2-1/2}.$$

We have $\sum_{i=1}^{i_n(m,t)}\mathbb{E}((D_i^n(N))^2) \leq K$ because the martingale N is bounded, so by Cauchy-Schwarz inequality we get

$$\mathbb{E}\Big(\Delta_n^{3/4-p/4}\sum_{i=1}^{i_n(m,t)}|\Psi_i'^n|\,|D_i^n(N)|\Big) \;\leq\; K\Delta_n^{3/4-p/4}\sqrt{\mathbb{E}\Big(\sum_{i=1}^{i_n(m,t)}(\Psi_i'^n)^2\Big)} \;\leq\; K\Delta_n^{1/4}.$$

Therefore it remains to prove that

$$a(N,n,t) \;=\; \Delta_n^{3/4-p/4}\sum_{i=1}^{i_n(m,t)}\mathbb{E}(\delta_i'^n D_i^n(N) \mid \mathcal{F}(m)_{i-1}^n) \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0. \tag{5.94}$$

Note that we always have

$$\left.\begin{array}{l} a(N,n,t) \;=\; \sum_{i=1}^{i_n(m,t)} b(n,i), \quad b(n,i) \;=\; \Delta_n^{3/4-p/4}\mathbb{E}(\delta_i'^n\,D_i^n(N) \mid \mathcal{F}(m)_{i-1}^m), \\ \text{and } \;|b(n,i)| \;\leq\; K\Delta_n^{1/4}. \end{array}\right\} \tag{5.95}$$

4) We have $\mathbb{E}(|\delta_i'^n|^2) \leq K\Delta_n^{p/2-1}$, hence by Cauchy-Schwarz inequality

$$\Delta_n^{3/4-p/4}\sum_{i=1}^{i_n(m,t)}\mathbb{E}\Big(|\delta_i'^n D_i^n(N)|\Big) \;\leq\; K\sqrt{\mathbb{E}(N_t^2)}.$$

It follows that the set of square-integrable martingales N satisfying (5.94) is closed under $\mathbb{L}^2$-convergence. This allows to use the following scheme for the proof:

a) Prove (5.94) when N is $(\mathcal{F}_t^{(0)})$-adapted and orthogonal to W;

b) Prove (5.94) when $N_t = \int_0^t \gamma_s dW_s$, where γ is $(\mathcal{F}_t^{(0)})$-adapted and constant in time over intervals $(t_{i-1}, t_i]$, with $t_0 = 0$ and $t_q = \infty$ for some q.

c) Conclude from the closeness proved before that (5.94) holds for all $N \in \mathcal{N}^0$, the set of all bounded $(\mathcal{F}_t^{(0)})$-martingales.

d) Prove (5.94) when N is in the set $\mathcal{N}^1$ of all martingales having $N_\infty = f(\chi_{t_1}, \cdots, \chi_{t_q})$, where f is any Borel bounded on $\mathbb{R}^q$ and $t_1 < \cdots < t_q$ and $q \geq 1$.

e) Since $\mathcal{N}^0 \cup \mathcal{N}^1$ is a total subset of the set of all square-integrable $(\mathcal{F}_t)$-martingales, conclude once more from the closeness that (5.94) holds for all such N.

5) We are thus left to proving (a), (b) and (d). The variable $\delta_i'^n$ is the sum of an $\mathcal{F}(m)_{i-1}^n$-measurable variable and of a martingale which is a stochastic integral with respect to W, by the representation theorem on the Wiener space. Then if N is an $(\mathcal{F}_t^{(0)})$-martingale orthogonal to W we have $b(n,i) = 0$, hence $a(N,n,t) = 0$ and (a) holds.

Next let N be as in (b). If the interval $[t(m,n,i), t(m,n,i+1)]$ is contained in one of the intervals $[t_j, t_{j+1}]$, then $\delta_i'^n \, D_i^n(N)$ is the product of an $\mathcal{F}(m)_{i-1}^n$-measurable variable, times a variable which is an odd function of the increments of the process W after the conditioning time $t(m,n,i)$, hence $b(n,i) = 0$. Thus $a(N,n,t)$ is the sum of at most q non-vanishing terms, all smaller than $K\Delta_n^{1/4}$ by (5.95): then $a(N,n,t) \to 0$.

This proves (b), hence (c), and the following extension of (c) is obvious: if $s \geq 0$ and N is a bounded martingale relative to the filtration $(\mathcal{F}_s \vee \mathcal{F}_t^{(0)})_{t\geq 0}$ and satisfying $N_r = 0$ if $r \leq s$, and if $a'(N,n,t)$ is associated with N by (5.95), except that we delete from the sum the term such that $t(n,m,i) \leq s < t(n,m,i+1)$, then we have $a'(N,n,t) \stackrel{\mathbb{P}}{\longrightarrow} 0$.

Finally we prove (d). Let $N \in \mathcal{N}^1$ be associated with q and f as in (d). Then (see [11]) we have $N_t = N_t^j$ for $t_j \leq t < t_{j+1}$ (by convention $t_0 = 0$ and $t_{q+1} = \infty$), where $N_t^j = M(j; \chi_{t_1}, \cdots, \chi_{t_j})_t$ and $M(j; z_1, \cdots, z_j)$ is a version of the martingale

$$M(j; z_1, \cdots, z_j)_t = \mathbb{E}^{(0)}\Big(\int \prod_{r=j+1}^{q} Q_{t_r}(dz_r) f(z_1, \cdots, z_j, z_{j+1}, \cdots, z_q) \mid \mathcal{F}_t^{(0)}\Big)$$

(with obvious conventions when $j = 0$ and $j = q$). We also set $N_t'^j = N_t^j - N_{t\wedge t_j}^j$. By the extension of (c) given above, we have $a'(N'^j, n, t) \stackrel{\mathbb{P}}{\longrightarrow} 0$ for all j. Furthermore $a(N,n,t)$ equals $\sum_{j=1}^q a'(N'^j, n, t)$, plus at most q terms, each being smaller than $K\Delta_n^{p/4-1/2}$ by (5.95). Therefore $a(N,n,t) \stackrel{\mathbb{P}}{\longrightarrow} 0$, and we are finished. □

5.12 Proof of Theorem 4.1.

By localization we can assume (SN-4p) and (SK). The following property is implicitly proved in the proof of Lemma 3.5 in Subsection 5.3 (with T playing the role of m here):

$$\overline{\mu}^m_{2p}(g,h;\eta,\zeta) \;\to\; \overline{\mu}_{2p}(g,h;\eta,\zeta), \quad \text{as } m\to\infty. \tag{5.96}$$

We write $A(g)_t = m_p\,(\theta\overline{g}(2))^{p/2}\int_0^t |\sigma_s|^p\,ds$. By (4.1) and (5.71) we have for each $m\geq 2$:

$$\widetilde{V}(g,p)^n \;=\; \Delta_n^{3/4-p/4}\,N(g,m)^n + Z(g,m)^n,$$

where

$$\begin{aligned}Z(g,m)^n \;=\;& \Delta_n^{3/4-p/4}\Big(U(g,m)^n + M(g,m)^n - N(g,m)^n + \overline{M}(g,m)^n - \overline{N}(g,m)^n\\ &+B(g,m)^n - D(g,m)^n + \overline{B}(g,m)^n - \overline{D}(g,m)^n\Big)\\ &+\frac{1}{\Delta_n^{1/4}}\Big(\Delta_n^{1-p/4}(D(g,m)^n + \overline{D}(g,m) - A(g)\Big) + \Delta_n^{3/4-p/4}\overline{N}(g,m).\end{aligned}$$

On the one hand, Lemmas 5.6, 5.7, 5.8, 5.9 and 5.10 yield

$$\lim_{m\to\infty}\ \limsup_{n\to\infty}\ \mathbb{P}\Big(|Z(g,m)^n_t|>\varepsilon\Big) \;=\; 0 \tag{5.97}$$

for all $\varepsilon>0$. On the other hand, we fix the d-dimensional Brownian motion B in (5.84) and (4.2) (the same in both). Using (5.96), we deduce that we can choose suitable versions for the square-roots ψ and ψ^m in such a way that $\psi^m(\eta,\zeta)\to\psi(\eta,\zeta)$ for all η,ζ. Then (5.84) converges in probability towards (4.2). The result then follows from Proposition 5.11 and from (5.97) in a standard way.

5.13 Theorem 4.4: a key decomposition.

Here we start the proof of Theorem 4.4, by providing a decomposition for the processes $\widetilde{V}^*(g,p)^n$ of (4.3). So we fix $p>3$, and assume α càdlàg. By localization we can and will assume (SN-2p) and (SK) without special mention.

As said before, in (4.6) one may take any sequence (T_m) which exhausts the jump times of X. A convenient choice is as follows: for any $q\geq 1$ we consider the successive jump times $(T(q,m): m\geq 1)$ of the Poisson process $\underline{\mu}((0,t]\times\{z: 1/q<\gamma(z)\leq 1/(q-1)\})$, where γ is the function occurring in (SH). Those stopping times have pairwise disjoint graphs as m and q vary, and $(T_m)_{m\geq 1}$ denotes any reordering of the double sequence $(T(q,m): q,m\geq 1)$. We complete this sequence by setting $T_0=0$.

Let P_q be the set of all $m\geq 1$ such that $T_m = T(q',m')$ for some $m'\geq 1$ and some $q'\leq q$. We consider the processes (compare with (5.5)):

$$\left.\begin{array}{llll} X^q = (\delta\,1_{\{z:\gamma(z)>1/q\}})*\underline{\mu}, & & M^q = (\delta\,1_{\{z:\gamma(z)\leq 1/q\}})*(\underline{\mu}-\underline{\nu}),\\ X'^q = X - X^q, & & X''^q = X'^q - M^q\\ Z'^q = X'^q+\chi, & & Z''^q = X''^q + \chi. \end{array}\right\} \tag{5.98}$$

So X''^q satisfies (2.14), with the same σ as in (2.13) and a bounded drift given by

$$b_t^q \;=\; b_t - \int_{\{z:\,\gamma(z)>1/q,|\delta(t,z)|\leq 1\}} \delta(t,z)\,\lambda(dz). \tag{5.99}$$

We denote by $\Omega_n(t,q)$ the set of all ω such that for any $m,m'\in P_q$ with $T_m(\omega)\leq t$, we have $2u_n < T_m(\omega)\leq t-4u_n$, and $|T_m(\omega)-T_{m'}(\omega)|>4u_n$, and also $T_m(\omega)/\Delta_n$ is not an integer. Since the set $\{T_m: m\in P_q\}$ is locally finite and $\mathbb{P}(T_m=t)=0$ for all m and $t\geq 0$, we have

$$\Omega_n(t,q)\;\to\;\Omega\quad\text{a.s., as}\quad n\to\infty. \tag{5.100}$$

Next, we denote by $\widetilde{V}^*(X,g,p)^n$ the process defined by (4.3) to emphasize the dependency on X, and likewise we have $\widetilde{V}^*(X'^q,g,p)^n$. Then a (relatively) simple computation shows the following key property, which holds on the set $\Omega_n(t,q)$:

$$\widetilde{V}^*(X,g,p)_t^n \;=\; \widetilde{V}^*(X'^q,g,p)_t^n + Y(q,g)_t^n,\quad Y(q,g)_t^n \;=\; \sum_{m\in P_q:\; T_m\leq t} \zeta(q,g)_m^n, \tag{5.101}$$

where, with the random integer $I_m^n=[T_m/\Delta_n]$, we have set

$$\begin{array}{rcl}\zeta(q,g)_m^n &=& \frac{1}{\Delta_n^{1/4}k_n}\Big(\sum_{j=1}^{k_n-1}\Big(\Big|\overline{Z'^q}(g)_{I_m^n+1-j}^n + g_j^n\Delta X_{T_m}\Big|^p \\ && -\Big|\overline{Z'^q}(g)_{I_m^n+1-j}^n\Big|^p - \Big|g_j^n\Delta X_{T_m}\Big|^p\Big) + (\overline{g}(p)_n - k_n\overline{g}(p))|\Delta X_{T_m}|^p\Big).\end{array} \tag{5.102}$$

(Note that $\overline{Z'^q}(g)_{I_m^n-j}^n$ possibly involves $\Delta_l^n Z'(q)$ for negative integers l, although this does not occur on the set $\Omega_n(t,q)$ when $m\in P_q$ and $j\leq 2k_n$; however, to have such variables defined everywhere, we make the convention $\Delta_i^n Y=0$ for any process Y when $i\leq 0$.)

5.14 The processes $Y(q,g)^n$.

In order to study the variables $Y(q,g)_t^n$ above, we set for $m\in P_q$:

$$\eta(q,g)_m^n \;=\; \frac{1}{\Delta_n^{1/4}k_n}\sum_{j=1}^{k_n-1}\{g_j^n\}^{p-1}\,\overline{Z'^q}(g)_{I_m^n-j+1}^n. \tag{5.103}$$

We have in fact a family $(g_l)_{1\leq l\leq d}$ of d weight functions, with the associated variables $(U_{m-},U_{m+},\overline{U}_{m-},\overline{U}_{m+})$ as before Theorem 4.4.

Lemma 5.13 *For any $q\geq 1$, the $(\mathbb{R}^d)^{\mathbb{N}^\star}$-valued variables $(\eta(q,g_l)_m^n)_{1\leq l\leq d,m\in P_q}$ converge stably in law, as $n\to\infty$, to $(\eta_m)_{m\in P_q}$, where η_m is the d-dimensional variable given by*

$$\eta_m \;=\; \sqrt{\theta}\,\sigma_{T_m-}U_{m-} + \sqrt{\theta}\,\sigma_{T_m}U_{m+} + \frac{\alpha_{T_m-}}{\sqrt{\theta}}\,\overline{U}_{m-} + \frac{\alpha_{T_m}}{\sqrt{\theta}}\,\overline{U}_{m+}. \tag{5.104}$$

Proof. As is well known, it is enough to prove the result for any finite subset of m's, say in a finite subset P'_q of P_q. Since q is fixed, we drop it from the notation, writing $Z' = Z'^q$, $Z'' = Z''^q$, $M = M^q$, $X' = X'^q$ and $X'' = X''^q$.

1) The stopping times $(T_m : m \in P_q)$ are independent of the Brownian motion W, and also of the restriction $\underline{\mu}(q)$ of the Poisson measure $\underline{\mu}$ to the set $\mathbb{R}_+ \times \{z : \gamma(z) \leq 1/q\}$. Therefore if $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(T_m : m \in P_q)$, the process W is a Brownian motion and the measure $\underline{\mu}(q)$ a Poisson measure with compensator $\underline{\nu}(q)$, the restriction of $\underline{\nu}$ to $\mathbb{R}_+ \times \{z : \gamma(z) \leq 1/q\}$ again, relative to the filtration $(\mathcal{H}_t)$. Thus X'' admits the same representation (2.14) and M has the same form (5.98) relatively to the two filtrations $(\mathcal{F}_t)$ and $(\mathcal{H}_t)$. Since the random integers I^n_m are $\mathcal{H}_0$-measurable, we deduce from (5.2) and (5.45) and (5.3) for M together with $|(g_l)_n(s)| \leq K$ that, for $v \in (0, 2p]$ and $j \in \mathbb{Z}$ and $i = l, \cdots, d$:

$$\left.\begin{array}{l} \mathbb{E}(|\Delta^n_{I^m_n+j} X''|^v) \;\leq\; K_{v,q} \Delta_n^{v/2}, \qquad \mathbb{E}(|\overline{M}(g_l)^n_{I^n_m+j}|^2) \;\leq\; K\sqrt{\Delta_n} \\ \mathbb{E}\Big(|\overline{X''}(g_l)^n_{I^n_m+j}|^v + |\overline{Z''}(g_l)^n_{I^n_m+j}|^v\Big) \;\leq\; K_{v,q} \Delta_n^{v/4}. \end{array}\right\} \tag{5.105}$$

Now, if f is a bounded function on $\mathbb{R}$, arguments similar to the one giving (5.47) (relative to the filtration $(\mathcal{H}_t)$ and using that σ is càdlàg and bounded and the drift b^q is also bounded), we obtain that if $2 \leq k'_n \leq 2k_n$,

$$\left.\begin{array}{l} \mathbb{E}\Big(\Big|\sum_{j=0}^{k'_n} f(j/k_n)\Delta^n_{I^m_n-j} X'' - \sigma_{T_m-} \sum_{j=0}^{k'_n} f(j/k_n)\Delta^n_{I^m_n-j} W\Big|^v\Big) \;=\; \mathrm{o}_u(\Delta_n^{v/4}) \\ \mathbb{E}\Big(\Big|\sum_{j=2}^{k'_n} f(j/k_n)\Delta^n_{I^m_n+j} X'' - \sigma_{T_m} \sum_{j=2}^{k'_n} f(j/k_n)\Delta^n_{I^m_n+j} W\Big|^v\Big) \;=\; \mathrm{o}_u(\Delta_n^{v/4}). \end{array}\right\} \tag{5.106}$$

Moreover $A_n = \sum_{j=0}^{k'_n} f(j/k_n)\Delta^n_{I^m_n-j} M$, say, can be written as $\delta^n \star (\underline{\mu}(q) - \underline{\nu}(q))_{T_m} - \delta^n \star (\underline{\mu}(q) - \underline{\nu}(q))_{T_m - 2u_n}$ for some predictable function δ^n satisfying $|\delta^n(t,z)| \leq K\gamma(z)$. Then a well known result (see e.g. Lemma 5.12 of [10], used with $2u_n$ instead of Δ_n and $\eta = \sqrt{u_n}$, and relative to the filtration $(\mathcal{H}_t)$) says that $A_n/\sqrt{u_n} \stackrel{\mathbb{P}}{\longrightarrow} 0$. The same holds if we take the indices $I^n_l + j$ instead of $I^n_m - j$, and thus

$$\frac{1}{\Delta_n^{1/4}} \sum_{j=0}^{k'_n} f(j/k_n)\Delta^n_{I^m_n-j} M \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0, \qquad \frac{1}{\Delta_n^{1/4}} \sum_{j=2}^{k'_n} f(j/k_n)\Delta^n_{I^m_n+j} M \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0. \tag{5.107}$$

2) We put for $i \geq 0$ and any weight function g:

$$\left.\begin{array}{ll} G(g)^n_{i-} = \frac{1}{k_n} \sum_{j=i+2}^{k_n-1} \Big\{g^n_j\Big\}^{p-1} g^n_{j-i-1}, & G(g)^n_0 = \frac{1}{k_n} \sum_{j=1}^{k_n-1} \Big\{g^n_j\Big\}^{p-1} g^n_j, \\ G(g)^n_{i+} = \frac{1}{k_n} \sum_{j=1}^{k_n-i} \Big\{g^n_j\Big\}^{p-1} g^n_{j+i-1}, & \\ \overline{G}(g)^n_{i-} = \sum_{j=i+1}^{k_n-1} \Big\{g^n_j\Big\}^{p-1} g'^n_{j-i}, & \overline{G}(g)^n_{i+} = \sum_{j=1}^{k_n-i} \Big\{g^n_j\Big\}^{p-1} g'^n_{j+i} \end{array}\right\} \tag{5.108}$$

Then a (tedious) computation shows that

$$\eta(q,g)^n_m \;=\; \frac{1}{\Delta_n^{1/4}} \Big(\sum_{i=0}^{k_n-3} G(g)^n_{i-}\ \Delta^n_{I^n_m-i} X' + G(g)^n_0\ \Delta^n_{I^n_m+1} X' + \sum_{i=2}^{k_n-1} G(g)^n_{i+}\ \Delta^n_{I^n_m+i} X' \Big)$$

$$-\frac{1}{\Delta_n^{1/4}k_n}\,\Big(\sum_{i=0}^{k_n-2}\overline{G}(g)^n_{i-}\chi^n_{I^n_m-i}+\sum_{i=1}^{k_n-1}\overline{G}(g)^n_{i+}\chi^n_{I^n_m+i}\Big).$$

Moreover if

$$\left.\begin{array}{llllll}H_-(g,t) &=& \int_t^1\{g(s)\}^{p-1}g(s-t)ds, & H_+(g,t) &=& \int_0^{1-t}\{g(s)\}^{p-1}g(s+t)ds,\\ \overline{H}_-(g,t) &=& \int_t^1\{g(s)\}^{p-1}g'(s-t)ds, & \overline{H}_+(g,t) &=& \int_0^{1-t}\{g(s)\}^{p-1}g'(s+t)ds,\end{array}\right\}$$

we have

$$G(g)^n_{i\pm}=H_\pm\Big(\frac{i}{k_n},g\Big)+\mathrm{O}_u(\sqrt{\Delta_n}),\qquad \overline{G}(g)^n_{i\pm}=\overline{H}_\pm\Big(\frac{i}{k_n},g\Big)+\mathrm{O}_u(\sqrt{\Delta_n}). \tag{5.109}$$

Using $|G^n_0|\leq K$ and $X'=M+X''$, (5.105), (5.106), (5.107) and (SN-2p), we deduce

$$\eta(q,g)^n_m\;=\;\sigma_{T_m-}\,\rho(g)^n_{m-}+\sigma_{T_m}\,\rho(g)^n_{m+}+\overline{\rho}(g)^n_{m-}+\overline{\rho}(g)^n_{m+}+\mathrm{o}_{Pu}(1), \tag{5.110}$$

where

$$\begin{array}{lll}\rho(g)^n_{m-} &=& \frac{1}{\Delta_n^{1/4}}\sum_{i=0}^{k_n-3}H_-\Big(\frac{i}{k_n},g\Big)\,\Delta^n_{I^n_m-i}W\\ \rho(g)^n_{m+} &=& \frac{1}{\Delta_n^{1/4}}\sum_{i=2}^{k_n-1}H_-\Big(\frac{i}{k_n},g\Big)\,\Delta^n_{I^n_m+i}W\\ \overline{\rho}(g)^n_{m-} &=& -\frac{1}{\Delta_n^{1/4}k_n}\sum_{i=0}^{k_n-2}\overline{H}_-\Big(\frac{i}{k_n},g\Big)\,\chi^n_{I^n_m-i}\\ \overline{\rho}(g)^n_{m+} &=& -\frac{1}{\Delta_n^{1/4}k_n}\sum_{i=1}^{k_n-1}\overline{H}_+\Big(\frac{i}{k_n},g\Big)\,\chi^n_{I^n_m+i}.\end{array}$$

3) At this stage, we use the same ideas than in Lemma 5.1. We denote by $\rho^n_{m\pm}$ and $\overline{\rho}^n_{m\pm}$ the d-dimensional variables with components $\rho(g_i)^n_{m\pm}$ and $\overline{\rho}(g_i)^n_{m\pm}$. First we argue at $\omega^{(0)}\in\Omega^{(0)}$ fixed. Under $\mathbb{Q}=\mathbb{Q}(\omega^{(0)},.)$ the variables $\overline{\rho}^n_{m-}$ and $\overline{\rho}^n_{m+}$ are independent one from the other, and also when m varies in P'_p as soon as n is large enough (so that $\omega^{(0)}\in\Omega_n(t,q)$). Moreover they are sums, normalized by $1/\Delta_n^{1/4}k_n$, of (approximately) k_n centered independent variables with a bounded fourth moment, and their covariance matrices are (approximately again) $\alpha^2_{T_m-}/\theta$ and $\alpha^2_{T_m}/\theta$ times Riemann approximations of the integrals defining $\overline{\Psi}_{p-}$ and $\overline{\Psi}_{p+}$ respectively. Then we prove exactly as for (5.27) (only the finite-dimensional convergence is needed here) that under $\mathbb{Q}$,

$$\Big(\overline{\rho}^n_{m-},\overline{\rho}^n_{m+}\Big)_{m\in P'_q}\;\stackrel{\mathcal{L}}{\longrightarrow}\;\Big(\frac{\alpha_{T_m-}(\omega^{(0)})}{\sqrt{\theta}}\;\overline{U}_{m-},\frac{\alpha_{T_m}(\omega^{(0)})}{\sqrt{\theta}}\;\overline{U}_{m+}\Big)_{m\in P'_q}. \tag{5.111}$$

(In fact we prove the convergence in (5.111) for each m first, and then we use the fact that the variables in the left side are independent for different values of m, under $\mathbb{Q}$, and as soon as $\omega^{(0)}\in\Omega_n(t,q)$.)

Second, exactly as for (5.36) (or as above for (5.111)), we get

$$\Big(\rho^n_{m-},\rho^n_{m+}\Big)_{m\in P'_q}\;\stackrel{\mathcal{L}}{\longrightarrow}\;\Big(\sqrt{\theta}\;U_{m-},\sqrt{\theta}\;U_{m+}\Big)_{m\in P'_q}. \tag{5.112}$$

Note that $(U_{m\pm},\overline{U}_{m\pm})$ are as described after (4.5). Then, as in Steps 2 and 4 of the proof of Lemma 5.1, we deduce from the convergences (5.111) under $\mathbb{Q}(\omega^{(0)},.)$ and (5.112) under $\mathbb{P}^{(0)}$, and also from (5.110), that $(\rho^n_{m-}, \rho^n_{m+}, \overline{\rho}^n_{m-}, \overline{\rho}^n_{m+})_{m \in P'_q}$ converges in law to $(\sqrt{\theta}\, U_{m-}, \sqrt{\theta}\, U_{m+}, \alpha_{T_m-}\overline{U}_{m-}/\sqrt{\theta}, \alpha_{T_m}\overline{U}_{m+}/\sqrt{\theta})_{m \in P'_q}$. Moreover this convergence in law is indeed a stable convergence, by exactly the same argument than for the same result in [8]. Finally by (5.104) and (5.110) and the definition of the stable convergence in law, we obtain the claim. □

Proposition 5.14 *If $q \geq 1$ and $t \geq 0$ are fixed, and in the same setting as before, we have (with $\stackrel{\mathcal{L}-(s)}{\longrightarrow}$ denoting the stable convergence in law):*

$$(Y(q, g_l)^n_t)_{1 \leq l \leq d} \quad \stackrel{\mathcal{L}-(s)}{\longrightarrow} \quad U(p, q)_t, \tag{5.113}$$

where $U(p, q)$ is the d-dimensional process associated with the functions (g_l) by (4.6), except that the sums is taken over $m \in P_q$ only.

Proof. With $\overline{g}_l(p)_n = \sum_{i=1}^{k_n} |g_l(i/k_n)|^p$, we have $|\overline{g}_l(p)_n - k_n \overline{g}_l(p)| \leq K$ by (2.10). Then, with the notation (5.103), a Taylor expansion plus the property $|\Delta X_{T_m}| \leq K$ yield

$$\Big|\zeta(q, g_l)^n_m - p\{\Delta X_{T_m}\}^{p-1} \eta(q, g_l)^n_m\Big| \leq$$
$$K\Delta_n^{1/4} \sum_{j=1}^{k_n-1} \Big((\overline{Z'^q}(g_l)^n_{I^n_m - j+1})^p + (\overline{Z'^q}(g_l)^n_{I^n_m - j+1})^2 \Big) + K\Delta_n^{1/4}.$$

Now if we apply (5.105) we see that the expectation of the sum in the right side above is bounded (recall $p > 3$). Therefore Lemma 5.13 implies

$$(\zeta(q, g_l)^n_m)_{1 \leq l \leq d, m \in P_q} \quad \stackrel{\mathcal{L}-(s)}{\longrightarrow} \quad \Big(p\{\Delta X_{T_m}\}^{p-1} \eta_m\Big)_{m \in P_q},$$

and (5.113) readily follows. □

5.15 The processes $\widetilde{V}^*(X'^q, g, p)^n$.

The aim of this subsection is to prove the following:

Proposition 5.15 *Under the same assumptions as before, and for all $\varepsilon > 0$, we have*

$$\lim_{q \to \infty} \limsup_n \mathbb{P}\Big(|\widetilde{V}^*(X'^q, g, p)^n_t| > \varepsilon\Big) = 0. \tag{5.114}$$

The proof is based on the following easy property (g is fixed throughout):

$$\widetilde{V}^*(X'^q, g, p)^n_t = \frac{1}{\Delta_n^{1/4} k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} \Gamma(q)^n_i + R(q)^n_t,$$

where

$$\Gamma(q)^n_i \;=\; |\overline{Z'^q}(g)^n_i|^p - \sum_{j=1}^{k_n-1} |g^n_j|^p\ \Delta^n_{i+j}\Sigma(q), \qquad \Sigma(q)_t = \sum_{s\leq t} |\Delta X'(q)_s|^p,$$

and where the remainder term satisfies

$$|R(q)^n_t| \;\leq\; \frac{K}{\Delta_n^{1/4}}\,\Big(\Sigma(q)_t\Big(\frac{\overline{g}(p)_n}{k_n} - \overline{g}(p)\Big) + \Sigma(q)_{u_n} + (\Sigma(q)_t - \Sigma(q)_{t-2u_n})\Big).$$

Lemma 5.16 *We can find a sequence η_q going to 0 as $q\to\infty$, with the following property: for any $q\geq 1$ and $i\geq 1$ we have a decomposition $\Gamma(q)^n_i = \Gamma'(q)^n_i + \Gamma''(q)^n_i$, where both $\Gamma'(q)^n_i$ and $\Gamma''(q)^n_i$ are $\mathcal{F}^n_{i+k_n}$-measurable and*

$$\left.\begin{array}{l} \mathbb{E}(|\Gamma'(q)^n_i|) \;\leq\; K_q\Delta_n^{1\wedge(p/4)} + \eta_q\Delta_n^{3/4}, \\ \mathbb{E}(\Gamma''(q)^n_i \mid \mathcal{F}^n_i) = 0, \\ \mathbb{E}(|\Gamma''(q)^n_i|^2) \;\leq\; K_q\Delta_n^{3/2} + \eta_q\Delta_n. \end{array}\right\} \tag{5.115}$$

Proof. 1) Let us fix i, q and n, which will be left out in most notation below. We consider the filtration $\mathcal{F}'_t = \mathcal{F}_{i\Delta_n+t}$, and associated with this filtration the Brownian motion $W'_t = W_{i\Delta_n+t} - W'_{i\Delta_n}$ and the Poisson random measure $\underline{\mu}'((0,t]\times A) = \underline{\mu}((i\Delta_n, i\Delta_n+t]\times A)$, whose compensator is still $\underline{\nu}$. Recalling (5.99), we set $b'_t = b^q_{i\Delta_n+t}$, and observe that $|b'_t|\leq Kq$ because b_t is bounded and $\int_{\{z:\,\gamma(z)>1/q\}} |\delta(t,z)|\lambda(dz) \leq q\int\gamma^2(z)\lambda(dz)$. With all this notation and (5.98), we have

$$X'^q_{i\Delta_n+t} \;=\; X'^q_{i\Delta_n} + \int_0^t b'_s ds + \int_0^t \sigma'_s dW'_s + (\delta' 1_{\{\gamma\leq 1/q\}})\star(\underline{\mu}'-\underline{\nu})_t. \tag{5.116}$$

Recalling g_n in (5.3), we then set

$$Y_t \;=\; \int_0^t b'_s\, g_n(s)\, ds + \int_0^t \sigma'_s\, g_n(s)\, dW'_s + (\delta' g_n 1_{\{\gamma\leq 1/q\}})\star(\underline{\mu}'-\underline{\nu})_t - \sum_{j=1}^{[t/\Delta_n]} g'^n_j \chi^n_{i+j-1}. \tag{5.117}$$

Then by (2.11) and (5.3) and (5.117), we see that $\overline{Z'(g)}^n_i = Y_{u_n}$. If we further set

$$Y'_t \;=\; (|\delta' g_n|^p\, 1_{\{\gamma\leq 1/q\}})\star\underline{\mu}'_t,$$

we obtain $Y'_{u_n} = \sum_{j=1}^{k_n-1} |g^n_j|^p\,\Delta^n_{i+j}\Sigma(q)$. Hence

$$\Gamma(q)^n_i \;=\; |Y_{u_n}|^p - Y'_{u_n}. \tag{5.118}$$

For simplicity of notation we write $f(x) = |x|^p$, which is C^2 (recall $p>3$), and we associate the functions

$$\begin{array}{rcl} F(x,y) &=& f(x+y) - f(x) - f'(x)y \\ G(x,y) &=& f(x+y) - f(x) - f(y) \\ H(x,y) &=& F(x,y) - f(y), \end{array}$$

which clearly satisfy

$$\left.\begin{array}{lcl} |F(x,y)| &\leq& K(|y|^p + y^2|x|^{p-2}), \\ |G(x,y)| &\leq& K(|x|\,|y|^{p-1} + |y|\,|x|^{p-1}), \\ |H(x,y)| &\leq& K(|x|\,|y|^{p-1} + y^2|x|^{p-2}). \end{array}\right\} \tag{5.119}$$

Then we apply Itô's formula and use (5.117) to obtain

$$|Y_t|^p - Y'_t \;=\; A_t + A'_t + N_t + N'_t,$$

where

$$A_t \;=\; \int_0^t a_s ds, \qquad A'_t \;=\; \sum_{j=1}^{[t/\Delta_n]} F(Y_{j\Delta_n-}, -g_j'^n \chi^n_{i+j-1}),$$

$$a_t \;=\; f'(Y_t) g_n(t) b'_t + \frac{1}{2}\, f''(Y_t) g_n(t)^2 \sigma_t'^2 + \int_{\{z:\gamma(z)\leq 1/q\}} H(Y_t, g_n(t)\delta'(t,z))\,\lambda(dz),$$

and N_t is a martingale with angle bracket $C = \langle N, N\rangle$ given by

$$C_t \;=\; \int_0^t c_s ds, \qquad c_t \;=\; f'(Y_t)^2 g_n(t)^2 \sigma_t'^2 + \int_{\{z:\gamma(z)\leq 1/q\}} G(Y_t, g_n(t)\delta'(t,z))^2\,\lambda(dz),$$

and finally

$$N'_t \;=\; -\sum_{j=1}^{[t/\Delta_n]} f'(Y_{j\Delta_n-})\, g_j'^n \chi^n_{i+j-1}$$

which is another martingale (because the χ_t's are centered) with square bracket

$$C'_t \;:=\; [N', N']_t \;=\; \sum_{j=1}^{[t/\Delta_n]} f'(Y_{j\Delta_n-})^2\, (g_j'^n)^2 (\chi^n_{i+j-1})^2.$$

2) The decomposition $\Gamma(q)^n_i = \Gamma'(q)^n_i + \Gamma''(q)^n_i$ is given by:

$$\Gamma'(q)^n_i \;=\; A_{u_n} + A'_{u_n}, \qquad \Gamma''(q)^n_i \;=\; N_{u_n} + N'_{u_n}.$$

The $\mathcal{F}^n_{i+k_n}$-measurability of $\Gamma'(q)^n_i$ and $\Gamma''(q)^n_i$ is obvious, as is the second part of (5.115). The rest of (5.115) will readily follows if we can find a sequence $\eta_q \to 0$ such that

$$\left.\begin{array}{ll} \mathbb{E}(|A_{u_n}|) \;\leq\; K_q \Delta_n^{1\wedge(p/4)} + \eta_q \Delta_n^{3/4}, & \mathbb{E}(|A'_{u_n}|) \;\leq\; K_q \Delta_n, \\ \mathbb{E}(C_{u_n}) \;\leq\; K_q \Delta_n^{3/2} + \eta_q \Delta_n, & \mathbb{E}(C'_{u_n}) \;\leq\; K_q \Delta_n^{3/2} + \eta_q \Delta_n. \end{array}\right\} \tag{5.120}$$

For this we need moment estimates for Y_t, as defined by (5.117). Recall that $|b'| \leq Kq$ and $|\sigma'| \leq K$, and also that $|g_n| \leq K$ and $|\delta(.,z)| \leq \gamma(z)$, and observe that $\eta'_q = \int_{\{z:\ \gamma(z)\leq 1/q\}} \gamma(z)^2\,\lambda(dz)$ goes to 0 as $q \to \infty$. In view of (SN-2p) for $q = 2p$, and since $|g_j'^n| \leq K\sqrt{\Delta_n}$, and using the Burholder-Davis-Gundy inequality for the martingale which is the last term in (5.117), we see that for all $r \in (0, 2p]$,

$$\mathbb{E}(|Y_t|^r) \;\leq\; Kq^r t^r + K t^{r/2} + K\eta'_q t^{1\wedge(r/2)}. \tag{5.121}$$

By $f(x) = |x|^p$ and (5.119), plus $p > 3$, we see that $|a_t| \leq K(q|Y_t|^{p-1} + |Y_t|^{p-2} + \eta'_q|Y_t|)$. Therefore (5.121) yields $\mathbb{E}(|a_t|) \leq Kq^p t^{1\wedge(p/2-1)} + K\eta'_q t^{1/2}$. In a similar way $c_t \leq K|Y_t|^{2p-2} + K\eta'_q Y_t^2$, hence $\mathbb{E}(c_t) \leq K(q^{2p-2}t^2 + \eta'_q t)$. Then the estimate for A_{u_n} and C_{u_n} in (5.120) follows upon taking $\eta_q = K\eta'_q$ for a K large enough.

For the same reasons, plus (SN-2p), the jth summand in the definition of A'_t has expectation smaller than $K\Delta_n^{p/2} + K\Delta_n(q^{p-2}(j\Delta_n)^{p-2} + (j\Delta_n)^{p/2-1} + \eta'_q(j\Delta_n))$, whereas the jth summand in the expression for C'_t has expectation smaller than $K\Delta_n(q^{2p-2}(j\Delta_n)^{2p-2} + (j\Delta_n)^{p-1} + \eta'_q(j\Delta_n))$. The two other estimates in (5.120) follow. □

Proof of Proposition 5.15. In view of (2.10) and of the fact that $\mathbb{E}(\Sigma(q)_{s+u} - \Sigma(q)_s) \leq K_q u$, we deduce that $R(q)^n_t \xrightarrow{\mathbb{P}} 0$ for all q. Hence it remains to prove that

$$\lim_{q\to\infty} \limsup_n \mathbb{P}\Big(\frac{1}{\Delta_n^{1/4} k_n} \Big| \sum_{i=0}^{[t/\Delta_n]-k_n} \Gamma(q)^n_i \Big| > \varepsilon\Big) = 0. \tag{5.122}$$

We set, with t fixed and the notation of the previous lemma:

$$L'(q)_n = \frac{1}{\Delta_n^{1/4} k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} \Gamma'(q)^n_i, \qquad L''(q)_n = \frac{1}{\Delta_n^{1/4} k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} \Gamma''(q)^n_i.$$

The first property in (5.115) yields $\mathbb{E}(|L'(q)_n|) \leq K_q \Delta_n^{(1/4)\wedge(p/4-3/4)} + \eta_q$, hence since $p > 3$:

$$\lim_{q\to\infty} \limsup_n \mathbb{E}(|L'_n|) = 0.$$

Next, the properties of $\Gamma''(q)^n_i$ in the Lemma 5.16 imply that $|\mathbb{E}(\Gamma''(q)^n_i \Gamma''(q)^n_j)|$ vanishes when $|j-i| > k_n$, and otherwise is smaller than $K_q\Delta_n^{3/2} + \eta_q\Delta_n$. Hence $\mathbb{E}((L''_n)^2) \leq K_q\Delta_n^{1/2} + \eta_q$, which yields

$$\lim_{q\to\infty} \limsup_n \mathbb{E}(|L''_n|^2) = 0.$$

Putting these two results together immediately yields (5.122). □

5.16 Proof of Theorem 4.4.

We start with the first claim, which easily follows from what precedes. The family (g_l) of weight functions is fixed. We observe that $U(p,q)_t$ converges to $U(p)_t$ in probability as $q \to \infty$. Then the result is a trivial consequence of (5.101) and Propositions 5.14 and 5.15.

It remains to prove the second claim, and we show that it can be reduced to the first claim. We take $p \geq 4$ an even integer, and it is enough to prove that $\frac{1}{\Delta_n^{1/4} k_n}(\overline{V}^*(g,p)^n - \widetilde{V}^*(g,p)^n) \xrightarrow{\text{u.c.p.}} 0$ for any weight function g. To see this we observe that the difference

$\overline{V}^*(g,p)^n_t - \widetilde{V}^*(g,p)^n_t$ is a linear combination of the processes (we omit to mention the function g below):

$$\frac{1}{\Delta_n^{1/4} k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} (\overline{Z}^n_i)^{p-2r}\ (\widehat{Z}^n_i)^r$$

for $r = 1, \cdots, p/2$. So it enough to prove that, for some $\rho > 3/4$ and all $r = 1, \cdots, p/2$,

$$\mathbb{E}\Big((\overline{Z}^n_i)^{p-2r}\ (\widehat{Z}^n_i)^r\Big) \ \leq\ K\Delta_n^\rho. \tag{5.123}$$

(SK) yields $\mathbb{E}(|\Delta^n_i X|^v \mid \mathcal{F}^n_{i-1}) \leq K\Delta_n^{(v\wedge 2)/2}$, so when $v \geq 1$, and because of (SN-4p) for the last estimate, we have

$$\left.\begin{array}{l} \mathbb{E}(|\overline{X}^n_i|^v \mid \mathcal{F}^n_i) \ \leq\ K_v\Delta_n^{(v/4)\wedge(1/2)}, \qquad \mathbb{E}(|\widehat{X}^n_i|^v \mid \mathcal{F}^n_i) \ \leq\ K_v\Delta_n^{1+v/2}, \\ v \leq 2p \ \Rightarrow\ \mathbb{E}\Big(\Big|\sum_{j=1}^{k_n}(g^n_j)^2\Delta^n_{i+j}X\Delta^n_{i+j}\chi\Big|^v \mid \mathcal{F}^n_i\Big) \ \leq\ K_v\Delta_n^{v/2+(v/2)\wedge 1}. \end{array}\right\} \tag{5.124}$$

Now $(\overline{Z}^n_i)^{p-2r}\ (\widehat{Z}^n_i)^r$ is a linear combination of terms of the form

$$a(u,v,w,s,t)^n_i \ =\ (\overline{X}^n_i)^u\ (\widehat{X}^n_i)^v\ (\overline{\chi}^n_i)^w\ (\widehat{\chi}^n_i)^s\ \Big(\sum_{j=1}^{k_n}(g^n_j)^2\Delta^n_{i+j}X\Delta^n_{i+j}\chi\Big)^t,$$

where u,v,w,s,t are integers with $u+w = p-2r$ and $v+s+t = r$. Using Hölder inequality, and taking advantage of (5.124) and of $\mathbb{E}(|\overline{\chi}^n_i|^l) \leq K_r\Delta_n^{l/4}$ and $\mathbb{E}(|\widehat{\chi}^n_i|^l) \leq K_r\Delta_n^{l/2}$, we see that for all $u', v', w', s', t' \geq 0$ such that $u' + v' + w' + s' + t' = 1$ and $u' = 0$ (resp. $v' = 0$, $w' = 0$, $s' = 0$, $t' = 0$) if and only if $u = 0$ (resp. $v = 0$, $w = 0$, $s = 0$, $t = 0$), and also $\frac{w}{w'} \vee \frac{2s}{s'} \vee \frac{t}{t'} \leq 2p$ (which is possible because $w + 2s + t \leq p$), we have $\mathbb{E}(|a(u,v,w,s,t)^n_i|) \leq K\Delta_n^\rho$, where

$$\rho \ =\ \frac{u}{4}\wedge\frac{u'}{2} + v'1_{v>0} + \frac{v}{2} + \frac{w}{4} + \frac{s}{2} + \frac{t}{2} + t'\wedge\frac{t}{2} \ =\ \frac{r}{2} + \frac{w}{4} + \frac{u}{4}\wedge\frac{u'}{2} + v'1_{v>0} + t'\wedge\frac{t}{2}.$$

Then $\rho > 3/4$ as soon as $r \geq 2$, or $r = 1$ and $w \geq 1$. The only other case is $r = 1$ and $w = 0$, so $u = p - 2 \geq 2$ and we have

$$\rho \ =\ \frac{1}{2} + \frac{u'}{2} + v'1_{v>0} + t'\wedge\frac{t}{2}.$$

Then we have three sub-cases:

(1) $v = 1$, hence $t = t' = s = s' = w' = 0$ and $\rho = \frac{1+u'}{2} + v'$ with the condition $u' + v' = 1$, so $u' = v' = 1/2$ yields $\rho > 3/4$;

(2) $s = 1$, hence $t = t' = v = v' = w' = 0$ and $\rho = \frac{1+u'}{2}$ with the conditions $u' + s' = 1$ and $s'p \geq 1$, so $s' = 1/3$ yields $\rho > 3/4$;

(3) $t = 1$, hence $v = v' = s = s' = w' = 0$ and $\rho = \frac{1}{2} + \frac{u'}{2} + t'\wedge\frac{1}{2}$ with the condition $u' + t'11$ and $2t'p \geq 1$, so $u' = t' = 1/2$ yield $\rho > 3/4$.

Hence in all cases (5.123) holds with some $\rho > 3/4$, and the proof is finished.

5.17 Proof of Theorem 4.5.

Here again the proof will be divided into several steps, and before proceeding we observe two preliminary facts. First, that $\overline{\mu}_4(g,g;\eta,\zeta)$ takes the form (4.9) results from a tedious but elementary calculation. Second, by localization we may assume (SN-4) and (SH).

We omit to mention the function g in $\overline{Y}_i^n$ and $\widehat{Y}_i^n$. We generally use the notation of the proof of Theorem 4.4, and in particular the stopping times $T(q,m)$ and T_m introduced in Subsection 5.13, the processes of (5.98), the sets $\Omega_n(t,q)$ satisfying (5.100), and the (random) integers I_m^n. In the sequel, we will vary the process X (but not the noise process χ), so the process $\overline{V}^n$ of (4.7) will be denoted by $\overline{V}(X)^n$. We also write $U'(\sigma)_t$ and $U(2,\sigma,\delta)_t$ for the two terms in (4.8), and $\overline{U}(\sigma,\delta)_t$ for their sum, to emphasize their dependency on the process σ and the function δ (through the jumps of X, for the latter).

Step 1. This step is devoted to proving the result when X satisfies the following two assumptions, in addition to (SH):

$$\gamma(z)\leq 1/q \quad\Rightarrow\quad \delta(\omega,t,z)=0, \tag{5.125}$$

$$\left.\begin{array}{lll} b'_s &=& b_s-\int\delta(t,x)1_{\{|\delta(y,z)|\leq 1\}}\,\lambda(dz) \;=\; \sum_{r\geq 1} b_{S_r}1_{[S_r,S_{r+1})}(t),\\ \sigma_s &=& \sum_{r\geq 1}\sigma_{S_r}1_{[S_r,S_{r+1})}(t), \end{array}\right\} \tag{5.126}$$

for some $q\geq 1$, and for a sequence of stopping times S_r, increasing strictly to ∞ and with $S_0=0$.

1) Under (5.125) and (5.126) we have $X_t^q=\sum_{s\leq t}\Delta X_s$, and $X'=X'^q$ is the continuous process given by the right side of (2.14), with b' instead of b. Similar to (5.101), we have on $\Omega_n(t,q)$:

$$\overline{V}(X)_t^n \;=\; \overline{V}(X')_t^n+Y_t^n-\frac{1}{2}\,Y_t'^n, \tag{5.127}$$

$$\begin{array}{lll} Y_t^n &=& \sum_{m\in P_q:\,T_m\leq t}\zeta_m^n, \qquad Y_t'^n \;=\; \sum_{m\in P_q:\,T_m\leq t}\zeta_m'^n,\\ \zeta_m^n &=& \frac{1}{\Delta_n^{1/4}k_n}\Big(\sum_{j=1}^{k_n-1}\Big(\Big|\overline{(X'+\chi)}_{I_m^n+1-j}^n+g_j^n\Delta X_{T_m}\Big|^2\\ && \qquad -\Big|\overline{(X'+\chi)}_{I_m^n+1-j}^n\Big|^2-\Big|g_j^n\Delta X_{T_m}\Big|^2\Big)+(\overline{g}(2)_n-k_n\overline{g}(2))|\Delta X_{T_m}|^2\Big),\\ \zeta_m'^n &=& \frac{1}{\Delta_n^{1/4}k_n}\sum_{j=1}^{k_n}(g_j'^n)^2\Big((\Delta X_{T_m})^2+2\Delta X_{T_m}\Delta_{I_m^n+1-j}^n(X'+\chi)\Big). \end{array}$$

Let $(\mathcal{H}_t)$ be the filtration defined in the proof of Lemma 5.13, and associated with our q. The same argument than in that lemma shows $\mathbb{E}(|\Delta_{I_m^n+1-j}^n(X'+\chi)|\mid\mathcal{H}_0)\leq K$, whereas $|\Delta X_{T_m}|\leq K$ by (SH). It follows that $\mathbb{E}(|\zeta_m'^n|)\leq K\Delta_n^{3/4}$, hence

$$Y_t'^n \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0. \tag{5.128}$$

2) Next we prove the stable convergence $\overline{V}(X')^n \stackrel{\mathcal{L}-(s)}{\longrightarrow} U'(\sigma)$ (in the functional sense). This looks the same as Theorem 4.1 for $p=2$, however we do not have (K) here. Now, a look at the proof of this theorem shows that (K) (instead of (H)) is used in two places

only, namely in Lemmas 5.8 and 5.9. Here, the proof of Lemma 5.9 goes through in an obvious way under (5.126), and we are left to showing that Lemma 5.8 holds, which we do for the first convergence in (5.75) only.

The process in the left side of (5.75), say at time t (and for $p = 2$), is the sum $\sum_{i=1}^{i_n(m,t)} \sum_{j=0}^{mk_n-1} \theta^n_{i,j}$, where

$$\theta^n_{i,j} \;=\; \Delta_n^{1/4}\,\mathbb{E}\Big(\phi(X'+\chi,g,2)_{I(m,n,i),j} - \phi(g,2)_{I(m,n,i)+j} \mid \mathcal{F}(m)^n_{i-1}\Big).$$

Let J_n be the set of all i such that $(i-1)(m+1)u_n < S_r \leq imu_n$ for some $r \geq 1$ (that is, the indices of those "big blocks" that contain at least one S_r), and consider the two processes

$$A^n_t \;=\; \sum_{i\in\{1,\cdots,i_n(m,t)\}\cap J_n} \sum_{j=0}^{mk_n-1} \theta^n_{i,j}, \quad A'^n_t \;=\; \sum_{i\in\{1,\cdots,i_n(m,t)\}\cap J_n^c} \sum_{j=0}^{mk_n-1} \theta^n_{i,j}.$$

Applying (5.51) with $u = 1$ (recall (SN-4)), we obtain $\mathbb{E}(|\theta^n_{i,j}|) \leq K\Delta_n^{3/4}$. Therefore $\mathbb{E}(\sup_{s\leq t\wedge S_r} |A'^n_s|)$ is obviously smaller than $Kr\Delta_n^{1/4}$ and, since $S_r \to \infty$ as $r \to \infty$, we deduce $A^n \stackrel{\text{u.c.p.}}{\longrightarrow} 0$ and it remains to prove the same for A'^n.

For this, and reproducing the proof of Lemma 5.8, we observe that (K) comes into play only to decompose the variables $\overline{\lambda}^n_{i+j}$ as $\xi^n_{i,j} + \xi'^n_{i,j}$. We easily deduce from (5.126) that when $i \notin J_n$ such a decomposition holds with $\xi^n_{i,j} = 0$ and $\xi'^n_{i,j} = b'_{i\Delta_n}\Delta_n$. Then the original proof goes through to show that $A'^n \stackrel{\text{u.c.p.}}{\longrightarrow} 0$, and thus Lemma 5.8 holds here.

3) We have $\overline{V}(X')^n \stackrel{\mathcal{L}-(s)}{\longrightarrow} U'(\sigma)$ from what precedes, and this gives the result (functional stable convergence in law) when X is continuous, in addition to satisfying (5.126). When X has jumps, the proof of Proposition 5.14 is valid when $p = 2$ (it only supposes the C^2 property of $x \mapsto |x|^p$), so we have $Y^n_t \stackrel{\mathcal{L}-(s)}{\longrightarrow} U(2,\sigma,\delta)_t$ (for t fixed, this is not a functional convergence).

Now, exactly as in the proof of Lemma 5.8 in [10], one can show that we have the *joint* stable convergence in law in Proposition 5.11 and Lemma 5.13, which results in the joint convergence

$$(\overline{V}(X')^n_t, Y^n_t) \;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\; (U'(\sigma)_t, U(2,\sigma,\delta)_t).$$

Then we easily deduce from (5.100), (5.127) and (5.128) that $\overline{V}(X)^n_t \stackrel{\mathcal{L}-(s)}{\longrightarrow} \overline{U}(\sigma,\delta)_t$.

Step 2. We turn to the general case, and we begin by constructing an approximation of X satisfying (5.125) and (5.126).

For $q \geq 1$ we recall the process b^q of (5.99). If further $r \geq 1$ we denote by $S(q,r)_r$ the strictly increasing rearrangement of the points in the set $\{k2^{-r} : k \geq 0\} \cup \{T(q,m) : m \geq 1\}$. By a classical density argument there are adapted processes $b(q,r)$ and $\sigma(q,r)$ with the following properties: they are bounded by the same bounds as b^q and σ respectively, constant over each interval $[(k-1)2^{-r}, k2^{-r})$ for $b(q,r)$ and each interval $[S(q,r)_{k-1}, S(q,r)_q)$

for σ, and such that for all $q,m\geq 1$ and $t\geq 0$:

$$\left.\begin{array}{rcl}\varepsilon(q,r)_t &=& \mathbb{E}\Big(\int_0^t(|b(q,r)_s-b^q_s|^2+|\sigma(q,r)_s-\sigma_s|^2)ds\Big)\ \to\ 0\\ \sigma(q,r)_{T(q,m)} &=& \sigma_{T(q,m)},\quad \sigma(q,r)_{T(q,m)-}\ \to\ \sigma_{T(q,m)-}\end{array}\right\}\tag{5.129}$$

as $r\to\infty$ (we use here the càdlàg property of σ). Next, we introduce a family of processes (recall (5.98) for M^q):

$$\left.\begin{array}{l}X(q,r)_t\ =\ X_0+\int_0^t b(q,r)_s ds+\int_0^t\sigma(q,r)_s dW_s+(\delta 1_{\{\gamma>1/q\}})*\underline{\mu}\\ X'(q,r)_t=X_t-X(q,r)_t=\int_0^t(b^q_s-b(q,r)_s)ds+\int_0^t(\sigma_s-\sigma(q,r)_s)dW_s+M^q_t.\end{array}\right\}\tag{5.130}$$

Finally, another notation will be

$$\left.\begin{array}{rcl}\varepsilon(q,r)^n_i &=& \mathbb{E}\Big(\int_{i\Delta_n}^{i\Delta_n+u_n}(|b(q,r)_s-b^q_s|^2+|\sigma(q,r)_s-\sigma_s|^2)ds\Big)\\ \varepsilon_q &=& \int_{\{z:\gamma(z)\leq 1/q\}}\gamma(z)^2\,\lambda(dz).\end{array}\right\}\tag{5.131}$$

By construction $X(q,r)$ satisfies (5.125) and (5.126), so Step 1 gives us that for any t and $q,r\geq 1$,

$$\overline{V}(X(q,r))^n_t\ \stackrel{\mathcal{L}-(s)}{\longrightarrow}\ \overline{U}(\sigma(q,r),\delta(q))_t,\tag{5.132}$$

where $\delta(q)(\omega,t,z)=\delta(\omega,t,z)\,1_{\{\gamma(z)>1/q\}}$, and the convergence even holds in the functional sense when X is continuous.

Note that, since σ and $\sigma(q,r)$ and α are uniformly bounded and the function $\overline{\mu}_4$ in (4.9) is locally Lipschitz in (η,ζ), we have

$$\mathbb{E}\Big(\sup_{s\leq t}|U'(\sigma)_s-U'(\sigma(q,r))_s|^2\Big)\ \leq\ K\mathbb{E}\Big(\int_0^t|\sigma_s-\sigma(q,r)_s|^2\,ds\Big)\ \leq\ K\varepsilon(q,r)_t.$$

On the other hand, since $\delta(q)$ is bounded, it follows from (4.6) that

$$\begin{array}{l}\mathbb{E}\Big(\sup_{s\leq t}|U(2,\sigma,\delta(q))_s-U(2,\sigma(q,r),\delta(q))_s|^2\Big)\\ \qquad\leq\ K\mathbb{E}\Big(\sum_{m\geq 1}(|\sigma_{T(q,m)-}-\sigma(q,r)_{T(q,m)-}|^2\,1_{\{T(q,m)\leq t\}}\Big),\end{array}$$

which goes to 0 as $r\to\infty$ by (5.129). Furthermore, we also have

$$\mathbb{E}\Big(\sup_{s\leq t}|U(2,\sigma,\delta(q))_s-U(2,\sigma,\delta)_s|^2\Big)\ \leq\ K\mathbb{E}\Big(\sum_{s\leq t}|\Delta X_s|^2\,1_{\{|\Delta X_s|\leq 1/q\}}\Big),$$

which goes to 0 as $q\to\infty$. So, summarizing those results, we end up with

$$\lim_{q\to\infty}\ \limsup_{r\to\infty}\ \mathbb{E}\Big(\sup_{s\leq t}|\overline{U}(\sigma,\delta)_s-\overline{U}(\sigma(q,r),\delta(q))_s|^2\Big)\ =\ 0.\tag{5.133}$$

Therefore, in order to get our theorem it remains to prove that for all $t, \eta > 0$ we have, where C refers to the case X is continuous and D to the general (discontinuous) case:

$$\left.\begin{array}{l} \text{C}: \ \lim_{q\to\infty} \limsup_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{s\leq t} |\overline{V}(X(q,r))^n_s - \overline{V}(X)^n_s| > \eta\Big) = 0 \\ \text{D}: \ \lim_{q\to\infty} \limsup_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(|\overline{V}(X(q,r))^n_t - \overline{V}(X)^n_t| > \eta\Big) = 0. \end{array}\right\} \tag{5.134}$$

Step 3. If $Z(q,r) = X(q,r) + \chi$, we have (for (q,r) fixed):

$$\phi(Z,g,2)^n_i - \phi(Z(q,r),g,2)^n_i = (\overline{X}^n_i)^2 - (\overline{X(q,r)}^n_i)^2 + 2\overline{\chi}^n_i(\overline{X}^n_i - \overline{X(q,r)}^n_i) - \tfrac{1}{2}\, v^n_i,$$
$$v^n_i = \sum_{j=1}^{k_n} (g'^n_j)^2 \Big((\Delta^n_{i+j}X)^2 - (\Delta^n_{i+j}X(q,r))^2 + 2\Delta^n_{i+j}\chi(\Delta^n_{i+j}X - \Delta^n_{i+j}X(q,r))\Big).$$

Therefore

$$\overline{V}(X)^n_t - \overline{V}(X(q,r))^n_t \ = \ G^1(q,r)^n_t + G^2(q,r)^n_t - \frac{1}{2}\, V^n_t,$$

where

$$V^n_t \ = \ \frac{1}{k_n \Delta_n^{1/4}} \sum_{i=0}^{[t/\Delta_n]-k_n} v^n_i,$$

$$G^1(q,r)^n_t = \frac{1}{\Delta_n^{1/4}}\Big(\frac{1}{k_n} \sum_{i=0}^{[t/\Delta_n]-k_n} \Big((\overline{X}^n_i)^2 - (\overline{X(q,r)}^n_i)^2\Big) - \overline{g}(2)\Big([X,X]_t - [X(q,r),X(q,r)]_t\Big)\Big),$$

$$G^2(q,r)^n_t \ = \ \frac{2}{k_n\Delta_n^{1/4}} \sum_{i=0}^{[t/\Delta_n]-k_n} \overline{\chi}^n_i \Big(\overline{X}^n_i - \overline{X(q,r)}^n_i\Big).$$

We obviously have $\mathbb{E}(|v^n_i|) \leq K\Delta_n$, so $V^n \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. Therefore, instead of (5.134), we are left to prove for $l = 1,2$:

$$\left.\begin{array}{l} \text{C}: \ \lim_{q\to\infty} \limsup_{r\to\infty} \limsup_{n\to\infty} \ \mathbb{P}\Big(\sup_{s\leq t} |G^l(q,r)^n_s| > \eta\Big) = 0 \\ \text{D}: \ \lim_{q\to\infty} \limsup_{r\to\infty} \limsup_{n\to\infty} \ \mathbb{P}\Big(|G^l(q,r)^n_t| > \eta\Big) = 0. \end{array}\right\} \tag{5.135}$$

Step 4. We begin by proving (5.135) for $l = 2$. We split the sum in the definition of $G^2(q,r)^n_t$ into two parts: $G^3(q,r)^n_t$ is the sum over those i's such that the fractional part of $i/2k_n$ is in $[0,1/2)$, and $G^4(q,r)^n_t$ which is the sum when the fractional part is in $[1/2,1)$, so it enough to show (5.135) for $l = 3$ and $l = 4$, and we will do it for $l = 3$ only. Now, $G^3(q,r)^n_t$ can be written as

$$\left.\begin{array}{l} G^3(q,r)^n_t = \sum_{j=0}^{J_n+1} \zeta(q,r)^n_i, \\ \zeta(q,r)^n_j \ = \ \frac{2}{k_n\Delta_n^{1/4}} \sum_{i=2jk_n}^{(2jk_n+k_n-1)\wedge([t/\Delta_n]-k_n)} \overline{\chi}^n_i \Big(\overline{X}^n_i - \overline{X(q,r)}^n_i\Big), \end{array}\right\} \tag{5.136}$$

where J_n is the integer part of $([t/\Delta_n] + 1 - 2k_n)/2k_n$ (J_n depends on t, and all $\zeta(q,r)^n_j$ have k_n summands, except the J_n'th one which may have less). Note that $\zeta(q,r)^n_j$ is

$\mathcal{F}^n_{2(j+1)k_n}$-measurable, and by successive conditioning we have $\mathbb{E}(\zeta(q,r)^n_j \mid \mathcal{F}^n_{2jk_n}) = 0$. Therefore by a martingale argument (5.135) will follow, if we prove

$$\lim_{q\to\infty} \limsup_{r\to\infty} \limsup_{n\to\infty} \mathbb{E}\Big(\sum_{j=0}^{J(n,t)} |\zeta(q,r)^n_j|^2\Big) = 0. \tag{5.137}$$

Now, recall (5.130) and (5.125). Then, by (5.3) and standard estimates, plus (5.129) and Cauchy-Schwarz inequality, plus (5.2) and successive conditioning, we get

$$\mathbb{E}\Big((\overline{\chi}^n_i)^2\ (\overline{X}^n_i - \overline{X(q,r)}^n_i)^2\Big) \leq K\Delta_n^{1/2}(\varepsilon(q,r)^n_i + u_n\varepsilon_q),$$

and so the expectation in (5.137) is smaller than $K(\varepsilon(q,r)_t + \varepsilon_q)$. Hence (5.137) holds.

Step 5. Now we turn to $l = 1$ in (5.135). We can write

$$G^1(q,r)^n_t = G^5(q,r)^n_t + G^6(q,r)^n_t,$$

where, with the notation $A(q,r) = [X,X] - [X(q,r),X(q,r)]$

$$\left.\begin{array}{lcl} G^5(q,r)^n_t &=& \sum_{i=0}^{[t/\Delta_n]-k_n} \vartheta(q,r)^n_i \\ \vartheta(q,r)^n_i &=& \frac{1}{k_n\Delta_n^{1/4}}\Big((\overline{X}^n_i)^2 - (\overline{X(q,r)}^n_i)^2 - \int_{i\Delta_n}^{i\Delta_n+u_n} g_n(s-i\Delta_n)^2\, dA(q,r)_s, \end{array}\right\} \tag{5.138}$$

$$G^6(q,r)^n_t = \frac{1}{\Delta_n^{1/4}}\Big(\frac{1}{k_n}\sum_{i=0}^{[t/\Delta_n]-k_n} \int_{i\Delta_n}^{i\Delta_n+u_n} g_n(s-i\Delta_n)^2 dA(q,r)_s - \overline{g}(2)A(q,r)_t\Big),$$

In this step we prove that $G^6(q,r)^n$ satisfies (5.135). A simple calculation shows that (recall the notation $\overline{g}(2)_n$ of (2.8)):

$$G^6(q,r)^n_t = \frac{1}{\Delta_n^{1/4}}\int_0^t \Big(\frac{\overline{g}(2)_n}{k_n} - \overline{g}(2)\Big) dA(q,r)_s + v(q,r)^n_t,$$

where because of (2.10) the remainder term $v(q,r)^n_t$ satisfies, with $A'(q,r)$ being the variation process of $A(q,r)$:

$$|v(q,r)^n_t| \leq \frac{K}{\Delta_n^{1/4}}\Big(A'(q,r)_{u_n} + (A'(q,r)_t - A'(q,r)_{t-2u_n})\Big).$$

In the continuous case C, we have $A'(q,r)_{s+u_n} - A'(q,r)_s \leq Ku_n$, hence $\sup_{s\leq t}|v(q,r)^n_s| \leq K\Delta_n^{1/4}$. In the discontinuous case D we only have $\mathbb{E}(A'(q,r)_{s+u_n} - A'(q,r)_s) \leq Ku_n$, so that $v(q,r)^n_t \stackrel{\mathbb{P}}{\longrightarrow} 0$ as $n\to\infty$. Then if we apply (2.10) we readily obtain (5.135) for $l = 6$.

Step 6. At this stage it remains to prove (5.135) for $l = 5$. For this we use (5.3) again, and Itô's formula, to get, with $Y^{n,i}_t = \int_{i\Delta_n}^t g_n(s-i\Delta_n)dY_s$ for any semimartingale Y and for $t \geq i\Delta_n$:

$$(\overline{X}^n_i)^2 - \int_{i\Delta_n}^{i\Delta_n+u_n} g_n(s-i\Delta_n)^2\,d[X,X]_s = 2\int_{i\Delta_n}^{i\Delta_n+u_n} X^{n,i}_s\,g_n(s-i\Delta_n)(b^q_s\,ds+\sigma_s\,dW_s)$$
$$+2\int_{i\Delta_n}^{i\Delta_n+u_n} X^{n,i}_{s-}\,dM^q_s + 2\int_{i\Delta_n}^{i\Delta_n+u_n}\int_{\{\gamma(z)>1/q\}} X^{n,i}_{s-}\,g_n(s-i\Delta_n)\,\delta(s,z)\,\underline{\mu}(ds,dz),$$

and a similar expression with (X, b^q, σ, δ) substituted with $(X(q,r), b(q,r), \sigma(q,r), \delta(q))$, so the second term on the right side above vanishes in this case (remember the last part of (5.130)). Therefore

$$\vartheta(q,r)^n_i \;=\; \frac{2}{k_n\Delta_n^{1/4}}\sum_{j=1}^6 \eta(q,r,j)^n_i,$$

where, using (5.99) and with the notation $I(n,i) = (i\Delta_n, i\Delta_n+u_n]$, we have

$$\begin{aligned}
\eta(q,r,1)^n_i &= \textstyle\int_{I(n,i)} X'(q,r)^{n,i}_s\,g_n(s-i\Delta_n)\,ds\,\Big(b_s+\int_{\{|\delta(s,z)|>1\}}\delta(s,z)\lambda(dz)\Big)\\
\eta(q,r,2)^n_i &= \textstyle\int_{I(n,i)} X(q,r)^{n,i}_s\,g_n(s-i\Delta_n)\,(b^q_s-b(q,r)_s)ds\\
\eta(q,r,3)^n_i &= \textstyle\int_{I(n,i)} X'(q,r)^{n,i}_s\,g_n(s-i\Delta_n)\,\sigma_s dW_s\\
\eta(q,r,4)^n_i &= \textstyle\int_{I(n,i)} X(q,r)^{n,i}_s\,g_n(s-i\Delta_n)\,(\sigma_s-\sigma(q,r)_s)dW_s\\
\eta(q,r,5)^n_i &= \textstyle\int_{I(n,i)} X^{n,i}_{s-}\,g_n(s-i\Delta_n)\,dM^q_s\\
\eta(q,r,6)^n_i &= \textstyle\int_{I(n,i)}\int_{\gamma(z)>1/q} X'(q,r)^{n,i}_{s-}\,g_n(s-i\Delta_n)\,\delta(s,z)\,(\underline{\mu}-\underline{\nu})(ds,dz).
\end{aligned}$$

Therefore, since moreover $\eta(q,r,j)^n_i$ for $j=3,4,5,6$ are martingale increments, (5.135) for $l=5$ will follow if we prove that for all $t>0$, and as $m\to\infty$:

$$j=1,2 \;\;\Rightarrow\;\; \lim_{q\to\infty}\;\limsup_{r\to\infty}\;\limsup_{n\to\infty}\;\Delta_n^{1/4}\,\mathbb{E}\Big(\sum_{i=0}^{[t/\Delta_n]-k_n}|\eta(q,r,j)^n_i|\Big)\;\to\;0, \tag{5.139}$$

$$j=3,4,5,6 \;\Rightarrow\; \lim_{q\to\infty}\;\limsup_{r\to\infty}\;\limsup_{n\to\infty}\;\Delta_n^{1/2}\,\mathbb{E}\Big(\sum_{i=0}^{[t/\Delta_n]-k_n}|\eta(q,r,j)^n_i|^2\Big)\;\to\;0. \tag{5.140}$$

Then, standard estimates yield for $s\in I(n,i)$ and $p\geq 2$ (recall $|b^q_t|+|b(q,r)_t|\leq Kq$):

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\leq s}|X'(q,r)^{n,i}_t|^2\Big) \;&\leq\; K(\varepsilon(q,r)^n_i+\Delta_n^{1/2}\,\varepsilon_q)\\
\mathbb{E}\Big(\sup_{t\leq s}|X(q,r)^{n,i}_t|^p\Big) \;&\leq\; K_p\Big(q^p\Delta_n^{p/2}+\Delta_n^{1/2}\Big)\\
\mathbb{E}\Big(\sup_{t\leq s}|X^{n,i}_t|^2\Big) \;&\leq\; K\Delta_n^{1/2}.
\end{aligned}$$

and it follows that, since $|g_n|\leq K$ and $\varepsilon(q,r)^n_i\leq K$ and $\varepsilon_q\leq K$ and $\int_{\{|\delta(s,z)|>1\}}|\delta(s,z)|\lambda(dz)\leq \int\gamma(z)^2\lambda(dz)<\infty$,

$$\begin{aligned}
j=1,2 \;\Rightarrow\; \mathbb{E}(|\eta(q,r,j)^n_i|) \;&\leq\; K\Delta_n^{1/2}\Big(q\sqrt{\varepsilon(q,r)^n_i}+\Delta_n^{1/4}\sqrt{\varepsilon_q}\Big)\\
j=3,4,5,6 \;\Rightarrow\; \mathbb{E}(|\eta(q,r,j)^n_i|^2) \;&\leq\; K\Delta_n^{1/2}\Big(q^2\Delta_n^{3/4}+\varepsilon(q,r)^n_i+\Delta_n^{1/2}\varepsilon_q\Big).
\end{aligned}$$

By Hölder's inequality

$$\Big(\Delta_n^{3/4}\sum_{i=0}^{[t/\Delta_n]-k_n}\sqrt{\varepsilon(q,r)_i^n}\Big)^2 \;\leq\; \Delta_n^{1/2}\sum_{i=0}^{[t/\Delta_n]-k_n}\varepsilon(q,r)_i^n \;\leq\; K\varepsilon(q,r)_t.$$

Since $\varepsilon(q,r)\to 0$ as $r\to\infty$, for each q, whereas $\varepsilon_q\to 0$ as $q\to\infty$. Therefore we readily obtain (5.139) and (5.140), and the proof is finished.

References

[1] Aït-Sahalia, Y., P.A. Mykland, and L. Zhang (2005): How often to sample a continuous-time process in the presence of market microstructure noise. *Review of Financial Studies* **18**, 351–416.

[2] Bandi, F.M. and J.R. Russell (2006): Separating microstructure noise from volatility. *Journal of Financial Economics* **79**, 655–692.

[3] Barndorff-Nielsen, O.E., and N. Shephard (2004): Power and bipower variation with stochastic volatility and jumps (with discussion). *Journal of Financial Econometrics* **2**, 1–48.

[4] Barndorff-Nielsen, O.E., S.E. Graversen, J. Jacod, M. Podolskij, N. Shephard (2006): A central limit theorem for realised power and bipower variations of continuous semimartingales. In: Yu. Kabanov, R. Liptser and J. Stoyanov (Eds.), ¿From Stochastic Calculus to Mathematical Finance. Festschrift in Honour of A.N. Shiryaev, Heidelberg: Springer, 2006, 33-68.

[5] Barndorff-Nielsen, O.E., P.R. Hansen, A. Lunde and N. Shephard (2006): Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise. Forthcoming in *Econometrica*.

[6] Gloter, A. and J. Jacod (2001): Diffusions with measurement errors. II-Optimal estimators. *ESAIM*, **5**, 243-260.

[7] Ibragimov, I.A. and R.Z. Has'minski (1981): *Statistical estimation: asymptotic theory.* Springer-Verlag: Berlin.

[8] Jacod, J. and Protter, P. (1998): Asymptotic error distributions for the Euler method for stochastic differential equations. *Ann. Probab.*, **26**, 267-307.

[9] Jacod, J. and A.N. Shiryaev (2003): *Limit Theorems for Stochastic Processes*, 2d ed., Springer-Verlag: Berlin.

[10] Jacod, J. (2008): Asymptotic properties of realized power variations and related functionals of semimartingales. *Stoch. Proc. Appl.*, **118**, 517-559.

[11] Jacod, J., Y. Li, P. Mykland, M. Podolskij and M. Vetter (2007): Microstructure Noise in the Continuous Case: The Pre-Averaging Approach. Forthcoming in *Stoch. Proc. Appl.*

[12] Li, Y. and P. Mykland (2007): Are volatility estimators robust with respect to modeling assumptions ?. *Bernoulli* **13** (3), 601-622.

[13] Podolskij, M. and M. Vetter (2006): Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. Forthcoming in *Bernoulli*.

[14] Podolskij, M. and M. Vetter (2008): Bipower-type estimation in a noisy diffusion setting. Preprint.

[15] L. Zhang, L., P.A. Mykland, Y. Aït-Sahalia (2005): A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association* **100**, 1394–1411.

[16] Zhang, L. (2006): Efficient estimation of volatility using noisy observations. *Bernoulli*, **12** (6), 1019-1043.