A nonparametric test for stationarity in functional time series

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Abstract

We propose a new measure for stationarity of a functional time series, which is based on an explicit representation of the L^2-distance between the spectral density operator of a non-stationary process and its best (L^2-)approximation by a spectral density operator corresponding to a stationary process. This distance can easily be estimated by sums of Hilbert-Schmidt inner products of periodogram operators (evaluated at different frequencies), and asymptotic normality of an appropriately standardised version of the estimator can be established for the corresponding estimate under the null hypothesis and alternative. As a result we obtain confidence intervals for the discrepancy of the underlying process from a functional stationary process and a simple asymptotic frequency domain level α test (using the quantiles of the normal distribution) for the hypothesis of stationarity of functional time series. Moreover, the new methodology allows also to test precise hypotheses of the form "the functional time series is approximately stationarity", which means that the new measure of stationarity is smaller than a given threshold. Thus in contrast to methods proposed in the literature our approach also allows to test for "relevant" deviations from stationarity.

We demonstrate in a small simulation study that the new method has very good finite sample properties and compare it with the currently available alternative procedures. Moreover, we apply our test to annual temperature curves.

Keywords: time series, functional data, spectral analysis, local stationarity, measuring stationarity, relevant hypotheses
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1 Introduction

In many applications of functional data analysis (FDA) data is recorded sequentially over time and naturally exhibits dependence. In the last years, an increasing number of authors have worked on analysing functional data from time series and we refer to the monographs of Bosq (2000) and Horváth and Kokoszka (2012) among others. An important assumption in most of the literature is stationarity, which allows a unifying development of statistical theory. For example, stationary processes with a linear representation have among others been investigated by Mas (2000), Bosq (2002) and Dehling and Sharipov (2005). Prediction methods (e.g., Antoniadis and Sapatinas, 2003; Aue et al., 2015; Bosq, 2000) and violation of the i.i.d. assumption in the context of change point detection have also received a fair amount of attention (e.g., Aue et al., 2009; Berkes et al., 2009; Horváth et al., 2010). Hörmann and Kokoszka (2010) provide a general framework to examine temporal dependence among functional observations of stationary processes. Frequency domain analysis of stationary functional time series has been considered by Panaretos and Tavakoli (2013) under the assumption of functional generalizations of cumulant-mixing conditions.

In many applications it is however not clear that the temporal dependence structure is constant and hence that stationarity is satisfied. It is therefore desirable to have tests for second order stationarity or measures for deviations from stationarity for data analysis of functional time series. In the context of Euclidean data (univariate and multivariate) there exists a considerable amount of literature on this problem. Early work can be found in Priestley and Subba Rao (1969) who proposed testing the "homogeneity" of a set of evolutionary spectra. Von Sachs and Neumann (2000) used coefficients with respect to a Haar wavelet series expansion of time-varying periodograms for this purpose, see also Nason (2013) who provided an important extension of their approach and Cardinali and Nason (2010) or Taylor et al. (2014) for further applications of wavelets in the problem of testing for stationarity. Paparoditis (2009, 2010) proposed to reject the null hypothesis of second order stationarity if the L^2-distance between a local spectral density estimate and an estimate derived under the assumption of stationarity is large. Dette et al. (2011) suggested to estimate this distance directly by sums of periodograms evaluated at the Fourier frequencies in order to avoid the problem of choosing additional bandwidths [see also Preuß et al. (2013) for an empirical process approach]. An alternative method to investigate second order stationarity can be found in Dwivedi and Subba Rao (2011) and Jentsch and Subba Rao (2015), who used the fact that the discrete Fourier transform (DFT) is asymptotically uncorrelated at the canonical frequencies if and only if the time series is second-order stationary. Recently, Jin et al. (2015) proposed a double-order selection test for checking second-order stationarity of a univariate time series, while Das and Nason (2016) investigated an experimental empirical measure of non-stationarity based on the mathematical roughness of the time evolution of fitted parameters of a dynamic linear model.

On the other hand – despite the frequently made assumption of second-order stationarity in functional data analysis – much less work has been done investigating the stationarity of functional data. A rigorous mathematical framework for locally stationary functional time series has only been recently developed by van Delft and Eichler (2016), who extended the concept of local stationarity introduced by Dahlhaus (1996, 1997) from univariate time series to functional data. To our best knowledge Aue and van Delft (2017) is the only reference that applies this framework to test for second-order stationarity of a functional time series against smooth alternatives. These authors follow the approach of Dwivedi and Subba Rao (2011) and show that the functional discrete Fourier transform (fDFT) is asymptotically uncorrelated at distinct Fourier frequencies if and only if the process is functional weakly stationary. This result is then used to construct a test statistic based on an empirical covariance operator of the fDFT's, which is subsequently projected to finite dimension. The asymptotic properties of the resulting quadratic form is demonstrated to be chi-square distributed both under the null and under the alternative of functional local stationarity. Although the authors thereby provide an explicit expression for the degree of departure from weak stationarity, the test requires the specification of the parameter M, the number of lagged fDFT's included. This can be seen as a disadvantage since it affects the power of the test.

In the present paper we propose a different test which is based on an explicit representation of the L^2-distance between the spectral density operator of a non-stationary process and its best (L^2-)approximation by a spectral density operator corresponding to a stationary process. This measure vanishes if and only if the time series is second order stationary, and consequently a test can be obtained by rejecting the hypothesis of stationarity for large values of a corresponding estimate. The L^2-distance is estimated by a functional of sums of integrated periodogram operators, for which (after appropriate standardisation) asymptotic normality can be established under the null hypothesis and any fixed alternative. While the proof of this statement is challenging and technically very difficult, the resulting test is extremely simple to use. Under the null hypothesis, the proposed test statistic is asymptotically (centred) normal distributed with a variance which is easy to estimate. Consequently the final test uses the quantiles of the standard normal distribution and the proposed test does neither require the choice of a bandwidth in order to estimate the time vary spectral density operators (we only estimate its integrals by summing with respect to the Fourier frequencies) nor bootstrap methods to obtain critical values. Therefore the proposed methodology is also very efficient from a computational point of view.

Moreover, as the asymptotic (normal) distribution is also available under any fixed alternative, our results can be used to construct (asymptotic) confidence intervals for the measure of stationarity. As other statistical applications we mention the problem of testing precise hypotheses [see Berger and Delampady (1987)], which means in the present context to test if the measure of stationarity exceeds a certain threshold. The formulation of a hypothesis in this form is motivated by the fact that in many applications it might be reasonable to tolerate small deviations from second order stationarity because the powerful methodology for stationary time series might be robust with respect to small deviations from stationarity. This requires the specification of a threshold, but we argue that in many applications it might be reasonable to think very carefully about the size of deviation from stationarity which one really wants to detect.

The rest of the paper is organised as follows. In Section 2 we introduce the main concept of local stationary functional time series, define a measure of stationarity for these processes and its corresponding estimates. Section 3 is devoted to the asymptotic properties of the proposed estimators and some statistical applications of the asymptotic theory. The finite sample properties of the new test are investigated in Section 4 by means of a small simulation study. In this section we also illustrate our method to temperature data. More specifically, we apply our test to annual temperature curves recorded at several measuring stations in Australia over the past 135 years. Finally, most of the proofs and technical arguments, which are rather complicated, can be found in Section 5.

2 A measure of stationarity on the function space

2.1 Notation and the functional setup

We begin with providing definitions and facts about operators used in the paper. Suppose that $\mathscr{H}$ is a separable Hilbert space. $\mathscr{L}(\mathscr{H})$ denotes the Banach space of bounded linear operators $A : \mathscr{H} \to \mathscr{H}$ with the operator norm given by $|||A|||_\infty = \sup_{\|x\|\le 1} \|Ax\|$. Each operator $A \in \mathscr{L}(\mathscr{H})$ has the adjoint operator $A^\dagger \in \mathscr{L}(\mathscr{H})$, which satisfies $\langle Ax, y\rangle = \langle x, A^\dagger y\rangle$ for each $x, y \in \mathscr{H}$. $A \in \mathscr{L}(\mathscr{H})$ is called self-adjoint if $A = A^\dagger$ and non-negative definite if $\langle Ax, x\rangle \ge 0$ for each $x \in \mathscr{H}$.

An operator $A \in \mathscr{L}(\mathscr{H})$ is called compact if it can be written in the form $A = \sum_{j\ge1} s_j(A)\langle e_j, \cdot\rangle f_j$, where $\{e_j : j \ge 1\}$ and $\{f_j : j \ge 1\}$ are orthonormal sets (not necessarily complete) of $\mathscr{H}$, $\{s_j(A) : j \ge 1\}$ are the singular values of A and the series converges in the operator norm. We say that a compact operator $A \in \mathscr{L}(\mathscr{H})$ belongs to the Schatten class of order $p \ge 1$ and write $A \in S_p(\mathscr{H})$ if $|||A|||_p^p = \sum_{j\ge1} s_j^p(A) < \infty$. The Schatten class of order $p \ge 1$ is a Banach space with the norm $|||\cdot|||_p$. A compact operator $A \in \mathscr{L}(\mathscr{H})$ is called Hilbert-Schmidt if $A \in S_2(\mathscr{H})$ and trace class if

$A \in S_1(\mathscr{H})$. The space of Hilbert-Schmidt operators $S_2(\mathscr{H})$ is also a Hilbert space with the inner product given by $\langle A, B\rangle_{\mathrm{HS}} = \sum_{j\geq 1}\langle Ae_j, Be_j\rangle$ for each $A, B \in S_2(\mathscr{H})$, where $\{e_j : j \geq 1\}$ is an orthonormal basis.

Let $L^2([0,1]^k, \mathbb{C})$ for $k \geq 1$ denote the Hilbert space of equivalence classes of square integrable measurable functions $f : [0,1]^k \to \mathbb{C}$ with the inner product given by

$$\langle f, g\rangle = \int_{[0,1]^k} f(x)\overline{g(x)}dx$$

for each $f, g \in L^2([0,1]^k, \mathbb{C})$, where $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$. We denote the norm of $L^2([0,1]^k, \mathbb{C})$ by $\|\cdot\|_2$. $L^2([0,1]^k, \mathbb{R})$ for $k \geq 1$ denotes the corresponding space of real functions.

An operator $A \in \mathscr{L}(L^2([0,1]^k, \mathbb{C}))$ is Hilbert-Schmidt if and only if there exists a kernel $k_A \in L^2([0,1]^k \times [0,1]^k, \mathbb{C})$ such that

$$Af(x) = \int_{[0,1]^k} k_A(x, y) f(y)dy$$

almost everywhere in $[0,1]^k$ for each $f \in L^2([0,1]^k, \mathbb{C})$ (see Theorem 6.11 of Weidmann (1980)). Furthermore,

$$|||A|||_2^2 = \|k_A\|_2^2 = \int_{[0,1]^k}\int_{[0,1]^k} |k_A(x, y)|^2 \, \mathrm{d}x\mathrm{d}y$$

and

$$\langle A, B\rangle_{\mathrm{HS}} = \langle k_A, k_B\rangle = \int_{[0,1]^k}\int_{[0,1]^k} k_A(x, y)\overline{k_B(x, y)}dxdy$$

for $A, B \in S_2(L^2([0,1]^k, \mathbb{C}))$ with the kernels $k_A \in L^2([0,1]^k \times [0,1]^k, \mathbb{C})$ and $k_B \in L^2([0,1]^k \times [0,1]^k, \mathbb{C})$ respectively. Finally, for $f, g \in L^2([0,1]^k, \mathbb{C})$, we define the tensor product $f \otimes g : \mathscr{L}(L^2([0,1]^k, \mathbb{C}))$ by setting $(f \otimes g)v = \langle v, g\rangle f$ for all $v \in L^2([0,1]^k, \mathbb{C})$. In particular, since the tensor product $L^2([0,1]^k, \mathbb{C}) \otimes L^2([0,1]^k, \mathbb{C})$ is isomorphic to $S_2(L^2([0,1]^k, \mathbb{C}))$, it defines a Hilbert-Schmidt operator with the kernel in $L^2([0,1]^k \times [0,1]^k, \mathbb{C})$ given by $(f \otimes g)(\tau, \sigma) = f(\tau)\overline{g}(\sigma)$ for each $\tau, \sigma \in [0,1]^k$.

2.2 Locally stationary functional time series

The second order dynamics of weakly stationary time series of functional data $\{X_h\}_{h\in\mathbb{Z}}$ can be completely described by the Fourier transform of the sequence of covariance operators, acting on $L^2([0,1], \mathbb{C})$, i.e.,

$$\mathscr{F}_\omega = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \mathbb{E}\big((X_h - \mu) \otimes (X_0 - \mu)\big)e^{-\mathrm{i}\omega h} \quad \omega \in [-\pi, \pi] \tag{2.1}$$

where $\mu = \mathbb{E}X_0$ denotes the mean function. We will assume our data are centered and hence $\mu = 0$. When stationarity is violated, we can no longer speak of a frequency distribution over all time and hence, if it exists, this object must become time-dependent. To allow for a meaningful definition of this object if stationarity is violated, we consider a triangular array $\{X_{t,T} : 1 \leq t \leq T\}_{T\in\mathbb{N}}$ as a doubly indexed functional time series, where $X_{t,T}$ is a random element with values in $L^2([0,1], \mathbb{R})$ for each $1 \leq t \leq T$ and $T \in \mathbb{N}$. The processes $\{X_{t,T} : 1 \leq t \leq T\}$ are extended on $\mathbb{Z}$ by setting $X_{t,T} = X_{1,T}$ for $t < 1$ and $X_{t,T} = X_{T,T}$ for $t > T$. Following van Delft and Eichler (2016), the sequence of stochastic processes $\{X_{t,T} : t \in \mathbb{Z}\}$ indexed by $T \in \mathbb{N}$ is called *locally stationary* if for all rescaled times $u \in [0,1]$ there exists an $L^2([0,1], \mathbb{R})$-valued strictly stationary process $\{X_t^{(u)} : t \in \mathbb{Z}\}$ such that

$$\left\| X_{t,T} - X_t^{(u)} \right\|_2 \leq \left(|\tfrac{t}{T} - u| + \tfrac{1}{T}\right) P_{t,T}^{(u)} \qquad a.s. \tag{2.2}$$

for all $1 \le t \le T$, where $P_{t,T}^{(u)}$ is a positive real-valued process such that for some $\rho > 0$ and $C < \infty$ the process satisfies $\mathbb{E}(|P_{t,T}^{(u)}|^{\rho}) < C$ for all t and T and uniformly in $u \in [0,1]$. If the second-order dynamics are changing gradually over time, the second order dynamics of the stochastic process $\{X_{t,T} : t \in \mathbb{Z}\}_{T\in\mathbb{N}}$ are then completely described by the *time-varying spectral density operator* given by

$$\mathscr{F}_{u,\omega} = \frac{1}{2\pi} \sum_{h\in\mathbb{Z}} \mathbb{E}\big(X_{t+h}^{(u)} \otimes X_t^{(u)}\big) e^{-\mathrm{i}\omega t_h}. \tag{2.3}$$

For each $u \in [0,1]$ and $\{X_t^{(u)} : t \in \mathbb{Z}\}$. Under the technical assumptions stated in Section 3, this object is a Hilbert-Schmidt operator and we shall denote its kernel function by $f_{u,\omega} \in L^2([0,1]^2, \mathbb{C})$, which is is twice-differentiable with respect to u and ω. Note that if the process is in fact second-order stationary, then (2.3) reduces to the form (2.1) and hence this framework lends itself in a natural way to test for changing dynamics in the second order structure.

In this paper, we are interested in testing the hypothesis

$$H_0 : \mathscr{F}_{u,\omega} \equiv \mathscr{F}_{\omega} \quad \text{a.e. on } [-\pi,\pi] \times [0,1] \tag{2.4}$$

versus

$$H_a : \mathscr{F}_{u,\omega} \neq \mathscr{F}_{\omega}, \text{on a subset of } [-\pi,\pi] \times [0,1] \text{ of positive Lebesgue measure}, \tag{2.5}$$

where $\mathscr{F}_{\omega}$ is an unknown non-negative definite Hilbert-Schmidt operator for each $\omega \in [-\pi,\pi]$, which does not depend on the rescaled time $u \in [0,1]$. We define the minimum distance

$$m^2 = \min_{\mathscr{G}} \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega} - \mathscr{G}_{\omega}|||_2^2 \, du\, d\omega, \tag{2.6}$$

where the minimum is taken over all mappings $\mathscr{G} : [-\pi,\pi] \to S_2(L^2([0,1],\mathbb{C}))$. Note that the hypotheses in (2.4) and (2.5) can be rewritten as

$$H_0 : m^2 = 0 \qquad \text{versus} \qquad H_a : m^2 > 0, \tag{2.7}$$

and a statistical test can be obtained by rejecting the null hypothesis H_0 for large values of an appropriate estimator of m^2. In order to construct such an estimator, we first derive an alternative representation of the minimum distance m^2.

Lemma 2.1. *The minimum distance m^2 defined in* (2.6) *can be expressed as*

$$m^2 = \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_{\omega}|||_2^2 \, du\, d\omega \tag{2.8}$$

where, for each $[-\pi,\pi]$ the operators $\widetilde{\mathscr{F}}_{\omega}$ are defined by

$$\widetilde{\mathscr{F}}_{\omega} := \int_0^1 \mathscr{F}_{u,\omega}\, du. \tag{2.9}$$

We refer to this operator $\widetilde{\mathscr{F}}_{\omega}$ as the time-integrated local spectral density operator as it is acts on $L^2([0,1],\mathbb{C})$ such that $\widetilde{\mathscr{F}}_{\omega}\phi$ no longer depends on $u \in [0,1]$ for each $\omega \in [-\pi,\pi]$.

Proof. Since $|||\cdot|||_2$ is induced by the Hilbert-Schmidt inner product, we have that

$$
\begin{aligned}
|||\mathscr{F}_{u,\omega} - \mathscr{G}_\omega|||_2^2 &= \langle \mathscr{F}_{u,\omega} - \mathscr{G}_\omega, \mathscr{F}_{u,\omega} - \mathscr{G}_\omega \rangle_{\mathrm{HS}} \\
&= \langle \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega + \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega, \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega + \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega \rangle_{\mathrm{HS}} \\
&= |||\mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega|||_2^2 + \langle \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega, \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega \rangle_{\mathrm{HS}} + \langle \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega, \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega \rangle_{\mathrm{HS}} + |||\widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega|||_2^2.
\end{aligned}
$$

By linearity and the definition of the Hilbert-Schmidt inner product,

$$
\int_0^1 \langle \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega, \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega \rangle_{\mathrm{HS}} du = \Big\langle \int_0^1 \mathscr{F}_{u,\omega} du - \widetilde{\mathscr{F}}_\omega, \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega \Big\rangle_{\mathrm{HS}} = 0.
$$

A similar argument shows that $\int_0^1 \langle \widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega, \mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega \rangle_{\mathrm{HS}} du = 0$. Hence,

$$
m^2 = \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega} - \widetilde{\mathscr{F}}_\omega|||_2^2 du d\omega + \min_{\mathscr{G}} \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega - \mathscr{G}_\omega|||_2^2 d\omega
$$

and the infimum of the second term is achieved at $\mathscr{G}_\omega \equiv \widetilde{\mathscr{F}}_\omega$. The proof is complete. □

Using the definition of the Hilbert-Schmidt norm, we can rewrite expression (2.8) in terms of $\mathscr{F}_{u,\omega}$

$$
m^2 = \int_{-\pi}^{\pi} \int_0^1 |||\mathscr{F}_{u,\omega}|||_2^2 du d\omega - \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_\omega|||_2^2 d\omega \tag{2.10}
$$

The two terms in (2.10) can now be easily estimated from the available data $\{X_{t,T} : 1 \le t \le T\}$ by sums of periodogram operators.

To be precise, suppose that the total sample size factorizes as $T = NM$, where $M, N \in \mathbb{N}$. Throughout the paper, we assume that

$$
\frac{\sqrt{T}}{N} \to 0 \quad \text{and} \quad \frac{N}{M^2} \to 0 \quad \text{as} \quad N, M \to \infty. \tag{2.11}
$$

For $u \in [0,1]$, $\omega \in [-\pi, \pi]$ and $N \ge 1$, the functional discrete Fourier transform (fDFT) is defined as a random function with values in $L^2([0,1], \mathbb{C})$ given by

$$
D_N^{u,\omega} := \frac{1}{\sqrt{2\pi N}} \sum_{s=0}^{N-1} X_{\lfloor uT \rfloor - N/2 + s + 1, T} e^{-\mathrm{i}\omega s}. \tag{2.12}
$$

The periodogram tensor is then defined by

$$
I_N^{u,\omega} := D_N^{u,\omega} \otimes D_N^{u,\omega}. \tag{2.13}
$$

Let $\omega_k = 2\pi k/N$, for $k = 1, \dots, N$ and $u_j = t_j/T = (N(j-1) + N/2)/T$, for $j = 1, 2, \dots, M$. We define functions $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$

$$
\begin{aligned}
\hat{F}_{1,T} &:= \frac{1}{T} \sum_{k=1}^{\lfloor N/2 \rfloor} \sum_{j=1}^{M} \langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}} \rangle_{HS}, & (2.14) \\
\hat{F}_{2,T} &:= \frac{1}{N} \sum_{k=1}^{\lfloor N/2 \rfloor} \Big|\Big|\Big| \frac{1}{M} \sum_{j=1}^{M} I_N^{u_j,\omega_k} \Big|\Big|\Big|_2^2. & (2.15)
\end{aligned}
$$

Let us observe that the function $\hat{F}_{1,T}$ is real-valued for each $T \in \mathbb{N}$ since

$$
\langle I_N^{u,\lambda}, I_N^{u,\omega} \rangle_{HS} = |\langle D_N^{u,\lambda}, D_N^{u,\omega} \rangle|^2
$$

for each $u \in [0,1]$, $\lambda, \omega \in [-\pi, \pi]$ and $N \in \mathbb{N}$. Finally, the estimator of the minimum distance m^2 in (2.10) is given by

$$\widehat{m}_T = 4\pi(\hat{F}_{1,T} - \hat{F}_{2,T}) \,. \tag{2.16}$$

The statistics (2.14) and (2.15) requires the choice of the number M of blocks, which determines the number N of observations in each block by the equation $T = MN$. As M and N correspond to the number of terms used in the Riemann sum approximating the integral with respect to du and $d\omega$ in (2.10) they have to be reasonable large. Some recommendations based on a small simulation study will be given in Section 4.

As in the case of a real-valued time series the periodogram tensor defined by (2.13) is not a consistent estimator. However, the estimators $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$ are consistent for the quantities appearing in the measure of stationarity defined in (2.10) as they are obtained by averaging periodogram tensors with respect to different Fourier frequencies. These heuristic arguments will be made more precise in the following section, where we state our main asymptotic results.

3 Asymptotic normality and statistical applications

In this section we establish asymptotic normality of an appropriately standardized version of the statistic $\widehat{m}_T$ defined in (2.16) and as a by-product its consistency for estimating the measure of stationarity m^2. For this purpose the functional process $\{X_{t,T} \colon t \in \mathbb{Z}\}_{T\in\mathbb{N}}$ is assumed to satisfy the following set of conditions.

Assumption 3.1. Assume $\{X_{t,T} \colon t \in \mathbb{Z}\}_{T\in\mathbb{N}}$ is locally stationary zero-mean stochastic process as introduced in Section 2 and let $\kappa_{k;t_1,\dots,t_{k-1}} : L^2([0,1]^{\lfloor k/2 \rfloor}) \to L^2([0,1]^{\lfloor \frac{k+1}{2} \rfloor})$ be a positive operator independent of T such that, for all $j = 1,\dots,k-1$ and some $\ell \in \mathbb{N}$,

$$\sum_{t_1,\dots,t_{k-1}\in\mathbb{Z}} (1 + |t_j|^{\ell}) \|\!|\kappa_{k;t_1,\dots,t_{k-1}}\|\!|_1 < \infty. \tag{3.1}$$

Let us denote

$$Y_t^{(T)} = X_{t,T} - X_t^{(t/T)} \qquad \text{and} \qquad Y_t^{(u,v)} = \frac{X_t^{(u)} - X_t^{(v)}}{(u-v)} \tag{3.2}$$

for $T \in \mathbb{N}$, $1 \le t \le T$ and $u, v \in [0,1]$ such that $u \ne v$. Suppose furthermore that k-th order joint cumulants satisfy

(i) $\|\mathrm{Cum}(X_{t_1,T},\dots,X_{t_{k-1},T}, Y_{t_k}^{(T)})\|_2 \le \frac{1}{T} \|\!|\kappa_{k;t_1-t_k,\dots,t_{k-1}-t_k}\|\!|_1$,

(ii) $\|\mathrm{Cum}(X_{t_1}^{(u_1)},\dots,X_{t_{k-1}}^{(u_{k-1})}, Y_{t_k}^{(u_k,v)})\|_2 \le \|\!|\kappa_{k;t_1-t_k,\dots,t_{k-1}-t_k}\|\!|_1$,

(iii) $\sup_u \|\mathrm{Cum}(X_{t_1}^{(u)},\dots,X_{t_{k-1}}^{(u)}, X_{t_k}^{(u)})\|_2 \le \|\!|\kappa_{k;t_1-t_k,\dots,t_{k-1}-t_k}\|\!|_1$,

(iv) $\sup_u \|\frac{\partial^\ell}{\partial u^\ell}\mathrm{Cum}(X_{t_1}^{(u)},\dots,X_{t_{k-1}}^{(u)}, X_{t_k}^{(u)})\|_2 \le \|\!|\kappa_{k;t_1-t_k,\dots,t_{k-1}-t_k}\|\!|_1$.

Note that for fixed u_0, the process $\{X_t^{(u_0)} \colon t \in \mathbb{Z}\}$ is strictly stationary and thus the results of van Delft and Eichler (2016) imply that the *local k-th order cumulant spectral kernel*

$$f_{u_0;\omega_1,\dots,\omega_{k-1}}(\tau_1,\dots,\tau_k) = \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\dots,t_{k-1}\in\mathbb{Z}} c_{u_0;t_1,\dots,t_{k-1}}(\tau_1,\dots,\tau_k) e^{-\mathrm{i}\sum_{l=1}^{k-1}\omega_l t_l} \tag{3.3}$$

is well-defined, where $\omega_1,\dots,\omega_{k-1} \in [-\pi,\pi]$ and

$$c_{u_0;t_1,\dots,t_{k-1}}(\tau_1,\dots,\tau_k) = \mathrm{Cum}\big(X_{t_1}^{(u_0)}(\tau_1),\dots,X_{t_{k-1}}^{(u_0)}(\tau_{k-1}), X_0^{(u_0)}(\tau_k)\big) \tag{3.4}$$

is the corresponding local cumulant kernel of order k at time u_0. We shall denote the corresponding operators acting on $L^2([0,1]^k, \mathbb{C})$ by $\mathscr{F}_{u_0,\omega_{k_1},\ldots,\omega_{2k-1}}$ and $C_{u_0,\omega_{k_1},\ldots,\omega_{2k-1}}$, respectively. For $k=2$ we obtain time-varying spectral density kernel $f_{u,\omega}(\tau_1,\tau_2)$ - the kernel of the operator defined in (2.3) - which is uniquely defined by the triangular array and twice-differentiable with respect to u and ω if assumption (iv) holds for $\ell=2$ (see also Lemma A.3 of Aue and van Delft (2017) for more details). The following two results establish the asymptotic normality of $\widehat{m}_T$ (appropriately standardized) under the null hypothesis of stationarity and any fixed alternative. The proof of Theorem 3.1 is postponed to Section 5.

Theorem 3.1. *Suppose that* (2.11) *and Assumption 3.1 hold. Then, under the null hypothesis H_0 we have*

$$\sqrt{T}\hat{m}_T \xrightarrow{d} N(0, v_{H_0}^2) \quad as \quad T \to \infty,$$

where the asymptotic variance $v_{H_0}^2$ is given by

$$v_{H_0}^2 = 4\pi \int_{-\pi}^{\pi} |||\widetilde{\mathscr{F}}_{\omega}|||_2^4 d\omega. \tag{3.5}$$

Observing the equivalent representation of the hypotheses in (3.12) it is reasonable to reject the null hypotheses (2.4) of a stationary functional process whenever

$$\widehat{m}_T > \frac{\widehat{v}_{H_0}}{\sqrt{T}} u_{1-\alpha}\,, \tag{3.6}$$

where $u_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution and $\widehat{v}_{H_0}^2$ is an appropriate estimator of the asymptotic variance $v_{H_0}^2$ given in (3.5). The asymptotic variance under the null hypothesis $v_{H_0}^2$ can be estimated by the statistic

$$\hat{v}_{H_0}^2 = \frac{16\pi^2}{N} \sum_{k=1}^{\lfloor N/2 \rfloor} \left[\frac{1}{M} \sum_{j=1}^{M} \langle I_N^{u_j,\omega_k}, I_N^{u_j,\omega_{k-1}} \rangle_{HS} \right]^2. \tag{3.7}$$

Theorem 3.1 and the following result show that the test defined by (3.6) is an asymptotic level α test.

Lemma 3.1. *Under the assumptions of Theorem 3.1, the estimator defined in* (3.7) *is consistent, that is*

$$\hat{v}_{H_0}^2 \to v_{H_0}^2$$

in probability as $T \to \infty$.

The final asymptotic result of this section establishes the consistency of the test (3.6). It states that asymptotic normality is still valid under any fixed alternative and therefore contains Theorem 3.1 as a special case.

Theorem 3.2. *Suppose that* (2.11) *and Assumption 3.1 hold. Then*

$$\sqrt{T}(\hat{m}_T - m^2) \xrightarrow{d} N(0, v^2) \quad as \quad T \to \infty,$$

where the asymptotic variance is given by

$$v^2 = 8\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}^{\dagger}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1} \otimes \mathscr{F}_{u,\omega_2} \rangle_{HS} du d\omega_1 d\omega_2 + 20\pi \int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}_{u,\omega} \otimes \mathscr{F}_{u,\omega}|||_2^2 du d\omega$$

$$+8\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle\widetilde{\mathscr{F}}^{\dagger}_{\omega_1,-\omega_1,-\omega_2},\widetilde{\mathscr{F}}_{\omega_1}\otimes\widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}d\omega_1 \mathrm{d}\omega_2+8\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega},\widetilde{\mathscr{F}}_{\omega}\rangle_{HS}\langle\widetilde{\mathscr{F}_{\omega}},\mathscr{F}_{u,\omega}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega$$
$$+8\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega}\otimes\mathscr{F}_{u,\omega},\widetilde{\mathscr{F}}_{\omega}\otimes\widetilde{\mathscr{F}}_{\omega}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega-16\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}^{\dagger}_{u,\omega_1,-\omega_1,-\omega_2},\mathscr{F}_{u,\omega_1}\otimes\widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega_1\mathrm{d}\omega_2$$
$$-16\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_1},\widetilde{\mathscr{F}}_{\omega_1}\otimes\mathscr{F}_{u,\omega_1}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega_1-8\pi\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_1}|||_2^2\mathrm{d}u\mathrm{d}\omega_1$$
$$-8\pi\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_1},\mathscr{F}_{u,\omega_1}\otimes\widetilde{\mathscr{F}}_{\omega_1}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega_1. \tag{3.8}$$

Besides consistency it follows from Theorem 3.2 that the probability of rejection of the test (3.6) can be calculated approximately by the formula

$$\mathbb{P}\big(\widehat{m}_T>\hat{v}_{H_0}u_{1-\alpha}/\sqrt{T}\big)\ \approx\ \Phi\Big(\sqrt{T}\ \frac{m^2}{v}-\frac{v_{H_0}}{v}\ u_{1-\alpha}\Big), \tag{3.9}$$

where $v_{H_0}^2$ and v^2 are defined in Theorem 3.1 and 3.2, respectively, and Φ is the distribution function of the standard normal distribution. We also briefly mention two other statistical applications of Theorem 3.2

(a) As Theorem 3.2 provides the asymptotic distribution at any point of the alternative it can be used to construct an asymptotic confidence interval for the measure of stationarity, that is

$$\Big[\max\Big\{0,\widehat{m}_T-\frac{\hat{v}_{H_1}}{\sqrt{T}}u_{1-\alpha/2}\Big\},\widehat{m}_T+\frac{\hat{v}_{H_1}}{\sqrt{T}}u_{1-\alpha/2}\Big]\ , \tag{3.10}$$

where $\hat{v}_{H_1}^2$ denotes an estimator of the variance in Theorem 3.2.

(b) Similarly, one can use Theorem 3.2 to construct a test for a *relevant deviation from stationarity*, that is

$$H_\Delta: m^2\le\Delta \qquad \text{vs.} \qquad K_\Delta: m^2>\Delta\ , \tag{3.11}$$

or for a test for the hypotheses of *similarity to stationarity*, that is

$$H_\Delta: m^2\ge\Delta \qquad \text{vs.} \qquad K_\Delta: m^2<\Delta\ . \tag{3.12}$$

Here Δ is a pre-specified constant such that for a value of m^2 larger (or smaller) than Δ the experimenter defines the second order properties to deviate relevantly from (or to be similar to) stationarity. Hypotheses of this type are called *precise hypotheses* and were considered by Berger and Delampady (1987), who recommended to use them instead of the "classical" hypotheses $H_0: m^2=0$ vs. $H_a: m^2\ne 0$.
The hypotheses in (3.11) and (3.12) require the specification of a threshold. In the classical case one simply uses $\Delta=0$, but we argue that in many applications it might be reasonable to think very carefully about the size of deviation from stationarity which one really wants to detect. For example, if the functional time series deviates only slightly from second order stationarity, it is often reasonable to work under the assumption of stationarity as many procedures are robust against small deviations from this assumption and procedures specifically adapted to non-stationarity usually have a larger variability.
In order to work under the assumption of "approximate second order stationarity" with a controlled type I error one can therefore investigate the hypotheses defined in (3.12). An

asymptotic level α test for these hypotheses is obtained by rejecting the null hypothesis, whenever

$$\widehat{m}_T - \Delta < \frac{\hat{\nu}_{H_1}}{\sqrt{T}} u_\alpha . \tag{3.13}$$

If $\hat{\nu}^2_{H_1}$ is a consistent estimator for the asymptotic variance in Theorem 3.2, then a straightforward calculation shows that under the assumptions of Theorem 3.2

$$\lim_{T\to\infty} \mathbb{P}\Big(\widehat{m}_T - \Delta < \frac{\hat{\nu}_{H_1}}{\sqrt{T}} u_\alpha\Big) = \begin{cases} 0 & \text{if } m^2 > \Delta \\ \alpha & \text{if } m^2 = \Delta \\ 1 & \text{if } m^2 < \Delta \end{cases}, \tag{3.14}$$

which means that the test (3.13) is a consistent and asymptotic level α test for the hypotheses (3.12).

4 Finite sample properties

In this section, we investigate the finite sample properties of the methods proposed in this paper by means of a simulation study and illustrate potential applications analysing annual temperature curves.

4.1 Simulation study

4.1.1 Tests for the classical hypothesis $H_0 : m^2 = 0$

For the investigation of the finite sample performance of the test (3.6) for the hypotheses in (2.7) with simulated data we consider a similar set-up as Aue and van Delft (2017), who used a Fourier basis representation on the interval $[0,1]$ to generate functional data. To be precise, let $\{\psi_l\}_{l=1}^\infty$ be the Fourier basis functions. Consider the p-th order time varying functional autoregressive process (tvFAR(p)), $(X_t, t \in \mathbb{Z})$ defined as

$$X_t = \sum_{t'=1}^{p} A_{t,t'}(X_{t-t'}) + \epsilon_t, \tag{4.1}$$

where $A_{t,1},\dots,A_{t,p}$ are time-varying auto-covariance operators and $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of mean zero innovations taking values in $L^2([0,1],\mathbb{R})$. We have

$$\langle X_t, \psi_l\rangle = \lim_{L_{max}\to\infty} \sum_{l'=1}^{L_{max}} \sum_{t'=1}^{p} \langle X_{t-t'}, \psi_l\rangle \langle A_{t,t'}(\psi_l), \psi_{l'}\rangle + \langle \epsilon_t, \psi_l\rangle \tag{4.2}$$

Therefore the first L_{max} Fourier coefficients of the process X_t are generated using the p-th order vector autoregressive, VAR(p), process

$$\widetilde{X}_t = \sum_{t'=1}^{p} \widetilde{A}_{t,t'} \widetilde{X}_{t-t'} + \widetilde{\epsilon}_t,$$

where $\widetilde{X}_t := \big(\langle X_t, \psi_1\rangle, \dots, \langle X_t, \psi_{L_{max}}\rangle\big)^T$ is the vector of Fourier coefficients, the (l,l')-th entry of $\widetilde{A}_{t,j}$ is given by $\langle A_{t,j}(\psi_l), \psi_{l'}\rangle$ and $\widetilde{\epsilon}_t := \big(\langle \epsilon_t, \psi_1\rangle, \dots, \langle \epsilon_t, \psi_{L_{max}}\rangle\big)^T$. The entries of the matrix $\widetilde{A}_{t,j}$ are generated as $N\big(0, \nu_{l,l'}^{(t,j)}\big)$ with $\nu_{l,l'}^{(t,j)}$ specified below. To ensure stationarity or existence of a causal solution the norms $\kappa_{t,j}$ of $A_{t,j}$ are required to satisfy certain conditions [see Bosq (2000) for stationary and van Delft and Eichler (2016) for local stationary time series, respectively].

If $A_{t,j} \equiv A_j$ for all t in (4.1) and the error sequence $(\epsilon_t, t \in \mathbb{Z})$ is an i.i.d. sequence, we obtain the stationary functional autoregressive (FAR) model of order p. In that case we generate the entries of the operator matrix from $N\big(0, \nu_{l,l'}^{(j)}\big)$ distributions. Functional white noise can be thought of as a FAR model of order 0.

Throughout this section the number of Monte Carlo replications is always 1000. We use the fda package from R to generate the functional data, where L_{max} is taken to be 15. The periodogram kernels are evaluated on a 1000×1000 grid on the square $[0,1]^2$ and their integrals are calculated by averaging the functional values at the grid points. The asymptotic variance under the null hypothesis is estimated by (3.7). In Table 1 we report the simulated nominal levels of the test (3.6) for the hypotheses in (2.7) for the sample sizes $T = 128$, 256, 512 and 1024, where we consider the following three (stationary) data generating processes:

(I) The functional white noise variables $\epsilon_1, \ldots, \epsilon_T$ i.i.d. with coefficient variances $\mathrm{Var}(\langle \epsilon_t, \psi_l \rangle) = \exp((l-1)/10)$.

(II) The FAR(2) variables $X_1, \ldots, X_T$ with operators specified by variances $\nu_{l,l'}^{(1)} = \exp(-l-l')$ and $\nu_{l,l'}^{(2)} = 1/(l + l'^{3/2})$ with norms $\kappa_1 = 0.75$ and $\kappa_2 = -0.4$ and the innovations $\epsilon_1, \ldots, \epsilon_T$ are as in (I).

(III) The FAR(2) variables $X_1, \ldots, X_T$ as in (II) but with $\kappa_1 = 0.4$ and $\kappa_2 = 0.45$.

Note that the test requires the choice of the number M of blocks, which determines the number N of observations in each block by the equation $T = MN$ and some combinations are displayed as well. As mentioned in Section 2 the quantities M and N have to be reasonable large, because they correspond to the number of terms used in the Riemann sum approximating the integral with respect to du and $d\omega$ in (2.10). Interestingly, the results reported in Table 1 are rather robust with respect to this choice and we observe a reasonable approximation of the nominal level in nearly all cases under consideration. Only for the sample size $T = 128$ the simulated level of the test (3.6) is substantially larger as required, if there is dependency in the data. From these results we recommend the choice $M = 16$ for sample sizes $T = 256$, 512, $M = 32$ if 1024 and $M = 8$ if $T = 128$.

Next we investigate the performance of the test (3.6) under the alternative, where we consider the (non-stationary) data generating processes:

(IV) The tvFAR(1) variables $X_1, \ldots, X_T$ with operator specified by variances $\nu_{l,l'}^{(t,1)} = \nu_{l,l'}^{(1)} = \exp(-l-l')$ and norm $\kappa_1 = 0.8$, and innovations are as in (I) with a multiplicative time-varying variance
$$\sigma^2(t) = \frac{1}{2} + \cos\left(\frac{2\pi t}{2048}\right) + 0.3 \sin\left(\frac{2\pi t}{2048}\right).$$

(V) The tvFAR(2) variables $X_1, \ldots, X_T$ with operators as in (IV), but with time-varying norm
$$\kappa_{1,t} = 1.8 \cos\left(1.5 - \cos\left(\frac{4\pi t}{T}\right)\right)$$
and constant norm $\kappa_2 = -0.81$ and innovations are as in (I).

(VI) The structural break FAR(2) variables $X_1, \ldots, X_T$ generated as follows

Table 1: *Empirical rejection probabilities (in percentage) of the test* (3.6) *for the hypotheses in* (2.7) *under the null hypothesis*

			I			II			III		
T	N	M	10%	5%	1%	10%	5%	1%	10%	5%	1%
128	32	4	11.2	5.9	2.4	18.9	8.7	2.8	19.3	9.0	3.6
128	16	8	12.5	6.1	2.2	20.1	8.8	3.2	18.2	7.1	2.9
256	32	8	11.4	5.3	1.8	14.3	7.3	2.1	12.3	7.3	2.1
256	16	16	10.3	4.8	0.9	12.6	6.9	2.9	13.1	8.1	3.1
512	64	8	10.9	4.7	1.1	9.7	4.2	0.8	11.0	5.9	1.8
512	32	16	11.3	6.1	1.6	8.9	4.1	0.6	8.8	4.1	0.7
1024	128	8	10.3	5.7	0.8	9.0	4.8	0.9	9.1	4.9	1.0
1024	64	16	9.2	4.8	0.3	8.1	4.2	0.4	8.3	4.1	0.3
1024	32	32	8.8	5.6	1.2	7.5	4.0	1.1	7.6	4.6	0.9

- for $t \leq 3T/8$, the operators are as in (II) with norms $\kappa_1 = 0.7$ and $\kappa_2 = 0.2$, with innovations as in (I).
- for $t > 3T/8$, the operators are as in (II) with norms $\kappa_1 = 0$ and $\kappa_2 = -0.2$, with innovations as in (I) but with coefficient variances $\text{Var}(\langle \epsilon_t, \psi_l \rangle) = 2\exp((l-1)/10)$.

The simulated power of the test (3.6) is displayed in Table 2 for the recommended number M of blocks. We observe that the test performs very well for models IV and VI, specially for larger values of T. For model V the power is lower and these results coincide with the findings of Aue and van Delft (2017). Moreover, in the three examples under consideration the test (3.6) is more powerful than the procedure suggested by these authors.

Table 2: *Empirical rejection probabilities (in percentage) of the test* (3.6) *for the hypotheses in* (2.7) *under the alternative hypothesis.*

			IV			V			VI		
T	N	M	10%	5%	1%	10%	5%	1%	10%	5%	1%
128	16	8	61.4	51.1	49.3	15.2	9.4	3.5	81.9	75.2	68.4
256	16	16	100.0	99.9	99.5	34.2	17.8	9.5	100.0	99.7	97.3
512	32	16	100.0	100.0	100.0	35.4	21.8	11.6	100.0	100.0	100.0
1024	32	32	100.0	100.0	100.0	46.4	30.2	21.1	100.0	100.0	100.0

4.1.2 Confidence intervals and tests for precise hypotheses

A particular nice by-product of our approach are the asymptotic confidence intervals for the measure of stationarity defined by (2.10). We now investigate the finite sample coverage probability of the confidence interval (3.10) considering the following time varying functional moving-average process or order 1:

$$X_{t,T} = A(\epsilon_t) - \frac{1}{2}\left(1 + b\cos\left(2\pi\frac{t}{T}\right)\right)B(\epsilon_{t-1}) \tag{4.3}$$

where A and B are operators with finite L_2 norm and $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a sequence of mean zero innovations taking values in $L^2([0,1],\mathbb{R})$. Note that in this model the measure of stationarity m^2 is an increasing function of the model parameter b.

As before we generate data from the model

$$\widetilde{X}_{t,T} = \widetilde{A}\widetilde{\epsilon}_t - \frac{1}{2}\left(1 + b\cos\left(2\pi\frac{t}{T}\right)\right)\widetilde{B}\widetilde{\epsilon}_{t-1}$$

where $\widetilde{X}_{t,T} = \left(\langle X_{t,T}, \psi_1\rangle, \dots, \langle X_{t,T}, \psi_{L_{max}}\rangle\right)^T$ is the vector of Fourier coefficients, the (l,l')-th entry of $\widetilde{A}$ and $\widetilde{B}$ are given by $\langle A(\psi_l), \psi_{l'}\rangle$ and $\langle B(\psi_l), \psi_{l'}\rangle$ respectively and $\widetilde{\epsilon}_t := \left(\langle \epsilon_t, \psi_1\rangle, \dots, \langle \epsilon_t, \psi_{L_{max}}\rangle\right)^T$. For simulation purposes both $\widetilde{A}_{l,l'}$ and $\widetilde{B}_{l,l'}$ are taken to be $\exp(-l-l')$ and the innovations $\widetilde{\epsilon}_t$ are generated as in Model I.

Note that the confidence intervals in (3.10) require an estimate $\hat{v}^2$ of the asymptotic variance v^2 of the statistic $\hat{m}_T$, which is defined in Theorem 3.2. Such an estimate can easily be obtained by plugging in estimates of each term of v^2 defined in Theorem 3.2. As all the data generating process considered in this section are Gaussian, the terms involving fourth order spectral density (1st, 4th and 6th term) do not contribute to the asymptotic variance (but they could be estimated by similar methods as described below, if necessary - see Section 4.2, where we use such an estimate analyzing a data example). The terms are estimated by taking sums over different frequencies and location for appropriate products of periodograms. For example the second term

$$20\pi \int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}_{u,\omega} \otimes \mathscr{F}_{u,\omega}|||_2^2 \, du d\omega$$

in the expression of v^2 is estimated by

$$\frac{20 \times 4\pi^2}{T} \sum_{k=4}^{\lfloor N/2 \rfloor} \sum_{j=1}^{M} \langle I_N^{u_j,\omega_k} \otimes I_N^{u_j,\omega_{k-1}}, I_N^{u_j,\omega_{k-2}} \otimes I_N^{u_j,\omega_{k-3}} \rangle_{HS},$$

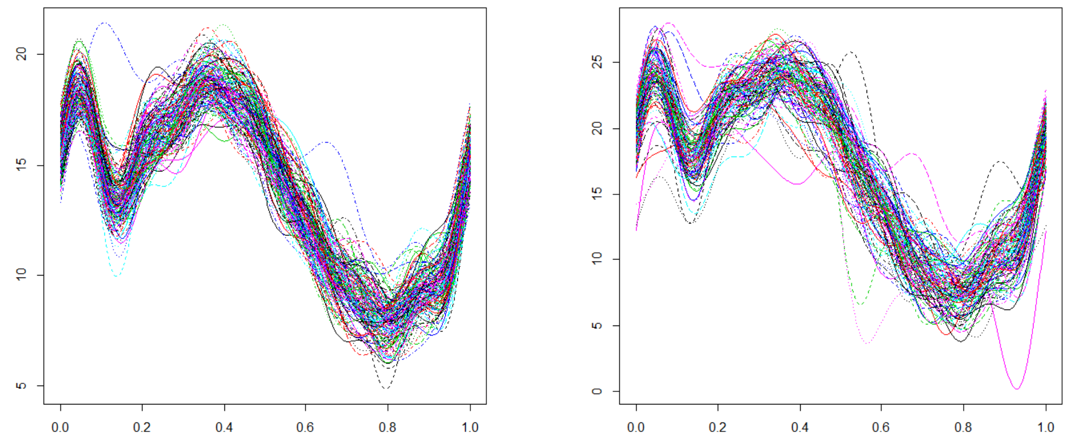
and the other terms are estimated similarly. Although the actual variance v^2 is always real-valued, the resulting estimator can be complex valued for small sample sizes. Since the imaginary part vanishes for increasing sample sizes, we use the real part of the calculated estimator as estimated variance $\hat{v}^2$. The coverage of the 95% and 90% confidence intervals for model (4.3) with $b = 0$, 0.25, 0.5 and , 0.75 are reported in Table 3 for different sample sizes and different numbers of blocks. We observe reasonable coverage probabilities in the cases $b = 0$, 0.25, 0.5, while the coverage probabilities are slightly too small if $b = 0.75$. Again we observe that the results are rather robust with respect to the choice of M and N as long as the number of blocks and the number of observations in each block are reasonable large compared to the sample size. Based on our numerical results we recommend $M = 8$ for sample sizes $T = 128$ and $T = 256$ and $M = 16$ for $T = 512$ and $T = 1024$.

Table 3: *Empirical coverage in percentage of the asymptotic confidence intervals of measures of stationarity in model* (4.3) *for different values of* b.

			b = 0		b = 0.25		b = 0.5		b = 0.75	
T	N	M	95%	90%	95%	90%	95%	90%	95%	90%
128	32	4	98.1	94.3	97.3	93.4	97.7	91.2	95.8	91.0
128	16	8	97.3	93.9	96.2	91.1	95.8	90.6	93.6	90.1
256	32	8	95.8	92.3	95.3	91.8	94.9	91.2	92.8	88.9
256	16	16	95.1	93.0	94.7	90.2	95.3	90.8	91.0	86.5
512	64	8	95.6	91.1	95.1	89.7	93.8	89.2	91.1	87.4
512	32	16	96.1	90.9	94.5	88.9	94.1	88.9	89.8	85.6
1024	128	8	95.2	90.7	95.8	91.3	96.1	92.5	90.9	88.6
1024	64	16	94.8	90.2	95.1	90.8	95.3	91.7	90.2	86.4
1024	32	32	94.3	91.0	94.5	89.3	93.2	90.3	88.8	85.7

Table 4: *Empirical rejection probabilities of the test* (3.13) *for the hypotheses* (3.12) *(similarity to stationarity) in model* (4.3) *for different values of* b.

			b = 0 (H_1)		b = 0.2 (H_1)		b = 0.4 (H_0)		b = 0.6 (H_0)		b = 0.8 (H_0)		b = 1 (H_0)	
T	N	M	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
256	32	8	15.4	22.3	11.8	18.9	8.9	11.7	5.3	9.8	4.8	8.7	3.6	7.4
256	16	16	13.8	15.7	11.3	13.9	9.1	10.8	5.0	8.8	3.8	7.9	2.9	6.2
512	64	8	22.6	33.1	19.9	28.0	9.0	13.3	5.6	11.2	4.9	8.9	4.1	8.2
512	32	16	20.5	28.9	13.1	21.2	7.8	11.6	4.9	7.8	3.8	7.0	2.8	5.9
1024	128	8	33.1	42.8	28.6	39.3	8.8	13.6	4.1	8.5	3.5	7.8	2.7	6.9
1024	64	16	30.6	37.9	27.1	35.8	8.0	14.3	3.7	8.6	2.8	7.3	2.1	6.2
1024	32	32	29.8	35.2	25.9	31.4	7.6	14.1	2.9	7.8	2.1	6.4	1.9	5.6



(a) Sydney

(b) Boulia Airport

Figure 1: *Time series of minimum temperature curves*

Finally we investigate the finite sample properties of the test (3.13) for the hypothesis of *similarity to stationarity* defined in (3.12). Note that for the model under consideration the measure m^2 is in fact an increasing function of the parameter b, for $b > 0$. The case $b = 0$ corresponds to the stationary case with $m^2 = 0$. And larger absolute values of b indicate larger departure from stationarity. In fact it can be shown that $m^2 = c_1 b^4 + c_2 b^2$, for constants $c_1, c_2 > 0$, which depends on the norm of the operators A and B in (4.3). We consider the case $\Delta = 0.0042$ which corresponds to the choice $b \approx 0.4$ in model (4.3). This means that values of b smaller than 0.4 correspond to the alternative, while larger values of b represent the null hypothesis. The simulation results are presented in Table 4. The results reflect the theoretical properties of the test described in Section 3 (see equation (3.14)).

4.2 Data example

We illustrate the new methodology proposed in this paper analyzing annual temperature curve data, recorded at several measuring stations across Australia. The recorded daily minimum temperatures for every year are treated as functional data. The locations of the measuring stations and lengths of the time series are reported in Table 5. The temperature curves of Sydney and Boulia airport are exemplarily presented in Figure 1.

We used the new test defined in (3.6) to investigate whether these temperature curves come from a stationary process or not. The test statistic and variance are estimated as in Section 4.1, where we used the recommendation made in Section4.1.1 for the choice of the number of blocks, that is $M = 8$ (and $M = 10$) as the sample size is closest to $T = 128$ (and often slightly larger). The corresponding p-values of the test (3.6) for the hypothesis of stationarity are reported in Table 5 and are very close to zero across all the stations. Thus, although for sample sizes the level of the test is slightly too large (see the discussion in Section 4.1.1), the results suggest strong evidence against the null hypothesis of stationarity for all the measuring stations.

The 95% confidence intervals for measures of stationarity are reported in Table 6 for $M =$ 8 and 10 (see the discussion in Section 4.1.2). Note that as the data is not guaranteed to be Gaussian, we need to estimate the terms involving fourth order spectral density in addition to

Table 5: *The p-values of the of the test* (3.6) *for the hypothesis of stationarity of the annual temperature curve data*

Measuring Station	T	M=8	M=10
Boulia Airport	120	0.04	0.00
Cape Otway	149	0.00	0.00
Gayndah Post Office	117	0.01	0.00
Gunnedah Pool	133	0.00	0.00
Hobart	121	0.00	0.00
Melbourne	158	0.00	0.00
Robe	129	0.01	0.00
Sydney	154	0.00	0.00

Table 6: *The* 95% *confidence intervals* (3.10) *for the measure* m^2 *of stationarity for the annual temperature curve data*

Measuring Station	T	M=8	M=10
Boulia Airport	120	(-0.23,0.53)	(-0.15,1.13)
Cape Otway	149	(0.62,0.86)	(0.83,1.11)
Gayndah Post Office	117	(0.18,0.63)	(0.16,0.44)
Gunnedah Pool	133	(9.45,12.08)	(3.48,9.33)
Hobart	121	(0.03,0.09)	(0.05,0.06)
Melbourne	158	(0.06,0.28)	(0.05,0.29)
Robe	129	(0.07,0.19)	(0.11,0.13)
Sydney	154	(0.02,0.07)	(0.02,0.07)

the estimate obtained in Section 4.1.2. These 4-th order spectral densities are estimated by 4-th order periodograms as described in formula (1.9) of Brillinger and Rosenblatt (1967). The results mostly agree with the results presented in Table 5. For Boulia Airport the 95% confidence interval contains 0, for all choices of M. Although the intervals are quite large compared to those of other stations. For all other measuring stations the confidence intervals suggest the measure to be strictly grater than 0. Specially for Cape Otway and Gunnedah Pool the departure from stationarity is quite high, compared to the other stations.

5 Proof of the main theorem

This section is devoted to prove Theorem 3.2. We recall that $T = NM$, where N defines the resolution in frequency of the local fDFT and M controls the number of nonoverlapping local fDFT's. To establish the convergence in distribution of $\sqrt{T}(\hat{m}_T - m^2)$ to a zero mean Gaussian random variable with limiting variance v^2 given by (3.8) we will show that

$$\sqrt{T}[\mathrm{E}\,\hat{m}_T - m^2] \to 0, \tag{4.4}$$

$$T\,\mathrm{Var}\,\hat{m}_T \to v^2, \tag{4.5}$$

and

$$T^{n/2}\,\mathrm{cum}_n(\hat{m}_T) \to 0 \tag{4.6}$$

for $n > 2$ as $T \to \infty$. As noted above, the estimator $\hat{m}$ is defined as

$$\widehat{m}_T = 4\pi(\hat{F}_{1,T} - \hat{F}_{2,T})$$

and therefore the distributional properties of $\sqrt{T}(\hat{m}_T - m^2)$ will follow from the joint distributional structure of $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$. In particular, multilinearity of cumulants implies that we have

$$\begin{aligned} T^{n/2}\operatorname{cum}_n(\hat{m}_T) &= T^{n/2}(4\pi)^n \operatorname{cum}_n(\hat{F}_{1,T} - \hat{F}_{2,T}) \\ &= T^{n/2}(4\pi)^n \sum_{x=0}^{n} (-1)^x \binom{n}{x} \operatorname{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T}), \end{aligned} \tag{4.7}$$

where $\operatorname{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T})$ denotes the joint cumulant

$$\operatorname{cum}(\underbrace{\hat{F}_{1,T},\ldots,\hat{F}_{1,T}}_{n \text{ times}}, \underbrace{\hat{F}_{2,T},\ldots,\hat{F}_{2,T}}_{n-x \text{ times}})$$

for $n, x \geq 0$.

The first two moments (4.4)-(4.5) can be determined by the cumulant structure of order $n = 1$ and $n = 2$ of $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$, respectively, while (4.6) will follow from showing that

$$T^{n/2}\operatorname{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T}) \to 0$$

as $T \to \infty$ for each $n > 2$ and $0 \leq x \leq n$. To ease readability, we relegate detailed derivations of technical propositions together with additional background material on cumulant tensors to the Appendix.

The main ingredient to our proof is the following result which allows us to re-express the cumulants of $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$, which consists of Hilbert-Schmidt inner products of local periodogram tensors, into the trace of cumulants of simple tensors of the local functional DFT's.

Theorem 5.1. *Let $\mathbb{E}|||I_N^{u,\omega}|||_2^{2n} < \infty$ for some $n \in \mathbb{N}$ uniformly in u and ω. Then*

$$\begin{aligned} &Cum\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_2}}\rangle_{HS}, \ldots, \langle I_N^{u_{j_{2n-1}},\omega_{k_{2n-1}}}, I_N^{u_{j_{2n}},\omega_{k_{2n}}}\rangle_{HS}\Big) \\ &= \operatorname{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G} (-1)^{G-1}(G-1)! \bigotimes_{g=1}^{G} \mathbb{E}\Big[\otimes_{p\in P_g} D_N^{u_{j_p},\omega_{k_p}}\Big]\Big) \end{aligned}$$

where the summation is over all indecomposable partitions $\boldsymbol{P} = P_1 \cup \ldots \cup P_G$ of the array

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ (n,1) & (n,2) & (n,3) & (n,4) \end{matrix} \tag{4.8}$$

where $p = (l,m)$ and $k_p = (-1)^m k_{2l-\delta\{m\in\{1,2\}\}}$ and $j_p = j_{2l-\delta\{m\in\{1,2\}\}}$ for $l \in \{1,\ldots,n\}$ and $m \in \{1,2,3,4\}$. Here the function $\delta_{\{A\}}$ equals 1 if event A occurs and 0 *otherwise.*

Proof of Theorem 5.1. First note that a sufficient condition for $\mathbb{E}|||I_N^{u,\omega}|||_2^p < \infty$ to exist is $\mathbb{E}\|D_N^{u,\omega}\|_2^{2p} < \infty$ or, in terms of moments of X, $\mathbb{E}\|X_{t,T}\|_2^{2p} < \infty$ for each $T \geq 1$, $1 \leq t \leq T$ and hence by Assumption 3.1

$$\operatorname{Cum}_n(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_2}}\rangle_{HS}) \leq \prod_{l=1}^{n} \sum_{k_l=1}^{\lfloor N/2\rfloor} \sum_{j_l=1}^{M} \sqrt{\mathbb{E}|||I_N^{u_{j_l},\omega_{k_l}}|||_2^2}\sqrt{\mathbb{E}|||I_N^{u_{j_2},\omega_{k_2}}|||_2^2} < \infty.$$

The definition of scalar cumulants, property A.1.1 and a basis expansion yield

$$
\begin{aligned}
&\mathrm{Cum}\Big(\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_2}}\rangle_{HS},\ldots,\langle I_N^{u_{j_{2n-1}},\omega_{k_{2n-1}}}, I_N^{u_{j_{2n}},\omega_{k_{2n}}}\rangle_{HS}\Big)\\
&\quad=\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\prod_{g=1}^{G}\mathbb{E}\prod_{l,m\in P_g}\langle I_N^{u_{j_l},\omega_{k_l}}, I_N^{u_{j_m},\omega_{k_m}}\rangle_{HS}\\
&\quad=\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\prod_{g=1}^{G}\mathbb{E}\prod_{(l,m)\in P_g}\mathrm{Tr}(I_N^{\dagger,u_{j_l},\omega_{k_l}} I_N^{u_{j_m},\omega_{k_m}})\\
&\quad=\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\prod_{g=1}^{G}\mathbb{E}\prod_{(l,m)\in P_g}\mathrm{Tr}\Big(I_N^{\dagger,u_{j_l},\omega_{k_l}}\otimes I_N^{u_{j_m},\omega_{k_m}}\Big).
\end{aligned}
$$

The result then follows from noting that the expectation commutes with the trace operation, an application of property A.1.4 and the product theorem for cumulant tensors (Appendix A.1.1). We remark that this can moreover be written as

$$
\begin{aligned}
&=\int_{[0,1]^{2n}}\Big[\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\prod_{g=1}^{G}\mathbb{E}\prod_{(l,m)\in P_g} I_N^{u_{j_l},\omega_{k_l}}(\tau_l,\sigma_l)I_N^{u_{j_m},\omega_{k_m}}(\sigma_l,\tau_l)\Big]\prod_{i=1}^{n}d\tau_i\prod_{i=1}^{n}d\sigma_i\\
&=\int_{[0,1]^{2n}}\mathrm{Cum}\Big(I_N^{u_{j_1},\omega_{k_1}}(\tau_1,\sigma_1)I_N^{u_{j_2},\omega_{k_2}}(\sigma_1,\tau_1),\ldots,I_N^{u_{j_{2n-1}},\omega_{k_{2n-1}}}(\sigma_n,\tau_n)I_N^{u_{j_{2n}},\omega_{k_{2n}}}(\sigma_n,\tau_n)\Big)\prod_{i=1}^{n}d\tau_i\prod_{i=1}^{n}d\sigma_i
\end{aligned}
$$

□

The following lemma shows that the cumulant tensor of the local fDFT's evaluated with the same midpoint u_i and on the manifold $\sum_{j=1}^{k}\omega_j\equiv 0 \mod 2\pi$ can in turn be expressed in terms of higher order cumulant spectral operators.

Lemma 5.1. *If Assumption 3.1 is satisfied and* $\sum_{j=1}^{k}\omega_j\equiv 0 \mod 2\pi$ *then*

$$
\Big|\Big|\Big|Cum\big(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\big)-\frac{(2\pi)^{1-k/2}}{N^{k/2-1}}\mathscr{F}_{u_i,\omega_1,\ldots,\omega_{k-1}}\Big|\Big|\Big|_1=O\Big(N^{-k/2}\times\frac{N}{M^2}\Big).
$$

When evaluated off the manifold, i.e.,$\sum_{j=1}^{k}\omega_j\neq 0 \mod 2\pi$ the above cumulant is of lower order (see Corollary A.1). Additionally, when the local fDFTS are evaluated on different midpoints then we have the following lemma.

Lemma 5.2. *If Assumption 3.1 is satisfied and* $|j_1-j_2|>1$ *for some midpoints* u_{j_1} *and* u_{j_2} *then*

$$
\Big|\Big|\Big|Cum\Big(D_N^{u_{j_1},\omega_1},\ldots,D_N^{u_{j_k},\omega_k}\Big)\Big|\Big|\Big|_1=O\Big(N^{-k/2}M^{-1}\Big)
$$

uniformly in $\omega_1,\ldots,\omega_k$.

Proofs of these statements are relegated to the Section A.2 of the Appendix, where the properties of cumulant tensors of the local fDFTs are investigated in more detail.

Using these results, (4.4) can now be established. More specifically, Theorem 5.1 for $n=1$ implies we can write

$$
\mathbb{E}\hat{F}_{1,T}=\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(\mathbb{E}\big[D_N^{u_{j_1},\omega_{k_1}}\otimes D_N^{u_{j_1},-\omega_{k_1}}\otimes D_N^{u_{j_1},-\omega_{k_1-1}}\otimes D_N^{u_{j_1},\omega_{k_1-1}}\big]\Big).
$$

Expressing this expectation in cumulant tensors, we get

$$
\begin{aligned}
\mathbb{E}\hat{F}_{1,T} &= \frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1234}\Big(\mathrm{Cum}\big((D_N^{u_{j_1},\omega_{k_1}}, D_N^{u_{j_1},-\omega_{k_1}}, D_N^{u_{j_1},-\omega_{k_1-1}}, D_N^{u_{j_1},\omega_{k_1-1}})\big)\Big)\\
&+\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1234}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_{k_1}}, D_N^{u_{j_1},-\omega_{k_1}})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_{k_1-1}}, D_N^{u_{j_1},\omega_{k_1-1}})\Big)\Big)\\
&+\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1324}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_{k_1}}, D_N^{u_{j_1},-\omega_{k_1-1}})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_{k_1}}, D_N^{u_{j_1},\omega_{k_1-1}})\Big)\Big)\\
&+\frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\mathrm{Tr}\Big(S_{1423}\Big(\mathrm{Cum}(D_N^{u_{j_1},\omega_{k_1}}, D_N^{u_{j_1},\omega_{k_1-1}})\otimes\mathrm{Cum}(D_N^{u_{j_1},-\omega_{k_1}}, D_N^{u_{j_1},-\omega_{k_1-1}})\Big)\Big)
\end{aligned}
$$

where S_{ijkl} denotes the permutation operator on $\otimes_{i=1}^4 L^2([0,1],\mathbb{C})$ that maps the components of the tensor according to the permutation $(1,2,3,4)\mapsto(i,j,k,l)$. By Corollary A.1 and Lemma 5.1, we thus find

$$
\mathbb{E}\hat{F}_{1,T} = \frac{1}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\langle\mathscr{F}_{u_j,\omega_{k-1}},\mathscr{F}_{u_j,\omega_k}\rangle_{HS} + O(\frac{1}{M^2}) + O(\frac{1}{N})
$$

A similar decomposition of the expectation of $\hat{F}_{2,T}$ together with Corollary A.1 and Lemma 5.2 gives

$$
\begin{aligned}
\mathbb{E}(\hat{F}_{2,T}) &= \frac{1}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\mathbb{E}\Big|\Big|\Big|\frac{1}{M}\sum_{j=1}^{M} I_N^{u_j,\omega_k}\Big|\Big|\Big|_2^2\\
&= \frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(\mathbb{E}\big[D_N^{u_{j_1},\omega_k}\otimes D_N^{u_{j_1},-\omega_k}\otimes D_N^{u_{j_2},-\omega_k}\otimes D_N^{u_{j_2},\omega_k}\big]\Big)\\
&= \frac{1}{NM^2}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\langle\mathscr{F}_{u_{j_1},\omega_k},\mathscr{F}_{u_{j_2},\omega_k}\rangle_{HS} + O(\frac{1}{T}) + O(\frac{1}{M^2}).
\end{aligned}
$$

Therefore,

$$
\sqrt{T}\Big[4\pi\,\mathbb{E}\,\hat{F}_{1,T} - \int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}_{u,\omega}|||_2^2\,du\,d\omega\Big]\to 0,
$$

and

$$
\sqrt{T}\Big[4\pi\,\mathbb{E}\,\hat{F}_{2,T} - \int_{-\pi}^{\pi}|||\widetilde{\mathscr{F}_\omega}|||_2^2\,d\omega\Big]\to 0
$$

as $T\to\infty$ provided (2.11) is satisfied.

In order to establish (4.5) and (4.6), it is of importance to be able to determine which indecomposable partitions of the array (5.9) are vanishing. For a fixed partition $P=\{P_1,\ldots,P_G\}$ of the array denote the size of the partition by G.

Lemma 5.3. *If Assumption 3.1 is satisfied then for finite n,*

$$
T^{n/2}\,\mathrm{cum}_{n-x,x}(\hat{F}_{1,T},\hat{F}_{2,T}) =
$$

$$
\frac{1}{T^{n/2}M^x}\sum_{k_1,\ldots,k_n=1}^{\lfloor N/2\rfloor}\sum_{\substack{j_1,\ldots,j_n\\ j_{n+1},\ldots,j_{n+x}=1}}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\bigotimes_{g=1}^{G}\mathbb{E}\Big[\otimes_{p\in P_g} D_N^{u_{j_p},\omega_{k_p}}\Big]\Big) = O(T^{1-n/2}N^{G-n-1})
$$

uniformly in $0\le x\le n$.

Proof of lemma 5.3. For a fixed partition $P = \{P_1, \ldots, P_G\}$, let the cardinality of set P_g be denoted by $|P_g| = \mathscr{C}_g$. By Corollary A.1 and Lemma 5.2 an upperbound of (5.6) is given by

$$O(T^{-n/2} M^{-x} \sum_{k_1,\ldots,k_n=1}^{\lfloor N/2 \rfloor} \sum_{\substack{j_1,\ldots,j_n \\ j_{n+1},\ldots,j_{n+x}=1}}^{M} \prod_{g=1}^{G} \frac{1}{N^{\mathscr{C}_g/2-1}} M^{-\delta_{\{\exists p_1,p_2 \in P_g : |j_{p_1} - j_{p_2}| > 1\}}}) \tag{4.9}$$

Note that $|\mathscr{C}_g| \geq 2$ and that the partition must be indecomposable. We can therefore assume, without loss of generality, that row l hooks with row $l+1$ for $l = 1, \ldots, n-1$, i.e., within each partition there must be at least one set P_g that contains an element from both rows. For fixed j_l, there are only finitely many possibilities, say E, for j_{l+1} (Lemma 5.2). If the set does not cover another row, then the fact that j_l is fixed and j_{l+1} are fixed, another set must contain at least an element from row l or $l+1$. But since the sets must communicate there are only finitely many options for j_{l+2}. If, on other hand, the same set covers elements from yet another row then given a fixed j_l, there are again finitely many options for j_{l+1} and for j_{l+2}. This argument can be continued inductively to find (4.9) is of order

$$O(N^{n/2} M^{-n/2-x} E^n M^{1+x} N^{-2n+G}) = O(T^{1-n/2} N^{G-n-1}). \qquad \square$$

Lemma 5.3 implies that for $n = 2$ partitions with $G \leq 2$ vanish, while for $n \geq 2$ all partitions of size $G \leq n+1$ will vanish. Moreover, for the partitions of larger size in case $n = 2$, only partitions for which all sets are such that $\sum_{k \in P_g} \omega_k \equiv 0 \mod 2\pi$ will not vanish. For $n > 2$, in decomposability of the partition and Corollary A.1 result in restrictions over frequencies $k_1, \ldots, k_n$. This is formalized in the following lemma.

Proposition 5.1. *For a partition of size $G = n + r_1 + 1$ for $r_1 \geq 1$ of the array* (5.9) *with $n > 2$, only partitions with at least r_1 restrictions in frequency direction are indecomposable.*

Proof of Proposition 5.1. We note that a minimal amount of restrictions will be given by those partitions in which frequencies and their conjugates are always part of the same set, i.e., in which for fixed row l, the first two columns are in the same set and the last two columns are in the same set. Given we need that $G \geq n+2$ and $\mathscr{C}_g \geq 2$, indecomposability of the array means that the smallest number of restrictions is given by partitions that have one large set that covers the first two or last two columns and $n - r_1$ rows and for the rest has $\frac{4n-2(n-r_1)}{2} = n + r_1$ sets with $\mathscr{C}_g = 2$. This means there are no constraints in frequency in $n - r_1 - 1$ rows but for the array to hook there must be r_1 constraints in terms of frequencies in rows $n - r_1 - 1$ to row n. $\square$

Together Lemma 5.3 and 5.1 allow to show (Appendix A.4) that

$$T^{n/2} \operatorname{cum}_{n-x,x}(\hat{F}_{1,T}, \hat{F}_{2,T}) \to 0$$

from which (4.6) follows and the asymptotic normality of our estimator is established. Finally, for the covariance structure of $\sqrt{T}\hat{F}_{1,T}$ and $\sqrt{T}\hat{F}_{2,T}$, we find in Appendix A.3

$$\begin{aligned}
\lim_{T\to\infty} T\operatorname{Cov}(\hat{F}_{1,T}, \hat{F}_{1,T}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1} \langle \mathscr{F}^{\dagger}_{u,\omega_1,-\omega_1,-\omega_2}, \mathscr{F}_{u,\omega_1} \otimes \mathscr{F}_{u,\omega_2} \rangle_{HS} du d\omega_1 d\omega_2 \\
&\quad + \frac{5}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1} |||\mathscr{F}_{u,\omega} \otimes \mathscr{F}_{u,\omega}|||_2^2 du d\omega \\
\lim_{T\to\infty} T\operatorname{Cov}(\hat{F}_{2,T}, \hat{F}_{2,T}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \langle \widetilde{\mathscr{F}}^{\dagger}_{\omega_1,-\omega_1,-\omega_2}, \widetilde{\mathscr{F}}_{\omega_1} \otimes \widetilde{\mathscr{F}}_{\omega_2} \rangle_{HS} d\omega_1 \mathrm{d}\omega_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega},\widetilde{\mathscr{F}_{\omega}}\rangle_{HS}\langle\widetilde{\mathscr{F}_{\omega}},\mathscr{F}_{u,\omega}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega}\otimes\mathscr{F}_{u,\omega},\widetilde{\mathscr{F}_{\omega}}\otimes\widetilde{\mathscr{F}_{\omega}}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega
\end{aligned}
$$

$$
\begin{aligned}
\lim_{T\to\infty} T\mathrm{Cov}_2(\hat{F}_{1,T},\hat{F}_{2,T}) =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}^{\dagger}_{u,\omega_1,-\omega_1,-\omega_2},\mathscr{F}_{u,\omega_1}\otimes\widetilde{\mathscr{F}_{\omega_2}}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega_1\mathrm{d}\omega_2\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_1},\widetilde{\mathscr{F}_{\omega_1}}\otimes\mathscr{F}_{u,\omega_1}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega_1\\
&+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^1 \left|\!\left|\!\left| \mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_1}\right|\!\right|\!\right|_2^2\mathrm{d}u\mathrm{d}\omega_1\\
&+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_1},\mathscr{F}_{u,\omega_1}\otimes\widetilde{\mathscr{F}_{\omega_1}}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega_1.
\end{aligned}
$$

A straightforward calculation then yields the asymptotic variance v^2 (3.8) and its kernel (section A.3.1).

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Appendix A Auxiliary results and proofs

A.1 Some properties of tensor products of operators

Let $\mathscr{H}_i$ for each $i = 1,\ldots,n$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The tensor of these is denoted by

$$
\mathscr{H} := \mathscr{H}_1\otimes\ldots\otimes\mathscr{H}_n = \bigotimes_{i=1}^{n}\mathscr{H}_i
$$

If $\mathscr{H}_i = \mathscr{H}\,\forall i$, then this is the n-th fold tensor product of $\mathscr{H}$. For $A_i \in H_i, 1\le i\le n$ the object $\bigotimes_{i=1}^n A_i$ is a multi-antilinear functional that generates a linear manifold, the usual algebraic tensor product of vector spaces $\mathscr{H}_i$, to which the scalar product

$$
\langle\bigotimes_{i=1}^{n}A_i,\bigotimes_{i=1}^{n}B_i\rangle = \prod_{i=1}^{n}\langle A_i,B_i\rangle
$$

can be extended to a pre-Hilbert space. The completion of the above algebraic tensor product is $\bigotimes_{i=1}^n \mathscr{H}_i$.

We shall use the following properties for Hilbert-Scmidt operators:

Properties A.1. *For $A_i, B_i, i = 1,\ldots,n$ be Hilbert-Schmidt operators on the Hilbert space $\mathscr{H} = L^2_{\mathbb{C}}([0,1])$, we have*

1. $\langle A,B\rangle_{HS} = \mathrm{Tr}(A^{\dagger}B)$
2. $\langle\bigotimes_{i=1}^n A_i,\bigotimes_{i=1}^n B_i\rangle_{HS} = \prod_{i=1}^n\langle A_i,B_i\rangle_{HS}$

3. $\langle A,B\rangle_{HS} = \int_0^1 a(\tau,\sigma)\overline{b(\sigma,\tau)}d\sigma d\tau$

4. *If* $A_i \in S_1(H)$, *then* $\prod_{i=1}^n \mathrm{Tr}(A_i) = \mathrm{Tr}(\bigotimes_{i=1}^n A_i)$

A.1.1 Cumulant tensors

Let X be a random element on a probability space $(\Omega,\mathscr{A},\mathbb{P})$ that takes values in a separable Hilbert space H. More precisely, we endow H with the topology induced by the norm on H and assume that $X : \Omega \to H$ is Borel-measurable. The *k-th order cumulant tensor* is defined by (van Delft and Eichler, 2016)

$$\mathrm{Cum}\big(X_1,\dots,X_k\big) = \sum_{l_1,\dots l_k\in\mathbb{N}} \mathrm{Cum}\Big(\prod_{t=1}^{k}\langle X_t,\psi_{l_t}\rangle\Big)(\psi_{l_1}\otimes\cdots\otimes\psi_{l_k}), \tag{5.2}$$

where the cumulants on the right hand side are as usual given by

$$\mathrm{Cum}\big(\langle X_1,\psi_{l_1}\rangle,\dots,\langle X_k,\psi_{l_k}\rangle\big) = \sum_{\nu=(\nu_1,\dots,\nu_p)} (-1)^{p-1}(p-1)!\prod_{r=1}^{p}\mathbb{E}\Big[\prod_{t\in\nu_r}\langle X_t,\psi_{l_t}\rangle\Big],$$

where the summation extends over all unordered partitions ν of $\{1,\dots,k\}$. The product theorem for cumulants (Brillinger, 1981, Theorem 2.3.2) can then be generalised (see e.g. Aue and van Delft, 2017, Theorem A.1) to simple tensors of random elements of H, i.e., $X_t = \otimes_{j=1}^{J_t} X_{tj}$ with $j = 1,\dots,J_t$ and $t = 1,\dots,k$. The joint cumulant tensor is then be given by

$$\mathrm{Cum}(X_1,\dots,X_k) = \sum_{\nu=(\nu_1,\dots,\nu_p)} S_\nu\Big(\otimes_{n=1}^{p}\mathrm{Cum}\big(X_{tj}|(t,j)\in\nu_n\big)\Big), \tag{5.3}$$

where S_ν is the permutation that maps the components of the tensor back into the original order, that is, $S_\nu(\otimes_{r=1}^{p}\otimes_{(t,j)\in\nu_r}X_{tj}) = X_{11}\otimes\cdots\otimes X_{kJ_t}$.

A.2 Bounds on cumulant tensors of local functional DFT

In this section, we prove Lemma 5.1, Lemma 5.2. A direct consequence of Lemma 5.1 is the following corollary

Corollary A.1. *We have*

$$\Big|\Big|\Big|Cum\big(D_N^{u_1,\omega_1},\dots,D_N^{u_k,\omega_k}\big)\Big|\Big|\Big|_p = O\Big(N^{1-k/2}\Big)$$

uniformly in $\omega_1,\dots,\omega_k$ *and* $u_1,\dots,u_k$. *Moreover, if* $\sum_{j=1}^k \omega_j \neq 0 \mod 2\pi$ *then*

$$\Big|\Big|\Big|Cum\big(D_N^{u_1,\omega_1},\dots,D_N^{u_k,\omega_k}\big)\Big|\Big|\Big|_p = O\Big(N^{-k/2}\Big).$$

Before we give the proofs, denote the function $\Delta^{(N)}(\omega) = \sum_{t=0}^{N-1} e^{-\mathrm{i}\omega t}$ for $\omega \in \mathbb{R}$. This function satisfies $|\Delta^{(N)}(\sum_{j=1}^k \omega_j)| = N$ for any $\omega_1,\dots,\omega_k$ for which their sum lies on the manifold $\omega \equiv 0 \mod 2\pi$, while it is of reduced magnitude off the manifold. For the canonical frequencies $\omega_k = 2\pi k/N$ with $k \in \mathbb{Z}$, we moreover have

$$\Delta^{(N)}(\omega_k) = \begin{cases} N, & k \in N\mathbb{Z}; \\ 0, & k \in \mathbb{Z}\setminus N\mathbb{Z}. \end{cases} \tag{5.4}$$

Proof of Lemma 5.1. Let $p \in \{1,2\}$. Using linearity of cumulants we write

$$\begin{aligned}
&\operatorname{Cum}\left(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\right)\\
=&\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\exp\left(-\mathrm{i}\sum_{j=1}^k s_j\omega_j\right)\operatorname{Cum}\left(X_{\lfloor u_iT\rfloor-N/2+s_1+1,T},\ldots,X_{\lfloor u_iT\rfloor-N/2+s_k+1,T}\right)\\
=&\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\exp\left(-\mathrm{i}\sum_{j=1}^k s_j\omega_j\right)C_{u_i-\frac{N/2-s_1-1}{T},s_1-s_k,s_2-s_k,\ldots,s_{k-1}-s_k}+R^1_{k,M,N} \qquad (5.5)
\end{aligned}$$

Using Lemma A.2 of Aue and van Delft (2017) and assumption 3.1 of

$$\begin{aligned}
\left\|\left|R^1_{k,M,N}\right\|\right|_p \leq&\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\Big(\frac{k}{T}+\sum_{j=1}^{k-1}\frac{|s_j-s_k|}{T}\Big)|||\kappa_{k,s_1-s_k,\ldots,s_{k-1}-s_k}|||_p\\
&\leq\frac{1}{(2\pi N)^{k/2}}\sum_{s_k=0}^{N-1}\frac{1}{T}\sum_{l_1,\ldots,l_{k-1}\in\mathbb{Z}}\Big(1+\sum_{j=1}^{k-1}|l_j|\Big)|||\kappa_{k,l_1,\ldots,l_{k-1}}|||_p=O\left(N^{-k/2}M^{-1}\right).
\end{aligned}$$

In addition, we can write the first term of (5.5) as

$$\begin{aligned}
=&\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\exp\left(-\mathrm{i}s_k\sum_j\omega_j\right)\exp\left(-\mathrm{i}\sum_{j=1}^{k-1}\omega_j(s_j-s_k)\right)C_{u_i-\frac{N/2-s_k-1}{T},s_1-s_k,s_2-s_k,\ldots,s_{k-1}-s_k}\\
=&\frac{(2\pi)^{1-k/2}}{N^{k/2}}\sum_{s=0}^{N-1}e^{-\mathrm{i}s\sum_j\omega_j}\mathscr{F}_{u_i-\frac{N/2-s-1}{T},\omega_1,\ldots,\omega_{k-1}}+R^2_{k,M,N}
\end{aligned}$$

where

$$\begin{aligned}
\left\|\left|R^2_{k,M,N}\right\|\right|_p\leq&\frac{1}{(2\pi N)^{k/2}}\sum_{s=0}^{N-1}\sum_{\substack{j:1,\ldots k-1:\\|s_j-s|\geq N-1}}\left\|\left|C_{u-\frac{N/2-s-1}{T},s_1-s,s_2-s,\ldots,s_{k-1}-s}\right\|\right|_p\\
\leq&\frac{1}{(2\pi N)^{k/2}}\sum_{s=0}^{N-1}\sum_{\substack{j:1,\ldots k-1:\\|s_j-s|\geq N-1}}|||\kappa_{k,s_1-s,s_2-s,\ldots,s_{k-1}-s}|||_p.\\
\leq&\frac{1}{(2\pi N)^{k/2}}\sum_{s=0}^{N-1}\frac{1}{N^2}\sum_{\substack{j:1,\ldots k-1:\\|l_j|>N}}N^2|||\kappa_{k,l_1,l_2,\ldots,l_{k-1}}|||_p\\
\leq&\frac{1}{(2\pi N)^{k/2}}\frac{1}{N}\sum_{\substack{j:1,\ldots k-1:\\|l_j|>N}}|l_j|^2|||\kappa_{k,l_1,l_2,\ldots,l_{k-1}}|||_p=o\left(N^{-k/2}\right).
\end{aligned}$$

Therefore, we have the cumulants satisfy

$$\operatorname{Cum}\left(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\right)=\frac{(2\pi)^{1-k/2}}{N^{k/2}}\sum_{s=0}^{N-1}e^{-\mathrm{i}s\sum_j\omega_j}\mathscr{F}_{u_i-\frac{N/2-s-1}{T},\omega_1,\ldots,\omega_{k-1}}+R^1_{k,M,N}+R^2_{k,M,N}.$$

On the manifold $\sum_{j=1}^k\omega_j\equiv 0 \mod 2\pi$ we have that $e^{-\mathrm{i}s\sum_j\omega_j}=1$. Assumption 3.1(iv) and a Taylor expansion yield therefore

$$\operatorname{Cum}\left(D_N^{u_i,\omega_1},\ldots,D_N^{u_i,\omega_k}\right)=\frac{(2\pi)^{k/2-1}}{N^{k/2-1}}\mathscr{F}_{u_i,\omega_1,\ldots,\omega_{k-1}}+R^1_{k,M,N}+R^2_{k,M,N}+R^3_{k,M,N},$$

where

$$\|\|R^3_{k,M,N}\|\|_p = \left\|\left\|\frac{(2\pi)^{k/2-1}}{N^{k/2}}\sum_{\ell=1}^{2}\sum_{s=0}^{N-1}\Big(\frac{1-N/2+s}{T}\Big)^{\ell}\frac{\partial^\ell}{\partial u^\ell}\mathscr{F}_{u,\omega_1,\ldots,\omega_k}\right\|\right\|_p$$
$$\leq \frac{(2\pi)^{k/2-1}}{N^{k/2}}O\Big(\frac{N}{T}+\frac{N}{M^2}\Big)\sum_{\ell=1}^{2}\sup_{u,\omega_1,\ldots,\omega_k}\left\|\left\|\frac{\partial^\ell}{\partial u^\ell}\mathscr{F}_{u,\omega_1,\ldots,\omega_k}\right\|\right\|_p = O\left(N^{-k/2}\Big(\frac{N}{T}+\frac{N}{M^2}\Big)\right)$$

which follows since $\sum_{s=0}^{N-1}\left(\frac{1-N/2+s}{T}\right) = \frac{(N-1)(1-N/2+N/2)}{T} = \frac{N}{2T}$ and similarly $\sum_{s=0}^{N-1}\left(\frac{1-N/2+s}{T}\right)^2 = O\big(\frac{N^3}{T^2}\big)$. We additionally note that, off manifold, the first term of (5.5) can be bounded in norm by

$$=\left\|\left\|\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\exp\left(-\mathrm{i}s_k\sum_j\omega_j\right)\exp\left(-\mathrm{i}\sum_{j=1}^{k-1}\omega_j(s_j-s_k)\right)C_{u_i-\frac{N/2-s_k-1}{T},s_1-s_k,s_2-s_k,\ldots,s_{k-1}-s_k}\right\|\right\|_p$$
$$\leq\frac{K}{(2\pi N)^{k/2}}\sum_{|l_1|\ldots,|l_{k-1}|<N}\left|\sum_{s_k=0}^{N-|l_j^*|}e^{-\mathrm{i}s_k\sum_j\omega_j}\right|\left\|\left\|C_{u_i-\frac{N/2-s_k-1}{T},l_1,l_2,\ldots,l_{k-1}}\right\|\right\|_p$$
$$\leq\frac{K}{(2\pi N)^{k/2}}\sum_{|l_1|\ldots,|l_{k-1}|<N}|l_j^*|\,\|\|\kappa_{k;t_1,\ldots,t_{l-1}}\|\|_p = O\left(N^{-k/2}\right),$$

for some constant K and where $j^* = \underset{j=1,\ldots,k-1}{\arg\max}|l_j|$. This finishes the proof of lemma 5.1. □

Proof of Lemma 5.2. Using again the linearity of cumulants we write

$$\mathrm{Cum}\left(D_N^{u_{j_1},\omega_1},\ldots,D_N^{u_{j_k},\omega_k}\right)$$
$$=\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\exp\left(-\mathrm{i}\sum_{v=1}^{k}s_v\omega_v\right)\mathrm{Cum}\left(X_{\lfloor u_{j_1}T\rfloor-N/2+s_1+1,T},\ldots,X_{\lfloor u_{j_k}T\rfloor-N/2+s_k+1,T}\right)$$
$$=\frac{1}{(2\pi N)^{k/2}}\sum_{s_1,\ldots,s_k=0}^{N-1}\exp\left(-\mathrm{i}\sum_{v=1}^{k}s_v\omega_v\right)C_{u'_k,\lfloor u_{j_1}T\rfloor-\lfloor u_{j_k}T\rfloor+s_1-s_k,\ldots,\lfloor u_{j_{k-1}}T\rfloor-\lfloor u_kT\rfloor+s_{k-1}-s_k}+R^1_{k,M,N}$$

where $u'_k = u_k-\frac{N/2-s_k-1}{T}$ and where $R^1_{k,M,N}$ is the error term derived in Lemma 5.1. Let

$$l_m = \lfloor u_{j_m}T\rfloor-\lfloor u_kT\rfloor+s_m-s_k \leftrightarrow s_m = t_{j_k}-t_{j_m}+l_m-s_k \qquad m=1,\ldots k-1$$

Similar to the proof of Lemma 5.1, we note that

$$\sum_{\substack{l_1,l_2,\ldots l_{k-1},\\|l_m|>N}}\|\|C_{u'_k,l_1,\ldots,l_m,\ldots,l_{k-1}}\|\|_p \leq \sum_{\substack{l_1,l_2,\ldots l_{k-1}\\,|l_m|>N}}\frac{|l_m|^2}{N^2}\|\|\kappa_{k,l_1,\ldots,l_{k-1}}\|\|_p = O(N^{-2}).$$

From which it follows that if $|l_m|>N$, the term

$$\left\|\left\|\frac{1}{(2\pi N)^{k/2}}\sum_{s_k=0}^{N-1}e^{-\mathrm{i}s_k\sum_{v=1}^{k}\omega_v}\sum_{\substack{l_1,l_2,\ldots l_{k-1},\\|l_m|>N}}e^{-\mathrm{i}\sum_{v=1}^{k-1}(t_{j_v}-t_{j_k}+l_v)\omega_v}C_{u'_k,l_1,\ldots,l_{k-1}}\right\|\right\|_p$$

is bounded by

$$\frac{1}{(2\pi N)^{k/2}}\sum_{s=0}^{N-1}\sum_{\substack{l_1,l_2,\ldots,l_{k-1}\\|l_m|\geq N}}\|\|\kappa_{k,l_1,\ldots,l_m,\ldots,l_{k-1}}\|\|_2$$
$$\leq\frac{1}{(2\pi N)^{k/2+2}}\sum_{s=0}^{N-1}\sum_{\substack{l_1,l_2,\ldots,l_{k-1}\\|l_1|\geq N}}|l_m|^2\|\|\kappa_{k,l_1,\ldots,l_m,\ldots,l_{k-1}}\|\|_p = O\left(N^{-k/2-1}\right).$$

□

A.3 Derivation of covariance and higher order cumulants

Covariance structure of $\sqrt{T}\hat{F}_{1,T}$

$$T\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{1,T}) = T\mathrm{Cum}(\frac{1}{T}\sum_{k_1=1}^{\lfloor N/2\rfloor}\sum_{j_1=1}^{M}\langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}, \frac{1}{T}\sum_{k_2=1}^{\lfloor N/2\rfloor}\sum_{j_2=1}^{M}\overline{\langle I_N^{u_{j_2},\omega_{k_2}}, I_N^{u_{j_2},\omega_{k_2-1}}\rangle}_{HS})$$

Using again Theorem 5.1

$$\mathrm{Cum}_2(\sqrt{T}\hat{F}_{1,T}) = \frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\bigotimes_{g=1}^{G}\mathbb{E}\Big[\otimes_{(l,m)\in P_g} D_N^{u_l,\omega_m}\Big]\Big)$$

where $p=(l,m)$ with $k_p=(-1)^{l-m}k_l-\delta_{\{m\in\{3,4\}\}}$ and $j_p=j_l$ for $l\in\{1,2\}$ and $m\in\{1,2,3,4\}$ and where $\delta_{\{A\}}$ equals 1 if event A occurs and 0 otherwise. In particular, we are interested in all indecomposable partitions of the array

$$\begin{array}{cccc}
\underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{A} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{B} & \underbrace{D_N^{u_{j_1},-\omega_{k_1-1}}}_{C} & \underbrace{D_N^{u_{j_1},\omega_{k_1-1}}}_{D} \\
\underbrace{D_N^{u_{j_2},-\omega_{k_2}}}_{E} & \underbrace{D_N^{u_{j_2},\omega_{k_2}}}_{F} & \underbrace{D_N^{u_{j_2},\omega_{k_2-1}}}_{G} & \underbrace{D_N^{u_{j_2},-\omega_{k_2-1}}}_{H}
\end{array}$$

By Lemma 5.3, all partitions of size $G<3$, will be of lower order. Moreover, partitions with sets consisting of 3 elements would imply a restriction in frequency direction (see the proof of Theorem A.1) and are therefore of lower order. The only partitions that remain are those that contain either one fourth-order cumulant and two second-order cumulants or those consisting only of second-order cumulants. Additionally, from these two structures Corollary A.1 and Lemma 5.2 indicate that for the partitions with structure $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$ to be indecomposable there must be at least one restriction in time. More restrictions in terms of frequency would mean the partition term is of lower order.

For the structure $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are therefore

$$\mathrm{Tr}\Big(S_{ABEFCDGH}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},\omega_{k_2-1}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(ABGH)(CD)(EF)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2-1}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(CDEF)(AB)(GH)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},\omega_{k_2-1}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(CDGH)(AB)(EF)}\Big(\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},\omega_{k_2-1}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2\Big]\Big).$$

For the partitions with structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, there must be at least one restriction in terms of time and frequency for the partition to be indecomposable. Those with more than the minimum restrictions are of lower order. For the structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, the significant indecomposable partitions are

$$\mathrm{Tr}\Big(S_{(AB)(CG)(EF)(DH)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2}[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2]\Big)$$
$$\mathrm{Tr}\Big(S_{(AB)(CF)(GH)(DE)}\Big(\delta_{j_1,j_2}\delta_{k_1-1,k_2}[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_2-1}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2]\Big)$$
$$\mathrm{Tr}\Big(S_{(AE)(BF)(CG)(DH)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2}[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2]\Big)$$

$$\mathrm{Tr}\Big(S_{(AE)(BF)(CD)(GH)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_2-1}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{(AH)(BG)(CD)(EF)}\Big(\delta_{j_1,j_2}\delta_{k_1,k_2-1}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\epsilon_2\big]\Big).$$

Therefore, we find

$$\begin{aligned}
&T\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{1,T})=\\
&\quad\frac{4}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}\frac{2\pi}{N}\langle\mathscr{F}^{\dagger}_{u,\omega_{k_1},-\omega_{k_1},-\omega_{k_2}},\mathscr{F}_{u,\omega_{k_1}}\otimes\mathscr{F}_{u,\omega_{k_2}}\rangle_{HS}+O(\frac{1}{NM^2}+\frac{1}{M^2})\\
&\quad+\frac{5}{T}\sum_{k=1}^{\lfloor N/2\rfloor}\sum_{j=1}^{M}|||\mathscr{F}_{u_j,\omega_k}\otimes\mathscr{F}_{u_j,\omega_k}|||_2^2+O(\frac{1}{M^2})
\end{aligned}$$

So that, as $T\to\infty$,

$$\begin{aligned}
&T\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{1,T})\to\\
&\quad=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\langle\mathscr{F}^{\dagger}_{u,\omega_1,-\omega_1,-\omega_2},\mathscr{F}_{u,\omega_1}\otimes\mathscr{F}_{u,\omega_2}\rangle_{HS}\,du\,d\omega_1\,d\omega_2+\frac{5}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}|||\mathscr{F}_{u,\omega}\otimes\mathscr{F}_{u,\omega}|||_2^2\,du\,d\omega
\end{aligned}$$

The corresponding kernel function is given by

$$\begin{aligned}
&\lim_{NM\to\infty}T\mathrm{Cov}\big(\hat{F}_{1,T}(\tau_1,\sigma_1),\hat{F}_{1,T}(\tau_2,\sigma_2)\big)=\\
&\qquad\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{0}^{1}f_{u,\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2)f_{u,-\omega_1}(\tau_1,\sigma_1)f_{u,\omega_2}(\tau_2,\sigma_2)\,du\,d\omega_1\,d\omega_2\\
&\qquad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{1}f_{u,\omega}(\tau_1,\sigma_1)f_{u,-\omega}(\tau_1,\tau_2)f_{u,-\omega}(\tau_2,\sigma_2)f_{u,\omega}(\sigma_1,\sigma_2)\,du\,d\omega\\
&\qquad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{1}f_{u,\omega}(\tau_1,\sigma_1)f_{u,-\omega}(\tau_1,\sigma_2)f_{u,\omega}(\tau_2,\sigma_2)f_{u,\omega}(\sigma_1,\tau_2)\,du\,d\omega\\
&\qquad+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{1}\big|f_{u,\omega}(\tau_1,\tau_2)f_{u,-\omega}(\sigma_1,\sigma_2)\big|^2\,du\,d\omega
\end{aligned}$$

Remark A.1. *To see why we only have the option $j_2=j_1$ and also $j_2\in\{-1,1\}$ will be of lower order, consider for example the decomposition*

$$\begin{aligned}
&\frac{1}{T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}(S_{AEBFCDGH)}(Cum(AE)\otimes Cum(BF)\otimes Cum(CD)\otimes Cum(GH))\\
&=\frac{1}{N^4T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\sum_{\substack{s_1,p_1,l_1,l_3,\\ r_2,p_2,l_2,l_4\\ 0\le t_{j,2}-t_{j,1}+l_1+r_2)<N\\ 0\le t_{j,1}-t_{j,2}-l_2+s_1)<N}}e^{-\mathrm{i}\omega_{k_1}(t_{j,2}-t_{j,1}+l_1+r_2-s_1-l_3)}e^{-\mathrm{i}\omega_{k_2}(-r_2+(t_{j,1}-t_{j,2}-l_2+s_1)+l_4)}\\
&\qquad\times\mathrm{Tr}(C_{u_{j_2,r_2},l_1}\otimes C_{u_{j_1,-l_2+s_1},l_2}\otimes C_{u_{j_1,q_1},l_3}\otimes C_{u_{j_2,q_2},l_4})
\end{aligned}$$

We can bound this by

$$\frac{1}{N^2T}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\sum_{\substack{s_1,r_2\\ 0\le t_{j,2}-t_{j,1}+l_1+r_2<N\\ 0\le t_{j,1}-t_{j,2}-l_2+s_1<N}}\sum_{l_1,l_2}e^{-\mathrm{i}(\omega_{k_1}-\omega_{k_2})(r_2-s_1+t_{j,1}-t_{j,2})}|||\kappa_{2;l_1}|||_p|||\kappa_{2;l_2}|||_p(\sum_{l}|||\kappa_{2;l}|||_p)^2$$

Since $|r_2 - s_1| \leq N-1$, *we find that if* $t_{j,2} - t_{j,1} \neq 0$, *then once* s_1 *is determined there is at most one option for* r_2 *in order for* $r_2 - s_1 + t_{j,1} - t_{j,2} \neq zN, z \in \mathbb{Z}$ *and hence such that the term*

$$\sum_{k_1=1}^{\lfloor N/2 \rfloor} e^{-\mathrm{i}(\omega_{k_1})(r_2-s_1+t_{j,1}-t_{j,2})}$$

will be of order O(N). Additionally, when $t_{j,2} - t_{j,1} \neq 0$, *then* $s_1 < |l_2|$. *Therefore, we obtain*

$$\frac{1}{N^2 T} \sum_{k_1,k_2=1}^{\lfloor N/2 \rfloor} \sum_{j_1,j_2=1}^{M} \sum_{l_1} |l_2| |||\kappa_{2;l_1}|||_p \sum_{l_2} |l_2| |||\kappa_{2;l_2}|||_p (\sum_{l} |||\kappa_{2;l}|||_p)^2 = O(\frac{1}{T} M) = O(\frac{1}{N}).$$

Showing that also if $j_2 \in \{-1,1\}$, *we will obtain a term of lower order due to the constraint over frequencies.*

Covariance structure of $\sqrt{T}\hat{F}_{2,T}$

$$T\mathrm{Cov}(\hat{F}_{2,T}, \hat{F}_{2,T}) = T\mathrm{Cum}(\frac{1}{NM^2} \sum_{k_1=1}^{\lfloor N/2 \rfloor} \sum_{j_1,j_2=1}^{M} \langle I_N^{u_{j_1},\omega_{k_1}}, I_N^{u_{j_2},\omega_{k_1-1}} \rangle_{HS}, \frac{1}{NM^2} \sum_{k_2=1}^{\lfloor N/2 \rfloor} \sum_{j_3,j_4=1}^{M} \overline{\langle I_N^{u_{j_3},\omega_{k_2}}, I_N^{u_{j_4},\omega_{k_2-1}} \rangle}_{HS})$$

Using again Theorem 5.1

$$\mathrm{Cum}_2(\hat{F}_{2,T}) = \frac{1}{N^2 M^4} \sum_{k_1,k_2=1}^{\lfloor N/2 \rfloor} \sum_{\substack{j_1,j_2,\\ j_3,j_4=1}}^{M} \mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1 \cup \ldots \cup P_G} (-1)^{G-1}(G-1)! \bigotimes_{g=1}^{G} \mathbb{E}\Big[\otimes_{(l,m)\in P_g} D_N^{u_l,\omega_m} \Big]\Big)$$

where $p=(l,m)$ with $k_p = (-1)^{l-m} k_l$ and $j_p = j_{2l-\delta_{\{m \in \{1,2\}\}}}$ for $l \in \{1,2\}$ and $m \in \{1,2,3,4\}$ and where $\delta_{\{A\}}$ equals 1 if event A occurs and 0 otherwise. . That is, we are interested in all indecomposable partitions of the array

$$\begin{array}{cccc}
\underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{A} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{B} & \underbrace{D_N^{u_{j_2},-\omega_{k_1}}}_{C} & \underbrace{D_N^{u_{j_2},\omega_{k_1}}}_{D} \\
\underbrace{D_N^{u_{j_3},-\omega_{k_2}}}_{E} & \underbrace{D_N^{u_{j_3},\omega_{k_2}}}_{F} & \underbrace{D_N^{u_{j_4},\omega_{k_2}}}_{G} & \underbrace{D_N^{u_{j_4},-\omega_{k_2}}}_{H}
\end{array}$$

For the same reason as above, we only have to consider the structures $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$ and $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$. For the structure $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are again

$$\mathrm{Tr}\Big(S_{ABEFCDGH}\Big(\delta_{j_1,j_3}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}} + \mathscr{E}_4) \otimes (\mathscr{F}_{u_{j_2},-\omega_{k_1}} + \mathscr{E}_2) \otimes (\mathscr{F}_{u_{j_4},\omega_{k_2}} + \mathscr{E}_2)\Big]\Big)$$

$$\mathrm{Tr}\Big(S_{(ABGH)(CD)(EF)}\Big(\delta_{j_1,j_4}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2}} + \mathscr{E}_4) \otimes (\mathscr{F}_{u_{j_2},-\omega_{k_1}} + \mathscr{E}_2) \otimes (\mathscr{F}_{u_{j_3},-\omega_{k_2}} + \mathscr{E}_2)\Big]\Big)$$

$$\mathrm{Tr}\Big(S_{(CDEF)(AB)(GH)}\Big(\delta_{j_2,j_3}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_2},-\omega_{k_1},\omega_{k_1},-\omega_{k_2}} + \mathscr{E}_4) \otimes (\mathscr{F}_{u_{j_1},\omega_{k_1}} + \mathscr{E}_2) \otimes (\mathscr{F}_{u_{j_4},\omega_{k_2}} + \mathscr{E}_2)\Big]\Big)$$

$$\mathrm{Tr}\Big(S_{(CDGH)(AB)(EF)}\Big(\delta_{j_2,j_4}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_2},-\omega_{k_1},\omega_{k_1},\omega_{k_2}} + \mathscr{E}_4) \otimes (\mathscr{F}_{u_{j_1},\omega_{k_1}} + \mathscr{E}_2) \otimes (\mathscr{F}_{u_{j_3},-\omega_{k_2}} + \mathscr{E}_2\Big]\Big)$$

For the structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are in this case

$$\mathrm{Tr}\Big(S_{(AB)(CG)(EF)(DH)}\big(\delta_{j_2,j_4}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}} \otimes \mathscr{F}_{u_{j_2},-\omega_{k_1}} \otimes \mathscr{F}_{u_{j_3},-\omega_{k_2}} \otimes \mathscr{F}_{u_{j_2},\omega_{k_1}} + \mathscr{E}_2\big]\big)\Big)$$

$$\mathrm{Tr}\Big(S_{(AB)(CF)(GH)(DE)}\big(\delta_{j_2,j_3}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}} \otimes \mathscr{F}_{u_{j_2},-\omega_{k_1}} \otimes \mathscr{F}_{u_{j_4},\omega_{k_2}} \otimes \mathscr{F}_{u_{j_2},\omega_{k_1}} + \mathscr{E}_2\big]\big)\Big)$$

$$\mathrm{Tr}\Big(S_{(AE)(BF)(CD)(GH)}\big(\delta_{j_1,j_3}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_4},\omega_{k_2}}+\mathscr{E}_2\big]\big)\Big)$$
$$\mathrm{Tr}\Big(S_{(AH)(BG)(CD)(EF)}\big(\delta_{j_1,j_4}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_3},-\omega_{k_2}}+\mathscr{E}_2\big]\big)\Big).$$

which converges to

$$\begin{aligned}\lim_{N,M\to\infty} NM\mathrm{Cov}\big(\hat{F}_{2,T},\hat{F}_{2,T}\big) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\langle\widetilde{\mathscr{F}}^{\dagger}_{\omega_1,-\omega_1,-\omega_2},\widetilde{\mathscr{F}}_{\omega_1}\otimes\widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}d\omega_1\mathrm{d}\omega_2\\ &+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\langle\mathscr{F}_{u,\omega},\widetilde{\mathscr{F}}_{\omega}\rangle_{HS}\langle\widetilde{\mathscr{F}_{\omega}},\mathscr{F}_{u,\omega}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega\\ &+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\langle\mathscr{F}_{u,\omega}\otimes\mathscr{F}_{u,\omega},\widetilde{\mathscr{F}}_{\omega}\otimes\widetilde{\mathscr{F}_{\omega}}\rangle_{HS}\mathrm{d}u\mathrm{d}\omega.\end{aligned}$$

The corresponding kernel is given by

$$\begin{aligned}&\lim_{N,M\to\infty} 2\pi T\,\mathrm{Cov}(\hat{F}_{2,T}(\tau_1,\sigma_1),\hat{F}_{2,T}(\tau_2,\sigma_2)) =\\ &\quad\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1\int_0^1\int_0^1 f_{u_1,\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2)f_{u_2,-\omega_1}(\tau_1,\sigma_1)f_{u_3,\omega_2}(\tau_2,\sigma_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}\omega_1\mathrm{d}\omega_2\\ &\quad+\int_{-\pi}^{\pi}\int_0^1\int_0^1\int_0^1 f_{u_1,\omega}(\tau_1,\sigma_1)f_{u_2,-\omega}(\tau_1,\tau_2)f_{u_3,-\omega}(\tau_2,\sigma_2)f_{u_2,\omega}(\sigma_1,\sigma_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}\omega\\ &\quad+\int_{-\pi}^{\pi}\int_0^1\int_0^1\int_0^1 f_{u_1,\omega}(\tau_1,\sigma_1)f_{u_2,-\omega}(\tau_1,\sigma_2)f_{u_3,\omega}(\tau_2,\sigma_2)f_{u_2,\omega}(\sigma_1,\tau_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}\omega.\end{aligned}$$

Cross-covariance $\hat{F}_{1,T}$ and $\hat{F}_{2,T}$

Using again Theorem 5.1

$$T\mathrm{Cov}_2(\hat{F}_{1,T},\hat{F}_{2,T}) = \frac{1}{N^2M^4}\sum_{k_1,k_2=1}^{\lfloor N/2\rfloor}\sum_{j_1,j_2,j_3=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\bigotimes_{g=1}^{G}\mathbb{E}\Big[\otimes_{(l,m)\in P_g}D_N^{u_l,\omega_m}\Big]\Big),$$

where this time we are interested in all indecomposable partitions of the array

$$\begin{array}{cccc}\underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{A} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{B} & \underbrace{D_N^{u_{j_1},-\omega_{k_1-1}}}_{C} & \underbrace{D_N^{u_{j_1},\omega_{k_1-1}}}_{D}\\ \underbrace{D_N^{u_{j_2},-\omega_{k_2}}}_{E} & \underbrace{D_N^{u_{j_2},\omega_{k_2}}}_{F} & \underbrace{D_N^{u_{j_3},\omega_{k_2}}}_{G} & \underbrace{D_N^{u_{j_3},-\omega_{k_2}}}_{H}\end{array}$$

For the partitions of the form $\mathrm{Cum}_4\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are again

$$\mathrm{Tr}\Big(S_{(ABEF)(CD)(GH)}\delta_{j_1,j_2}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_3},\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(ABGH)(CD)(EF)}\delta_{j_1,j_3}\Big[\frac{2\pi}{N}\mathscr{F}_{u_{j_1},\omega_{k_1},-\omega_{k_1},\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(CDEF)(AB)(GH)}\delta_{j_1,j_2}\Big[\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},-\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_3},\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$
$$\mathrm{Tr}\Big(S_{(CDGH)(AB)(EF)}\delta_{j_1,j_3}\Big[(\frac{2\pi}{N}\mathscr{F}_{u_{j_1},-\omega_{k_1-1},\omega_{k_1-1},\omega_{k_2}}+\mathscr{E}_4)\otimes(\mathscr{F}_{u_{j_1},\omega_{k_1}}+\mathscr{E}_2)\otimes(\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2)\Big]\Big)$$

For the structure $\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2\mathrm{Cum}_2$, the only significant terms are

$$\mathrm{Tr}\Big(S_{(AB)(CG)(EF)(DH)}\delta_{j_1,j_3}\delta_{k_1-1,k_2}\delta_{j_1,j_3}\delta_{k_1-1,k_2}\big(\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2\big]\Big)$$

$$\mathrm{Tr}\Big(S_{(AB)(CF)(GH)(DE)}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\delta_{j_1,j_2}\delta_{k_1-1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\mathscr{F}_{u_{j_3},\omega_{k_2}}\otimes\mathscr{F}_{u_{j_1},\omega_{k_1-1}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{(AE)(BF)(CD)(GH)}\delta_{j_1,j_2}\delta_{k_1,k_2})^2\delta_{j_1,j_2}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},\omega_{k_2}}+\mathscr{E}_2\big]\Big)$$
$$\mathrm{Tr}\Big(S_{((AH)(BG)(CD)(EF)}\delta_{j_1,j_3}\delta_{k_1,k_2})^2\delta_{j_1,j_3}\delta_{k_1,k_2}\big[\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2}}+\mathscr{E}_2\big]\Big)$$

Collecting terms, the cross covariance converges to

$$\begin{aligned}\lim_{N,M\to\infty} NM\mathrm{Cov}(\hat{F}_{1,T},\hat{F}_{2,T}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1 \langle \mathscr{F}^{\dagger}_{u_1,\omega_1,-\omega_1,-\omega_2},\mathscr{F}_{u_1,-\omega_1}\otimes\widetilde{\mathscr{F}}_{\omega_2}\rangle_{HS}\mathrm{d}u_1\mathrm{d}\omega_1\mathrm{d}\omega_2\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 \langle(\mathscr{F}_{u_1,\omega_1}\otimes\mathscr{F}_{u_1,-\omega_1})^{\dagger},\widetilde{\mathscr{F}}_{\omega_1}\otimes\mathscr{F}_{u_1,\omega_1}\rangle_{HS}\mathrm{d}u_1\mathrm{d}\omega_1\\
&\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^1 |||\mathscr{F}_{u_1,\omega_1}\otimes\mathscr{F}_{u_1,\omega_1}|||_2^2\mathrm{d}u_1\mathrm{d}\omega_1\\
&+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^1 \langle(\mathscr{F}_{u_1,\omega_1}\otimes\mathscr{F}_{u_1,-\omega_1})^{\dagger},\mathscr{F}_{u_1,-\omega_1}\otimes\widetilde{\mathscr{F}}_{-\omega_1}\rangle_{HS}\mathrm{d}u_1\mathrm{d}\omega_1,\end{aligned}$$

and the kernel function is given by

$$\begin{aligned}&\lim_{N,M\to\infty} T\,\mathrm{Cov}(\hat{F}_{1,T}(\tau_1,\sigma_1),\hat{F}_{2,T}(\tau_2,\sigma_2))\\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1\int_0^1 f_{u_1,\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2)f_{u_1,-\omega_1}(\tau_1,\sigma_1)f_{u_2,\omega_2}(\tau_2,\sigma_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}\omega_1\mathrm{d}\omega_2\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\int_0^1 f_{u_1,\omega}(\tau_1,\sigma_1)f_{u_1,-\omega}(\tau_1,\tau_2)f_{u_2,-\omega}(\tau_2,\sigma_2)f_{u_1,\omega}(\sigma_1,\sigma_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}\omega\\
&+\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1\int_0^1 f_{u_1,\omega}(\tau_1,\sigma_1)f_{u_1,-\omega}(\tau_1,\sigma_2)f_{u_2,\omega}(\tau_2,\sigma_2)f_{u_1,\omega}(\sigma_1,\tau_2)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}\omega.\end{aligned}$$

A.3.1 Limiting Variance of $\widehat{M_T^2}$

The limiting variance of $\widehat{M_T^2}$ is then simply $\mathrm{Var}(\hat{F}_{1,T}-\hat{F}_{2,T})$ and its expression in (3.8) is easily derived. For the expression of its kernel, denote

$$\sigma^2((\tau_1,\sigma_1),(\tau_2,\sigma_2)) := T\mathrm{Cov}\big(\hat{F}_{1,T}(\tau_1,\sigma_1)-\hat{F}_{2,T}(\tau_1,\sigma_1),\hat{F}_{1,T}(\tau_2,\sigma_2)-\hat{F}_{2,T}(\tau_2,\sigma_2)\big),$$

then the previous results of this section show that we find

$$\begin{aligned}\nu^2 := T\mathrm{Var}(\widehat{M_T^2}) &= 16\pi^2\int_{[0,1]^4}\sigma^2((\tau_1,\sigma_1),(\tau_2,\sigma_2))d\tau_1 d\tau_2 d\sigma_1 d\sigma_2\\
=&8\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^1\int_{[0,1]^4} f_{u,-\omega_1}(\tau_1,\sigma_1)f_{u,\omega_2}(\tau_2,\sigma_2)f_{u,\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2)d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 du d\omega_1 d\omega_2\\
&+16\pi\int_{-\pi}^{\pi}\int_0^1\int_{[0,1]^4} f_{u,\omega}(\tau_1,\sigma_1)f_{u,\omega}(\sigma_1,\sigma_2)f_{u,-\omega}(\tau_2,\sigma_2)f_{u,-\omega}(\tau_1,\tau_2)d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 du d\omega\\
&+4\pi\int_{-\pi}^{\pi}\int_0^1\left(\int_{[0,1]^2}|f_{u,\omega}(\tau,\sigma)|^2 d\tau d\sigma\right)^2 du d\omega\\
&+8\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{[0,1]^3}\int_{[0,1]^4} f_{u_1,\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2)f_{u_2,-\omega_1}(\tau_1,\sigma_1)f_{u_3,\omega_2}(\tau_2,\sigma_2)\tau_1 d\tau_2 d\sigma_1 d\sigma_2\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}\omega_1\mathrm{d}\omega_2\\
&+16\pi\int_{-\pi}^{\pi}\int_{[0,1]^3}\int_{[0,1]^4} f_{u_1,\omega}(\tau_1,\sigma_1)f_{u_2,-\omega}(\tau_1,\tau_2)f_{u_3,-\omega}(\tau_2,\sigma_2)f_{u_2,\omega}(\sigma_1,\sigma_2)d\tau_1 d\tau_2 d\sigma_1 d\sigma_2\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}\omega\end{aligned}$$

$$-16\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{[0,1]^2}\int_{[0,1]^4} f_{u_1,\omega_1,-\omega_1,-\omega_2}(\tau_1,\sigma_1,\tau_2,\sigma_2) f_{u_1,-\omega_1}(\tau_1,\sigma_1) f_{u_2,\omega_2}(\tau_2,\sigma_2) d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 du_1 du_2 d\omega_1 d\omega_2$$
$$-32\pi\int_{-\pi}^{\pi}\int_{[0,1]^2}\int_{[0,1]^4} f_{u_1,\omega}(\tau_1,\sigma_1) f_{u_1,-\omega}(\tau_1,\sigma_2) f_{u_2,\omega}(\tau_2,\sigma_2) f_{u_1,\omega}(\sigma_1,\tau_2) d\tau_1 d\tau_2 d\sigma_1 d\sigma_2 du_1 du_2 d\omega.$$

Under H_0 this reduces to

$$\nu_{H_0}^2 = 4\pi\int_{-\pi}^{\pi}\left(\int_{[0,1]^2} |f_\omega(\tau,\sigma)|^2 d\tau d\sigma\right)^2 d\omega.$$

A.4 Higher order cumulants

A.4.1 $n-$th order cumulants of $\hat{F}_{1,T}$ for finite n

Theorem A.1. *For finite n,*

$$T^{n/2} Cum\left(\hat{F}_{1,T},\dots,\hat{F}_{1,T}\right) = O(T^{1-n/2})$$

Proof of Theorem A.1. By Theorem 5.1

$$\begin{aligned}
&T^{n/2}\mathrm{Cum}\left(\hat{F}_{1,T},\dots,\hat{F}_{1,T}\right)\\
=&\frac{1}{T^{n/2}}\sum_{k_1,\dots,k_n=1}^{\lfloor N/2\rfloor}\sum_{j_1,\dots,j_n=1}^{M}\mathrm{Cum}\Big(\mathrm{Tr}(D_N^{u_{j_1},\omega_{k_1}}\otimes D_N^{u_{j_1},-\omega_{k_1}}\otimes D_N^{u_{j_1},-\omega_{k_1-1}}\otimes D_N^{u_{j_1},\omega_{k_1-1}}),\dots\\
&\qquad\dots,\mathrm{Tr}(D_N^{u_{j_n},-\omega_{k_n}}\otimes D_N^{u_{j_n},\omega_{k_n}}\otimes D_N^{u_{j_n},\omega_{k_n-1}}\otimes D_N^{u_{j_n},-\omega_{k_n-1}}\Big)\\
=&\frac{1}{T^{n/2}}\sum_{k_1,\dots,k_n=1}^{\lfloor N/2\rfloor}\sum_{j_1,\dots,j_n=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\dots\cup P_G}(-1)^{G-1}(G-1)!\bigotimes_{g=1}^{G}\mathbb{E}\Big[\otimes_{p\in P_g} D_N^{u_{j_p},\omega_{k_p}}\Big]\Big)
\end{aligned}\tag{5.6}$$

where the summation is over all indecomposable partitions $\boldsymbol{P} = P_1\cup\dots\cup P_G$ of the array

$$\begin{matrix}
(1,1) & (1,2) & (1,3) & (1,4)\\
(2,1) & (2,2) & (2,3) & (2,4)\\
\vdots & & & \vdots\\
\vdots & & & \vdots\\
(n,1) & (n,2) & (n,3) & (n,4)
\end{matrix}\tag{5.7}$$

and, for $p=(l,m)$, $k_p=(-1)^m k_l-\delta_{\{m\in\{3,4\}\}}$ and $j_p=j_l$ for $l\in\{1,\dots,n\}$ and $m\in\{1,2,3,4\}$ and where $\delta_{\{A\}}$ equals 1 if event A occurs and 0 otherwise. For a fixed partition $P=\{P_1,\dots,P_G\}$, let the cardinality of set P_g be given by $|P_g|=\mathscr{C}_g$. By Lemma 5.3 with $x=0$, we find the following bound

$$\frac{1}{T^{n/2}}\sum_{k_1,\dots,k_n=1}^{\lfloor N/2\rfloor}\sum_{j_1,\dots,j_n=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\dots\cup P_G}(-1)^{G-1}(G-1)!\bigotimes_{g=1}^{G}\mathbb{E}\Big[\otimes_{p\in P_g} D_N^{u_{j_p},\omega_{k_p}}\Big]\Big) = O(T^{1-n/2}N^{G-n-1})$$

and therefore partitions of size $G-n-1<0\Leftrightarrow G\le n$, will be of lower order and we only have to consider partitions of size $G\ge n+1$. This means only those indecomposable partitions will remain for which there must at least be one more set within each partition than that there are rows. For $n>2$, the lemma moreover implies we can restrict ourselves to the case $G>n+1$. Hence, to those partitions where there are two more sets in the partition than there are rows. We remark this holds uniformly over all combinations of possible frequencies. However, by Corollary A.1, a partition $P = \{P_1, \ldots, P_G\}$ that contains a set such that $\sum_{j \in P_g} \omega_j \neq 0 \mod 2\pi$ is of lower order and leads the bound in Lemma 5.3 to be multiplied by an order N^{-1}. Proposition 5.1 therefore implies that we find (5.6) becomes

$$O(T^{1-n/2} N^{G-n-1} N^{-r_1}) = O(T^{1-n/2} N^{r_1 - r_1}) = O(T^{1-n/2})$$

and asymptotic normality of $\hat{F}_{1,T}$ is established. □

A.4.2 $n-$th order cumulants of $\hat{F}_{2,T}$ for finite n

Theorem A.2. *For finite n,*

$$T^{n/2} Cum\big(\hat{F}_{2,T}, \ldots, \hat{F}_{2,T}\big) = O(T^{1-n/2})$$

Proof of Theorem A.2. By Theorem 5.1

$$\begin{aligned} & T^{n/2} \mathrm{Cum}\big(\hat{F}_{2,T}, \ldots, \hat{F}_{2,T}\big) \\ & = \frac{1}{T^{n/2} M^n} \sum_{k_1, \ldots, k_n = 1}^{\lfloor N/2 \rfloor} \sum_{j_1, \ldots, j_{2n} = 1}^{M} \mathrm{Tr}\Big(\sum_{\boldsymbol{P} = P_1 \cup \ldots \cup P_G} (-1)^{G-1} (G-1)! \bigotimes_{g=1}^{G} \mathbb{E}\Big[\otimes_{p \in P_g} D_N^{u_{j_p}, \omega_{k_p}} \Big] \Big) \end{aligned} \tag{5.8}$$

where the summation is over all indecomposable partitions $\boldsymbol{P} = P_1 \cup \ldots \cup P_G$ of the array

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ (n,1) & (n,2) & (n,3) & (n,4) \end{matrix} \tag{5.9}$$

and, for $p = (l, m)$, $k_p = (-1)^m k_l$ and $j_p = j_{2l - \delta_{\{m \in \{1,2\}\}}}$ for $l \in \{1, \ldots, n\}$ and $m \in \{1,2,3,4\}$. By Lemma 5.3 with $x = n$ this is of order

$$O(N^{n/2} M^{-3n/2} E^n M^{n+1} N^{-2n+G}) = O(T^{1-n/2} N^{G-n-1}).$$

Proposition 5.1 yields that $G \geq n + r_1 + 1$ for $r_1 \geq 1$

$$O(T^{1-n/2} N^{G-n-1} N^{-r_1}) = O(T^{1-n/2} N^{r_1 - r_1}) = O(T^{1-n/2})$$

and asymptotic normality of $\hat{F}_{2,T}$ is established. □

A.5 Estimation of Variance

Proof of Lemma 3.1: We write

$$\begin{aligned} \mathbb{E}(\hat{\upsilon}^2_{H_0}) = & \frac{16\pi^2}{N} \sum_{k=2}^{\lfloor N/2 \rfloor} \mathbb{E}\Big[\frac{1}{M} \sum_{j=1}^{M} \langle I_N^{u_j, \omega_k}, I_N^{u_j, \omega_{k-1}} \rangle_{HS} \Big]^2 \\ = & \frac{16\pi^2}{N} \sum_{k=2}^{\lfloor N/2 \rfloor} \mathrm{Var}\Big[\frac{1}{M} \sum_{j=1}^{M} \langle I_N^{u_j, \omega_k}, I_N^{u_j, \omega_{k-1}} \rangle_{HS} \Big] + \frac{16\pi^2}{N} \sum_{k=2}^{\lfloor N/2 \rfloor} \Bigg(\mathbb{E}\Big[\frac{1}{M} \sum_{j=1}^{M} \langle I_N^{u_j, \omega_k}, I_N^{u_j, \omega_{k-1}} \rangle_{HS} \Big] \Bigg)^2 \end{aligned}$$

$$=\frac{16\pi^2}{NM^2}\sum_{k=2}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Cov}\Big[\langle I_N^{u_{j_1},\omega_k},I_N^{u_{j_1},\omega_{k-1}}\rangle_{HS},\langle I_N^{u_{j_2},\omega_k},I_N^{u_{j_2},\omega_{k-1}}\rangle_{HS}\Big]$$
$$+\frac{16\pi^2}{N}\sum_{k=2}^{\lfloor N/2\rfloor}\left(\frac{1}{M}\sum_{j=1}^{M}\mathbb{E}\Big[\langle I_N^{u_j,\omega_k},I_N^{u_j,\omega_{k-1}}\rangle_{HS}\Big]\right)^2$$

Using Theorem 5.1 we can write the first term as

$$\frac{16\pi^2}{NM^2}\sum_{k=2}^{\lfloor N/2\rfloor}\sum_{j_1,j_2=1}^{M}\mathrm{Tr}\Big(\sum_{\boldsymbol{P}=P_1\cup\ldots\cup P_G}(-1)^{G-1}(G-1)!\bigotimes_{g=1}^{G}\mathbb{E}\Big[\otimes_{p\in P_g}D_N^{u_{j_p},\omega_{k_p}}\Big]\Big)$$

where $p=(l,m)$ and $k_p=(-1)^m k_{2l-\delta\{m\in\{1,2\}\}}$ and $j_p=j_{2l-\delta\{m\in\{1,2\}\}}$ for $l\in\{1,\ldots,n\}$ and $m\in\{1,2,3,4\}$ and the sum is over all indecomposible partitions of the array

$$\begin{array}{cccc}\underbrace{D_N^{u_{j_1},\omega_{k_1}}}_{A} & \underbrace{D_N^{u_{j_1},-\omega_{k_1}}}_{B} & \underbrace{D_N^{u_{j_2},-\omega_{k_1}}}_{C} & \underbrace{D_N^{u_{j_2},\omega_{k_1}}}_{D}\\ \underbrace{D_N^{u_{j_3},-\omega_{k_2}}}_{E} & \underbrace{D_N^{u_{j_3},\omega_{k_2}}}_{F} & \underbrace{D_N^{u_{j_4},\omega_{k_2}}}_{G} & \underbrace{D_N^{u_{j_4},-\omega_{k_2}}}_{H}.\end{array}$$

Because of the indecomposability of there must be at least one set in the partition with an element from both the rows. Therefore using Corollary A.1 and Remark A.1 if $j_1\neq j_2$, the order of the term inside the summation is at most $O((NM)^{-1})$ and otherwise the order of the aforementioned term is $O(1)$. In both cases, the summation is of $o(1)$ as $N,M\to\infty$. By Lemma 5.1 and Lemma 5.2 and the fact that $\mathscr{F}_{\omega,u}\equiv\widetilde{\mathscr{F}_\omega}$ under H_0, the second term converges to $4\pi\int_{-\pi}^{\pi}\|\!|\widetilde{\mathscr{F}_\omega}|\!\|_2^4 d\omega$. Therefore under H_0 we have $\mathbb{E}(\hat{v_{H_0}}^2)\to v_{H_0}^2$ as $N,M\to\infty$.

To calculate the variance of the estimator $\hat{v}_{H_0}^2$, we first calculate

$$E\Big((\hat{v}_{H_0}^2)^2\Big)=\frac{2^8\pi^2}{N^2}\sum_{k_1,k_2}\mathbb{E}\left[\frac{1}{M^2}\sum_{j_1,j_2}\langle I_N^{u_{j_1},\omega_{k_1}},I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}},I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\right]^2$$
$$=\frac{2^8\pi^4}{N^2}\sum_{k_1,k_2}\mathrm{Var}\left[\frac{1}{M^2}\sum_{j_1,j_2}\langle I_N^{u_{j_1},\omega_{k_1}},I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}},I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\right]$$
$$+\frac{2^8\pi^4}{N^2}\sum_{k_1,k_2}\left[\mathbb{E}\left(\frac{1}{M^2}\sum_{j_1,j_2}\langle I_N^{u_{j_1},\omega_{k_1}},I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}},I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\right)\right]^2$$
$$=\frac{2^8\pi^4}{N^2M^4}\sum_{k_1,k_2}\sum_{j_1,j_2,j_3,j_4}\mathrm{Cov}\Big(\langle I_N^{u_{j_1},\omega_{k_1}},I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}},I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS},$$
$$\langle I_N^{u_{j_3},\omega_{k_1}},I_N^{u_{j_3},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_4},\omega_{k_2}},I_N^{u_{j_4},\omega_{k_2-1}}\rangle_{HS}\Big)$$
$$+\frac{2^8\pi^4}{N^2}\sum_{k_1,k_2}\left[\frac{1}{M^2}\sum_{j_1,j_2}\mathbb{E}\Big[\langle I_N^{u_{j_1},\omega_{k_1}},I_N^{u_{j_1},\omega_{k_1-1}}\rangle_{HS}\langle I_N^{u_{j_2},\omega_{k_2}},I_N^{u_{j_2},\omega_{k_2-1}}\rangle_{HS}\Big]\right]^2$$

The first term can be calculated using the product theorem for cumulant tensors (5.3) and interchanging cumulant and trace operators (the interchange is justified because the covariance is finite due to Assumption 3.1). As before to ensure the indecomposibility of the partitions the covariance inside the summation is bounded above by $O(1)$ in general, but the upper bound is $O(T^{-1})$ if $\{j_1,j_2\}\cap\{j_3,j_4\}=\emptyset$. In either case the first term converges to 0 as $N,M\to\infty$. Similar calculation of the second term yields

$$\frac{2^8\pi^2}{N^2M^2}\sum_{k_1,k_2}\sum_{j_1 j_2}\Big[\langle(\mathscr{F}_{u_{j_1},\omega_{k_1}}\otimes\mathscr{F}_{u_{j_1},-\omega_{k_1-1}})^\dagger,\mathscr{F}_{u_{j_2},\omega_{k_2}}\otimes\mathscr{F}_{u_{j_2},-\omega_{k_2-1}}\rangle_{HS}\Big]$$

Using the fact that under H_0, $\mathscr{F}_{u,\omega} \equiv \widetilde{\mathscr{F}}_\omega$ for all $u \in [0,1]$ and $\omega \in [-\pi,\pi]$, we finally get $\mathbb{E}\left((\hat{v}^2_{H_0})^2\right) \to \mathbb{E}^2(\hat{v}^2_{H_0})$ and consequently $\mathrm{Var}(\hat{v}^2_{H_0}) \to 0$ as $N, M \to \infty$. Hence $\hat{v}^2_{H_0} \xrightarrow{p} v^2_{H_0}$.

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