Distributions on unbounded moment spaces and random moment sequences

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Abstract

In this paper we define distributions on moment spaces corresponding to measures on the real line with an unbounded support. We identify these distributions as limiting distributions of random moment vectors defined on compact moment spaces and as distributions corresponding to random spectral measures associated with the Jacobi, Laguerre and Hermite ensemble from random matrix theory. For random vectors on the unbounded moment spaces we prove a central limit theorem where the centering vectors correspond to the moments of the Marchenko-Pastur distribution and Wigner's semi-circle law.

Keyword and Phrases: Gaussian ensemble, Laguerre ensemble, Jacobi ensemble, random matrix, moment spaces, canonical moments, random moment sequences, Wigner's semicircle law, Marcenko-Pastur distribution

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1 Introduction

For a set $T \subset \mathbb{R}$ let $\mathcal{P}(T)$ denote the set of all probability measures on the Borel field of T with existing moments. For a measure $\mu \in \mathcal{P}(T)$ we denote by

$$m_k(\mu) = \int_T x^k \mu(dx) \ ; \qquad k = 0, 1, 2, \ldots$$

the k-th moment and define

$$\mathcal{M}(T) = \left\{ \boldsymbol{m}(\mu) = (m_1(\mu), m_2(\mu), \dots)^T \,|\, \mu \in \mathcal{P}(T) \right\} \subset \mathbb{R}^{\mathbb{N}}. \tag{1.1}$$

as the set of all moment sequences. We denote by Π_n ($n \in \mathbb{N}$) the canonical projection onto the first n coordinates and call

$$\mathcal{M}_n(T) = \Pi_n\left(\mathcal{M}(T)\right) \subset \mathbb{R}^n. \tag{1.2}$$

the n-th moment space. Moment spaces of the form (1.1) and (1.2) have found considerable interest in the literature [see Karlin and Studden (1966)]. Most authors concentrate on the "classical" moment space corresponding to measures on the interval $[0, 1]$ [see Karlin and Shapeley (1953), Krein and Nudelman (1977), among others]. Chang et al. (1993) equipped the n-th moment space $\mathcal{M}_n([0, 1])$ with a uniform distribution in order to understand more fully its shape and the structure. In particular, these authors proved asymptotic normality of an appropriately standardized version of a projection $\Pi_k(\boldsymbol{m}_n)$ of a uniformly distributed vector $\boldsymbol{m}_n$ on $\mathcal{M}_n([0, 1])$. Since this seminal paper, several authors have extended these investigations in various directions. Gamboa and Lozada-Chang (2004) considered large deviation principles for random moment sequences on the space $\mathcal{M}_n([0, 1])$, while Lozada-Chang (2005) investigated similar problems for moment spaces corresponding to more general functions defined on a bounded set. More recently, Gamboa and Rouault (2009) discussed random spectral measures related to moment spaces of measures on the interval $[0, 1]$ and moment spaces related to measures defined on the unit circle.

The present paper is devoted to the problem of defining probability distributions on unbounded moment spaces. We will investigate these distributions from several perspectives. In Section 2 we introduce a class of general distributions on the moment space corresponding to measures defined on a compact interval. By a limiting argument we will derive canonical distributions on the moment spaces corresponding to measures on the unbounded intervals $[0, \infty)$ and $\mathbb{R}$, respectively. In Section 3 we show that these distributions appear naturally in the study of random spectral measures of the classical Jacobi, Laguerre and Gaussian ensemble. Finally, in Section 4 we consider random moment sequences distributed according to the new probability distributions on the unbounded moment spaces. In particular, we prove weak convergence of a centered random moment vector, where the centering vector corresponds to the moments of the Marchenko-Pastur law (in the case of the moment space $\mathcal{M}([0, \infty))$) and to the semi-circle law (for the moment space $\mathcal{M}(\mathbb{R})$).

2 Distributions on unbounded moment spaces

2.1 Canonical moments and recurrence coefficients of orthogonal polynomials

Chang et al. (1993) considered random vectors on the moment space $\mathcal{M}_n([0, 1])$ governed by a uniform distribution. In the present section we will define a class of more general distributions on the n-th moment space $\mathcal{M}_n([a,b])$ corresponding to the set $\mathcal{P}([a,b])$ of all probability measures on the interval $[a,b]$. The motivation for considering this class is twofold. One the one hand we want introduce distributions on the moment space $\mathcal{M}_k([a,b])$, which are different from the uniform distribution. On the other hand we want to define distributions on unbounded moment spaces as limits of distributions on $\mathcal{M}_k([a,b])$, when $b-a\to\infty$. For these purposes we will make extensive use of the theory of canonical moments, which is briefly defined here for the sake of a self contained presentation. For details we refer to the monograph of Dette and Studden (1997). Let $\boldsymbol{m}_{k-1}=(m_1,\dots,m_{k-1})^T\in\mathcal{M}_{k-1}([a,b])$ be a given vector of moments of a probability measure on the interval $[a,b]$, then these first $k-1$ moments impose bounds on the k-th moment m_k such that the moment vector $\boldsymbol{m}_k=(m_1,\dots,m_{k-1},m_k)^T$ is an element of the k-th moment space $\mathcal{M}_k([a,b])$. More precisely, define for $\boldsymbol{m}_{k-1}\in\mathcal{M}_{k-1}([a,b])$

$$m_k^- = \min\left\{ m_k(\mu) \,\middle|\, \mu\in\mathcal{P}([a,b]) \text{ with } \int_a^b t^i d\mu(t) = m_i \text{ for } i=1,\dots,k-1 \right\},$$
$$m_k^+ = \max\left\{ m_k(\mu) \,\middle|\, \mu\in\mathcal{P}([a,b]) \text{ with } \int_a^b t^i d\mu(t) = m_i \text{ for } i=1,\dots,k-1 \right\},$$

then it follows that $\boldsymbol{m}_k=(m_1,\dots,m_k)^T\in\operatorname{Int}\mathcal{M}_k([a,b])$ if and only if $m_k^-<m_k<m_k^+$, where $\operatorname{Int}C$ denotes the interior of a set $C\in\mathbb{R}^k$. Consequently, we define for a point $\boldsymbol{m}_k\in\operatorname{Int}\mathcal{M}_k([a,b])$ the canonical moment of order $l=1,\dots,k$ as

$$p_l = p_l(\boldsymbol{m}_k) = \frac{m_l-m_l^-}{m_l^+-m_l^-}\ ; \qquad l=1,\dots,k. \tag{2.1}$$

Note that for $\boldsymbol{m}_k\in\operatorname{Int}\mathcal{M}_k([a,b])$ we have $p_l\in(0,1)$; $l=1,\dots,k$; and that p_k describes the relative position of the moment m_k in the set of all possible k-th moments with fixed moments $m_1,\dots,m_{k-1}$. It can also be shown that the canonical moments do not depend on the interval $[a,b]$, that is they are invariant under linear transformations of the measure [see Dette and Studden (1997)]. Moreover, the definition (2.1) defines a one-to one mapping

$$\varphi_n : \begin{cases} \operatorname{Int}\mathcal{M}_n^{[a,b]} \longrightarrow (0,1)^n \\ \boldsymbol{m}_n \mapsto \boldsymbol{p}_n = (p_1,\dots,p_n)^T \end{cases} \tag{2.2}$$

from the interior of the moment space $\mathcal{M}_n^{[a,b]}$ onto the open cube $(0,1)^n$. It can be shown that for a point $(m_1,\dots,m_{2n-1})\in\operatorname{Int}\mathcal{M}_{2n-1}([a,b])$ the canonical moments appear in the three-term recurrence relation

$$xP_k(x) = P_{k+1}(x) + b_{k+1}P_k(x) + a_kP_{k-1}(x) \qquad k=1,\dots,n-1, \tag{2.3}$$

$(P_0(x)=1, P_1(x)=x-b_1)$ of the monic orthogonal polynomials

$$P_k(x) = \begin{vmatrix} m_0 & \cdots & m_{k-1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ m_m & \cdots & m_{2k-1} & x^k \end{vmatrix} \Big/ \begin{vmatrix} m_0 & \cdots & m_{k-1} \\ \vdots & \ddots & \vdots \\ m_{k-1} & \cdots & m_{2k-2} \end{vmatrix} \ ; \quad k=1,\dots,n \tag{2.4}$$

associated with the vector $(m_1, \dots, m_{2n-1})$ [see Chihara (1978)]. These polynomials are orthogonal with respect to every measure with first moments $m_1, \dots, m_{2n-1}$ and the recursion coefficients in (2.3) are given by

$$
\begin{aligned}
&(2.5) \qquad b_{k+1} = a + (b-a)((1-p_{2k-1})\,p_{2k} + (1-p_{2k})\,p_{2k+1}) ; \qquad k = 0, \dots, n-1 \\
&(2.6) \qquad a_k = (b-a)^2 (1-p_{2k-2})\,p_{2k-1}\,(1-p_{2k-1})\,p_{2k} ; \qquad k = 1, \dots, n-1
\end{aligned}
$$

where we put $p_{-1} = p_0 = 0$ (note that $a_k > 0$; $k = 1, \dots, n$). In the case $T = [0, \infty)$ the upper bound m_k^+ does not exist, but we can still define for a point $\boldsymbol{m}_{k-1} \in \operatorname{Int} \mathcal{M}_{k-1}([0,\infty))$ the lower bound

$$
m_k^- = \min \left\{ m_k(\mu) \,\middle|\, \mu \in \mathcal{P}([0,\infty)) \text{ with } \int_0^\infty t^i d\mu(t) = m_i \text{ for } i = 1, \dots, k-1 \right\},
$$

where $\boldsymbol{m}_k = (m_1, \dots, m_k)^T \in \operatorname{Int} \mathcal{M}_k([0,\infty))$ if and only if $m_k > m_k^-$. In this case, the analogues of the canonical moments are defined by the quantities

$$
(2.7) \qquad z_l = \frac{m_l - m_l^-}{m_{l-1} - m_{l-1}^-} \qquad l = 1, \dots, k
$$

(with $m_0^- = 0$) and related to the coefficients in the three-term recurrence relation (2.3) for the monic orthogonal polynomials by

$$
\begin{aligned}
&(2.8) \qquad a_k = z_{2k-1} z_{2k}, \\
&(2.9) \qquad b_k = z_{2k-2} + z_{2k-1}.
\end{aligned}
$$

Note that (2.7) defines a one to one mapping

$$
(2.10) \qquad \psi_n : \begin{cases} \operatorname{Int} \mathcal{M}_n([0,\infty)) \longrightarrow (0,\infty)^n \\ \boldsymbol{m}_n \mapsto \boldsymbol{z}_n = (z_1, \dots, z_n)^T \end{cases}
$$

from the interior of the moment space $\mathcal{M}_n([0,\infty))$ onto $(\mathbb{R}^+)^n$. Finally in the case $T = \mathbb{R}$ neither m_k^- nor m_k^+ can be defined. Nevertheless, there exists also a one to one mapping

$$
(2.11) \qquad \xi_n : \begin{cases} \operatorname{Int} \mathcal{M}_{2n-1}(\mathbb{R}) \longrightarrow (\mathbb{R} \times \mathbb{R}^+)^{n-1} \times \mathbb{R} \\ \boldsymbol{m}_{2n-1} \mapsto (b_1, a_1, \dots, a_{n-1}, b_n)^T. \end{cases}
$$

from the interior of the $(2n-1)$th moment space onto the space of coefficients in the three term recurrence relation (2.3), which can be considered as the analogue of (2.10) and is defined by

$$
\begin{aligned}
&(2.12) \qquad \int_{\mathbb{R}} x^k P_k(x) d\mu(x) = a_1 \dots a_k \qquad k = 1, \dots, n-1, \\
&(2.13) \qquad \int_{\mathbb{R}} x^{k+1} P_k(x) d\mu(x) = a_1 \dots a_k (b_1 + \dots + b_{k+1}) \qquad k = 0, \dots, n-1,
\end{aligned}
$$

[see for example Wall (1948)]. It should be mentioned that the relations for the coefficients in the three term recurrence relation (2.3) could be read backwards, because any measure with support on an interval $[a, b]$ or on the interval $[0, \infty)$ is also a measure on the real line. In other words, a measure on the real line is supported on the interval $[0, \infty)$ if and only if the coefficients in the recurrence relation (2.3) for the corresponding monic orthogonal polynomials allow a representation of the form (2.8) and (2.9) with non-negative values z_k. Similarly, the measure is supported on a compact interval if and only if a representation of the form (2.5) and (2.6) holds, where the quantities p_k vary in the interval $[0, 1]$. In the following sections we will use the canonical moments and corresponding quantities on the interval $[0, \infty)$ and the real line for the definition of distributions on the corresponding moment spaces.

2.2 Distributions on unbounded moment spaces

In Section 4 we will show that some of the results of Chang et al. (1993) and Gamboa and Lozada-Chang (2004) hold for a rather broad class of distributions on the moment space $\mathcal{M}_n([a, b])$. For the definition of this class, let for $k \geq 1$

$$f_k : (0, 1) \longrightarrow \mathbb{R}$$

be a non-negative integrable function with $\int_0^1 f_k(x)dx > 0$, then a distribution on the interior of the moment space $\mathcal{M}_n([a, b])$ is defined by

$$f_n(\boldsymbol{m}_n) = \prod_{k=1}^{n} c_{k,n} f_k(p_k(\boldsymbol{m}_n)) \mathbb{1}_{\{m_k^- < m_k < m_k^+\}}, \tag{2.14}$$

where $p_k(\boldsymbol{m}_n)$ is the k-th canonical moment defined in (2.1) and

$$c_{k,n} = \Big((b - a)^{n(n+1)/2} \int_0^1 f_k(x)(x - x^2)^{n-k} dx \Big)^{-1} \quad (k = 1, \ldots, n)$$

are normalizing constants such that f_n is a density on $\mathcal{M}_n([a, b])$ (see the proof of the following Theorem 2.1). Our first Theorem gives the distribution of the canonical moments corresponding to the random vector $\boldsymbol{m}_n$ with density f_n defined in (2.14).

Theorem 2.1. *Suppose that $\boldsymbol{m}_n$ is a random vector on the moment space $\mathcal{M}_n([a, b])$ with density f_n defined in* (2.14). *Then the canonical moments $p_1(\boldsymbol{m}_n), \ldots, p_n(\boldsymbol{m}_n)$ are independent and $p_k(\boldsymbol{m}_n)$ has the density*

$$c_{k,n}(b - a)^{n(n+1)/2} f_k(x)(x - x^2)^{n-k} \mathbb{1}_{\{0<x<1\}}$$

for $1 \leq k \leq n$.

Proof: The k-th canonical moment p_k depends only on the first k moments $m_1, \dots, m_k$, which implies for the Jacobian determinant of the mapping φ_n defined in (2.2)

$$\Big|\frac{\partial \varphi_n}{\partial \boldsymbol{m}_n}\Big| = \prod_{k=1}^{n} \frac{\partial p_k(\boldsymbol{m}_n)}{\partial m_k} = \prod_{k=1}^{n} (m_k^+ - m_k^-)^{-1} = \prod_{k=1}^{n} (b-a)^{-k} \prod_{i=1}^{k-1} (p_i(1-p_i))^{-1}, \tag{2.15}$$

where we have used Theorem 1.4.9 and equation (1.3.6) in Dette and Studden (1997) for the last equality. Therefore, the Jacobian determinant in (2.15) simplifies as

$$\Big|\frac{\partial \varphi_n}{\partial \boldsymbol{m}_n}\Big| = \prod_{k=1}^{n} \Big((b-a)^k \prod_{i=1}^{k-1} p_i(1-p_i)\Big)^{-1} = (b-a)^{-n(n+1)/2} \prod_{k=1}^{n} \Big(p_k(1-p_k)\Big)^{-(n-k)}$$

considering the product structure of f_n, this gives the asserted distribution. □

In Section 4 we will show that the first k components of a random vector with density (2.14) converge weakly (after appropriate standardization) to a normal distribution. This generalizes the results of Chang et al. (1993), who considered the case $f_k \equiv 1$ for all $k \geq 1$. For the construction of distributions on the unbounded moment space $\mathcal{M}_n([0,\infty))$, a special case will be of particular interest, that is $f_k(x) = x^{\gamma_k}(1-x)^{\delta_k}$, where $\gamma = (\gamma_k)_{k\in\mathbb{N}}$, $\delta = (\delta_k)_{k\in\mathbb{N}}$ are sequences of real parameters, such that $\gamma_k, \delta_k > -1$ for all $k \geq 1$. In this case the density on the moment space $\mathcal{M}_n([a,b])$ is given by

$$f_n^{(\gamma,\delta)}(\boldsymbol{m}_n) \;=\; c_n^{[a,b]} \prod_{k=1}^{n} \left(\frac{m_k - m_k^-}{m_k^+ - m_k^-}\right)^{\gamma_k} \left(\frac{m_k^+ - m_k}{m_k^+ - m_k^-}\right)^{\delta_k} \mathbb{1}_{\{m_k^- < m_k < m_k^+\}}, \tag{2.16}$$

where

$$c_n^{[a,b]} = \Big\{(b-a)^{n(n+1)/2} \int_0^1 x^{n-k+\gamma_k}(1-x)^{n-k+\delta_k} dx\Big\}^{-1} \tag{2.17}$$

is the normalizing constant. The choice of the density (2.16) is motivated by results of Dette and Studden (1995) who showed that the empirical distribution of the (appropriately normalized) roots of the Jacobi polynomials $P_k^{(\gamma_k,\delta_k)}(x)$ converges weakly to a distribution with unbounded support if $\gamma_k \to \infty$ or $\delta_k \to \infty$. Note that for the density $f_n^{(\gamma,\delta)}$ the canonical moment p_k has a Beta distribution $Beta(\gamma_k + n - k + 1, \delta_k + n - k + 1)$. In the following we use densities of the form (2.16) to construct a distribution on the unbounded moment space

$$\mathcal{M}_n([0,\infty)) \;=\; \big\{\boldsymbol{m}_n(\mu) = (m_1(\mu), \dots, m_n(\mu))^T | \mu \in \mathcal{P}([0,\infty))\big\}\,. \tag{2.18}$$

For this purpose, recall that the relation (2.7) defines a one to one mapping between the moment space $\operatorname{Int}\mathcal{M}_n([0,\infty))$ and $(\mathbb{R}^+)^n$.

Theorem 2.2. *Let $f_n^{(\gamma^{(d)},\delta^{(d)})}$ denote the density defined in (2.16) on the moment space $\mathcal{M}_n([0,d])$ corresponding to the probability measures on the interval $[0,d]$, where the parameter sequences $\gamma^{(d)} = (\gamma_k^{(d)})_{k\in\mathbb{N}}, \delta^{(d)} = (\delta_k^{(d)})_{k\in\mathbb{N}}$ depend on length d and satisfy*

$$\gamma_k^{(d)} \xrightarrow[d\to\infty]{} \gamma_k > -1, \qquad \frac{\delta_k^{(d)}}{d} \xrightarrow[d\to\infty]{} \delta_k \in \mathbb{R}^+$$

for all $k \geq 1$. Then for $d \to \infty$ the density $f_n^{(\gamma^{(d)},\delta^{(d)})}$ converges point-wise to the function

$$\begin{aligned}(2.19)\qquad g_n^{(\gamma,\delta)}(\boldsymbol{m}_n) &= c_n^{[0,\infty)} \prod_{k=1}^n \left(\frac{m_k - m_k^-}{m_{k-1} - m_{k-1}^-}\right)^{\gamma_k} \exp\left(-\delta_k \frac{m_k - m_k^-}{m_{k-1} - m_{k-1}^-}\right) \mathbb{1}_{\{m_k > m_k^-\}} \\ &= c_n^{[0,\infty)} \prod_{k=1}^n z_k(\boldsymbol{m}_n)^{\gamma_k} \exp(-\delta_k z_k(\boldsymbol{m}_n)) \mathbb{1}_{\{z_k(\boldsymbol{m}_n)>0\}},\end{aligned}$$

where the constant $c_n^{[0,\infty)}$ is given by

$$c_n^{[0,\infty)} = \prod_{k=1}^n \frac{\delta_k^{\gamma_k+n-k+1}}{\Gamma(\gamma_k + n - k + 1)}.$$

Moreover, $g_n^{(\gamma,\delta)}$ defines a density on the unbounded moment space $\mathcal{M}_n([0,\infty))$.

Proof: The fact that $g_n^{(\gamma,\delta)}$ is a density is evident from the transformation in the proof of Theorem 2.3 below, we prove here only the convergence. For a fixed point $\boldsymbol{m}_n \in \mathcal{M}_n([0,\infty))$, there exists a $d_0 \in \mathbb{N}$ with $\boldsymbol{m}_n \in \mathcal{M}_n([0,d])$ for all $d \geq d_0$. Let $\boldsymbol{p}_n(\boldsymbol{m}_n)$ denote the vector of canonical moments corresponding to the vector $\boldsymbol{m}_n$ in the moment space $\mathcal{M}_n([0,d])$. We will show at the end of this proof that

$$(2.20)\qquad p_k(\boldsymbol{m}_n) = \frac{z_k(\boldsymbol{m}_n)}{d}(1 + o(1)), \qquad k = 1, \ldots, n,$$

where the quantities $z_k(\boldsymbol{m}_n)$ are defined in (2.7). Observing this representation and the definition (2.16), it follows for $d \to \infty$

$$\begin{aligned}f_n^{(\gamma^{(d)},\delta^{(d)})}(\boldsymbol{m}_n) &= c_n^{[0,d]} \prod_{k=1}^n \left(\frac{z_k(\boldsymbol{m}_n)}{d}\right)^{\gamma_k^{(d)}} \left(1 - \frac{z_k(\boldsymbol{m}_n)}{d}\right)^{\delta_k^{(d)}} (1 + o(1)) \\ &= d^{-(\gamma_1^{(d)}+\cdots+\gamma_n^{(d)})} c_n^{[0,d]} \prod_{k=1}^n z_k(\boldsymbol{m}_n)^{\gamma_k} \exp(-\delta_k z_k(\boldsymbol{m}_n))(1 + o(1)).\end{aligned}$$

Finally, we obtain from (2.17) for the normalizing constant

$$\begin{aligned}d^{-(\gamma_1^{(d)}+\cdots+\gamma_n^{(d)})} c_n^{[0,d]} &= d^{-n(n+1)/2-(\gamma_1^{(d)}+\cdots+\gamma_n^{(d)})} \prod_{k=1}^n \frac{\Gamma(\gamma_k^{(d)} + \delta_k^{(d)} + 2n - 2k + 2)}{\Gamma(\gamma_k^{(d)} + n - k + 1)\Gamma(\delta_k^{(d)} + n - k + 1)} \\ &= d^{-n(n+1)/2-(\gamma_1^{(d)}+\cdots+\gamma_n^{(d)})} \prod_{k=1}^n \frac{(\delta_k^{(d)})^{\gamma_k+n-k+1}}{\Gamma(\gamma_k + n - k + 1)}(1 + o(1)) \\ &= c_n^{[0,\infty)}(1 + o(1)),\end{aligned}$$

which proves the assertion of the Theorem. For the remaining proof of the representation (2.20), let μ be a measure on the interval $[0, d]$ with first moments given by $\boldsymbol{m}_n$ and let ν denote the measure on the interval $[0, 1]$ obtained from μ by the linear transformation $x \mapsto x/d$. We write $p_k(\mu)$ for $p_k(\boldsymbol{m}_n)$ and $z_k(\mu)$ for $z_k(\boldsymbol{m}_n)$. Invariance of the canonical moments under linear transformations yields $p_k(\mu) = p_k(\nu)$. The recursion variables of the measure ν can be decomposed as

$$z_k(\nu) = (1 - p_{k-1}(\nu))p_k(\nu). \tag{2.21}$$

The Stieltjes-transform of μ has a continued fraction expansion

$$\int_0^d \frac{d\mu(x)}{\zeta - x} = \frac{1}{\left| \zeta - z_1(\mu) \right.} - \frac{z_1(\mu)z_2(\mu)}{\left| \zeta - (z_2(\mu) + z_3(\mu)) \right.} - \frac{z_3(\mu)z_4(\mu)}{\left| \zeta - (z_4(\mu) + z_5(\mu)) \right.} - \cdots \tag{2.22}$$

for $\zeta \in \mathbb{C} \setminus [0, d]$. Since the measure μ is obtained from ν by linear transformation, the continued fraction expansion of the Stieltjes-transform of μ can be written in terms of the recursion coefficients of ν,

$$\int_0^d \frac{d\mu(x)}{\zeta - x} = \frac{1}{\left| \zeta - dz_1(\nu) \right.} - \frac{d^2 z_1(\nu)z_2(\nu)}{\left| \zeta - d(z_2(\nu) + z_3(\nu)) \right.} - \frac{d^2 z_3(\nu)z_4(\nu)}{\left| \zeta - d(z_4(\nu) + z_5(\nu)) \right.} - \cdots$$

[see Theorem 3.3.3 in Dette and Studden (1997)]. A continued fraction expansion as in (2.22) is unique, which yields $dz_k(\nu) = z_k(\mu)$. With equation (2.21) we obtain

$$p_k(\mu) = p_k(\nu) = \frac{z_k(\nu)}{1 - p_{k-1}(\nu)} = \frac{1}{d} \frac{z_k(\mu)}{1 - p_{k-1}(\mu)}$$

for $k > 1$. The first canonical moment is given by

$$p_1(\boldsymbol{m}_n) = \frac{m_1 - m_1^-}{m_1^+ - m_1^-} = \frac{m_1}{d} = \frac{z_1(\boldsymbol{m}_n)}{d}.$$

and equation (2.20) follows by an induction argument. □

The following Theorem is essential for the asymptotic investigations in Section 4 and gives the distribution of the the vector $\boldsymbol{z_n} = (z_1, \ldots, z_n)^T$ corresponding to a random vector point on the moment space $\mathcal{M}_n([0, \infty))$.

Theorem 2.3. *Let $\boldsymbol{m}_n \in \mathcal{M}_n([0, \infty))$ be governed by a law with density $g_n^{(\gamma,\delta)}$, then the recursion variables $\boldsymbol{z}_n = \psi_n(\boldsymbol{m}_n)$ defined by (2.7) are independent and gamma distributed, that is*

$$z_k \sim Gamma(\gamma_k + n - k + 1, \delta_k) \qquad k = 1, \ldots, n.$$

Proof: By its definition in (2.7), the random variable z_k depends only on the moment $m_1, \dots, m_k$, therefore the Jacobi matrix of the mapping ψ_n is a lower triangular matrix. We obtain for the Jacobian determinant

$$\left|\frac{\partial \boldsymbol{m}_n}{\partial \boldsymbol{z}_n}\right| = \prod_{k=1}^{n} \left|\frac{\partial m_k}{\partial z_k}\right| = \prod_{k=1}^{n} (m_{k-1} - m_{k-1}^-) = \prod_{k=2}^{n} z_1 \dots z_{k-1} = \prod_{k=1}^{n} z_k^{n-k},$$

where the third identity follows from the definition of the z_i in (2.7). Considering the second representation of the density in (2.19), this yields the claimed distribution. □

We will show in Section 3 that densities of the form (2.16) arise naturally as distributions of moments corresponding to random spectral measures. We conclude this section with a discussion of distributions on the moments space corresponding to measures on $\mathbb{R}$. For the sake of brevity we restrict ourselves to moment spaces of odd dimension, that is

$$\mathcal{M}_{2n-1}(\mathbb{R}) \quad = \quad \left\{ \boldsymbol{m}_{2n-1}(\mu) = (m_1(\mu), \dots, m_{2n-1}(\mu))^T |\ \mu \in \mathcal{P}(\mathbb{R}) \right\}.$$

To derive a class of distributions on $\mathcal{M}_{2n-1}(\mathbb{R})$ we consider the moment space $\mathcal{M}_{2n-1}([-s, s])$ with $s \to \infty$ and a density of the form (2.16) with parameters varying with s. The proof of the following result is similar to the proof of Theorem 2.2 and therefore omitted.

Theorem 2.4. *Denote by* $f_{2n-1}^{(\gamma^{(s)}, \delta^{(s)})}$ *the density defined in* (2.16) *on the moment space* $\mathcal{M}_{2n-1}([-s, s])$, *where the parameters satisfy*

$$\begin{aligned} \gamma_{2k-1}^{(s)} &= \delta_{2k-1} s^2 + o(1), \qquad \delta_{2k-1}^s = \delta_{2k-1} s^2 + o(1), \\ \gamma_{2k}^{(s)} &= \gamma_k + o(1), \qquad \delta_{2k}^{(s)} = \delta_{2k} s^2 + o(s^2) \end{aligned}$$

with $\gamma_k > -1,\ \delta_k > 0$. *Then* $f_{2n-1}^{(\gamma^{(s)}, \delta^{(s)})}$ *converges point-wise to the function*

$$(2.23) \quad h_{2n-1}^{(\gamma,\delta)}(\boldsymbol{m}_{2n-1}) = \prod_{k=1}^{n} \sqrt{\frac{\delta_{2k-1}}{\pi}} \exp\left(-\delta_{2k-1} b_k^2(\boldsymbol{m}_{2n-1})\right) \times \prod_{k=1}^{n-1} \frac{{\delta_{2k}}^{\gamma_k + 2n - 2k}}{\Gamma(\gamma_k + 2n - 2k)} a_k^{\gamma_k}(\boldsymbol{m}_{2n-1}) \exp\left(-\delta_{2k} a_k(\boldsymbol{m}_{2n-1})\right) \mathbb{1}_{\{a_k(\boldsymbol{m}_{2n-1}) > 0\}}.$$

Moreover, the function $h_{2n-1}^{(\gamma,\delta)}$ *defines a density on the moment space* $\mathcal{M}_{2n-1}(\mathbb{R})$.

The following result is the analogue of Theorem 2.3.

Theorem 2.5. *Let* $\boldsymbol{m}_{2n-1} \in \mathcal{M}_{2n-1}(\mathbb{R})$ *be a random vector with density* $h_{2n-1}^{(\gamma,\delta)}$ *defined in* (2.23). *Then the random recursion coefficients* $(b_1, a_1 \dots, a_{n-1}, b_n)^T = \xi_{2n-1}(\boldsymbol{m}_{2n-1})$ *in the recurrence relation* (2.3) *for the orthogonal polynomials associated with* $\boldsymbol{m}_{2n-1}$ *by* (2.4) *are independent and*

$$\begin{aligned} b_k &\sim \mathcal{N}(0, \tfrac{1}{2\delta_{2k-1}}), \\ a_k &\sim \mathit{Gamma}(\gamma_k + 2n - 2k, \delta_{2k}). \end{aligned}$$

Proof: Suppose μ is a measure with first moments given by $\boldsymbol{m}_{2n-1}$ and let $P_1(x), \ldots, P_n(x)$ denote the corresponding monic polynomials. The recursion coefficients can be calculated by (2.12) and (2.13). This implies that the coefficient b_k depends only on the moments $m_1, \ldots, m_{2k-1}$, while a_k depends only on $m_1, \ldots, m_{2k}$. Consequently, the Jacobi matrix of ξ_{2n-1} is a lower triangular matrix. From (2.12) we obtain

$$\frac{\partial m_{2k}}{\partial a_k} = \frac{\partial}{\partial a_k} \int_{\mathbb{R}} x^k P_k(x) d\mu(x) = a_1 \ldots a_{k-1}$$

and by the identity (2.13) it follows

$$\frac{\partial m_{2k-1}}{\partial b_k} = \frac{\partial}{\partial b_k} \int_{\mathbb{R}} x^k P_{k-1}(x) d\mu(x) = a_1 \ldots a_{k-1}.$$

The Jacobian determinant is therefore given by

$$\begin{aligned} \left| \frac{\partial \boldsymbol{m}_{2n-1}}{\partial \xi_{2n-1}(\boldsymbol{m}_{2n-1})} \right| &= \left| \Big(\prod_{k=1}^{n-1} \frac{\partial m_{2k-1}}{\partial b_k} \frac{\partial m_{2k}}{\partial a_k} \Big) \frac{\partial m_{2n-1}}{\partial b_n} \right| \\ &= \Big(\prod_{k=2}^{n-1} (a_1 \ldots a_{k-1})^2 \Big) (a_1 \ldots a_{k-1}) = \prod_{k=1}^{n-1} a_k^{2n-2k-1}. \end{aligned}$$

Considering the product structure in the density $h_{2n-1}^{(\gamma,\delta)}$ in terms of the recursion coefficients, the assertion of the theorem follows. □

3 Random spectral measures

Let A be a linear, self-adjoint operator on a n-dimensional Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and cyclic vector $e_1 \in \mathcal{H}$ (i.e., $e_1, Ae_1, \ldots, A^{n-1}e_1$ are linearly independent). The spectral Theorem yields the existence of a unique probability measure μ on the real Borel field, such that

$$\langle e_1, A^n e_1 \rangle = m_n(\mu) \tag{3.1}$$

for all $n \geq 1$ [see Dunford and Schwartz (1963)]. This measure defines the unitarily equivalent L^2-space in which the operator A is represented by the multiplication $f(x) \mapsto xf(x)$. We call μ spectral measure of the matrix A. If $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A and $u_1, \ldots, u_n$ is a corresponding system of orthonormal eigenvectors, the spectral measure can be written as

$$\mu = \sum_{i=1}^{n} w_i \delta_{\lambda_i} \tag{3.2}$$

where the weights are given by $w_i = |\langle u_i, e_1 \rangle|^2$ and δ_x denotes the Dirac measure in the point x. The identity (3.2) follows easily from the fact that the matrix with columns $u_1, \ldots, u_n$

diagonalizes A. If the eigenvalues are distinct, lie in the interior of $T \in \{[a,b],[0,\infty),\mathbb{R}\}$ and the vector of weights $(w_1,\dots,w_n)^T$ is contained in the simplex

$$Sim_n = \Big\{ (w_1,\dots,w_n)^T \in \mathbb{R}^n \Big| w_i > 0, \sum_{i=1}^n w_i = 1 \Big\}$$

then the first $2n-1$ moments of the spectral measure satisfy $\boldsymbol{m}_{2n-1}(\mu) \in \operatorname{Int} \mathcal{M}_{2n-1}(T)$ which follows by an application of Theorem 1.4.1 in Dette and Studden (1997).

We consider in this section random spectral measures associated with three central distributions in random matrix theory: the Hermite (or Gaussian), the Laguerre and the Jacobi ensemble. These classical ensembles are distributions on the space of self-adjoint matrices $A \in \mathbb{K}^{n\times n}$ with real ($\mathbb{K}=\mathbb{R}$), complex ($\mathbb{K}=\mathbb{C}$) or quaternion entries ($\mathbb{K}=\mathbb{H}$). The Hermite ensemble arises in physics and is the distribution of the matrix

$$A = \frac{1}{\sqrt{2}}(X + X^*),$$

where all real entries in $X \in \mathbb{K}^{n\times n}$ are independent and standardnormal distributed and X^* denotes the adjoint of X. The eigenvalues of the Hermite ensemble have the joint density

$$f_G(\lambda) = c_G |\Delta(\lambda)|^\beta \prod_{i=1}^n e^{-\lambda_i^2/2} \tag{3.3}$$

where $\beta = \dim_{\mathbb{R}} \mathbb{K}$, $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant and c_G is a normalizing constant [see Mehta (2004)]. The Laguerre ensemble appears in the study of singular values of a Gaussian matrix, it has the eigenvalue density

$$f_L(\lambda) = c_L^a |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^a e^{-\lambda_i} \mathbb{1}_{\{\lambda_i > 0\}}, \tag{3.4}$$

with parameter $a > 1$ and a normalizing constant c_L^a. Finally, the Jacobi ensemble is motivated by multivariate analysis of variance [MANOVA; see Muirhead (1982)] and is defined by the eigenvalue density

$$f_J(\lambda) = c_J^{a,b} |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^a (1-\lambda_i)^b \mathbb{1}_{\{0<\lambda_i<1\}}. \tag{3.5}$$

with $a, b > -1$ and a normalizing constant $c_J^{a,b}$. It is a common feature of all three classical ensembles that the matrix of the orthonormal eigenvectors is Haar distributed on the group of orthogonal (unitary/symplectic) matrices and independent from the eigenvalues [see Dawid (1977)]. As a consequence, the matrix distribution is uniquely determined by the eigenvalue density. Since the eigenvector matrix is Haar distributed, the first row of the matrix is Haar distributed on the unit sphere in $\mathbb{K}^n$ and the weights $w_1, \dots, w_n$ follow a Dirichlet distribution with density

$$\frac{\Gamma(\frac{n\beta}{2})}{\Gamma(\frac{\beta}{2})^n} w_1^{\beta/2-1} \dots w_n^{\beta/2-1} \mathbb{1}_{\{w \in Sim_n\}}. \tag{3.6}$$

The identity for the spectral measure in (3.2) motivates the following definition of distributions on the set of probability measures.

Definition 3.1. *Let*

$$\mu = w_1 \delta_{\lambda_1} + \dots + w_n \delta_{\lambda_n}$$

denote a probability measure with random support points $\lambda = (\lambda_1, \dots, \lambda_n)$ *and random weights with* $w_1, \dots, w_n$ *with Dirichlet density* (3.6)*, where the weights and (random) support points are independent.*

(i) If the density of λ *is given by* (3.3)*, we call* μ *random spectral measure of the Gaussian ensemble, i.e.* $\mu \sim G\beta E_n$*.*

(ii) If the density of λ *is given by* (3.4)*, we call* μ *random spectral measure of the Laguerre ensemble, i.e.* $\mu \sim L\beta E_n(a)$*.*

(iii) If the density of λ *is given by* (3.5)*, we call* μ *random spectral measure of the Jacobi ensemble, i.e.* $\mu \sim J\beta E_n(a, b)$*.*

Note that these distributions are well defined for all $\beta > 0$. If $\beta \in \{1, 2, 4\}$, we obtain the spectral measures of matrices from the classical ensembles. The results in this section show that there is a connection between classical ensembles and the distributions on moment spaces as given in Section 2. More precisely, the moments (3.1) of the spectral measures are distributed according to the densities defined on $\mathcal{M}_n([0, 1])$ and obtained on the unbounded moment spaces by Theorem 2.2 and Theorem 2.4. Our first Theorem considers the spectral measure of the Jacobi ensemble. The proof relies on a tridiagonal matrix model for the $J\beta E_n(a, b)$ distribution for all values of $\beta > 0$, which was proved in a recent paper by Killip and Nenciu (2004).

Theorem 3.2. *Let* $\mu \sim J\beta E_n(\gamma_0, \delta_0)$ *denote a random spectral measure of the Jacobi ensemble, then the distribution of the corresponding random moment vector* $\boldsymbol{m}_{2n-1}(\mu)$ *on* $\mathcal{M}_{2n-1}([0, 1])$ *is absolute continuous with density* $f_{2n-1}^{(\gamma,\delta)}$ *defined in* (2.16) *where the parameters of the density are given by*

$$\gamma_{2k-1} = (\tfrac{\beta}{2} - 2)(n - k) + \gamma_0 \ ; \qquad \delta_{2k-1} = (\tfrac{\beta}{2} - 2)(n - k) + \delta_0$$

for $1 \le k \le n$ *and*

$$\gamma_{2k} = (\tfrac{\beta}{2} - 2)(n - k) \ ; \qquad \delta_{2k} = (\tfrac{\beta}{2} - 2)(n - k - 1) + \gamma_0 + \delta_0$$

for $1 \le k \le n - 1$*.*

Proof: It follows from Theorem 2.2 and Proposition 5.3 in Killip and Nenciu (2004) that μ is the spectral measure of the tridiagonal matrix

$$J_n = \begin{pmatrix} d_1 & c_1 & & \\ c_1 & d_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ & & c_{n-1} & d_n \end{pmatrix},$$

where the entries are given by

$$\begin{aligned} d_k =& p_{2k-2}(1-p_{2k-3}) + p_{2k-1}(1-p_{2k-2}) \\ c_k =& \sqrt{p_{2k-1}(1-p_{2k-2})p_{2k}(1-p_{2k-1})} \end{aligned}$$

with $p_{-1} = p_0 = 0$ and $p_1, \dots, p_{2n-1}$ are independent random variables distributed as

$$p_k \sim \begin{cases} Beta\left(\frac{2n-k}{4}\beta, \frac{2n-k-2}{4}\beta + \gamma_0 + \delta_0 + 2\right), & k \text{ even}, \\ Beta\left(\frac{2n-k-1}{4}\beta + \gamma_0 + 1, \frac{2n-k-1}{4}\beta + \delta_0 + 1\right), & k \text{ odd}. \end{cases}$$

Note that Killip and Nenciu (2004) define the Jacobi ensemble by the eigenvalue density

$$c|\Delta(\lambda)|^\beta \prod_{i=1}^n (2-\lambda_i)^{\gamma_0}(2+\lambda_i)^{\delta_0} \mathbb{1}_{\{-2<\lambda_i<2\}}$$

and therefore the matrix found in their paper is the transformed matrix $4J_n - 2I_n$. Additionally, they work with a beta distribution on the interval $[-1, 1]$ which is obtained from the usual beta distribution on $[0, 1]$ by the transformation $x \mapsto 1 - 2x$.
The tridiagonal matrix J_n defines monic polynomials $P_1(x), \dots, P_n(x)$ via a recursion (2.3) with recursion coefficients

$$b_k = d_k \ (1 \le k \le n), \qquad a_k = c_k^2 \ (1 \le k \le n-1).$$

Indeed, the polynomial $P_k(x)$ is the characteristic polynomial of the upper left $(k \times k)$-subblock of the matrix J_n. The orthogonality measure of these polynomials is precisely the spectral measure μ [see Deift (2000) for the corresponding statement for orthonormal polynomials]. Therefore, the recursion coefficients in the recursion (2.3) of the monic polynomials orthogonal with respect to the measure μ are given by

$$\begin{aligned} b_k =& p_{2k-2}(1-p_{2k-3}) + p_{2k-1}(1-p_{2k-2})\ ; \quad k = 1, \dots, n, \\ a_k =& p_{2k-1}(1-p_{2k-2})p_{2k}(1-p_{2k-1})\ ; \qquad k = 1, \dots, n-1. \end{aligned}$$

By identity (2.5) and (2.6), $\boldsymbol{p}_{2n-1} = (p_1, \ldots, p_{2n-1})^T$ is exactly the vector of canonical moments of the spectral measure μ and by definition, their joint density is given by

$$\begin{aligned} f_{\boldsymbol{p}}(\boldsymbol{p}_{2n-1}) =& c\prod_{k=1}^{n} p_{2k-1}^{(2n-(2k-1)-1)\beta/4+\gamma_0}(1-p_{2k-1})^{(2n-(2k-1)-1)\beta/4+\delta_0} \\ &\times \prod_{k=1}^{n-1} p_{2k}^{(2n-2k)\beta/4-1}(1-p_{2k})^{(2n-2k-2)\beta/4+\gamma_0+\delta_0+1} \\ =& c\prod_{k=1}^{n} p_{2k-1}^{(n-k)\beta/2+\gamma_0}(1-p_{2k-1})^{(n-k)\beta/2+\delta_0} \\ &\times \prod_{k=1}^{n-1} p_{2k}^{(n-k)\beta/2-1}(1-p_{2k})^{(n-k-1)\beta/2+\gamma_0+\delta_0+1}. \end{aligned}$$

Since the eigenvalues of the matrix J_n are contained in the interval $(0,1)$, the moments $\boldsymbol{m}_{2n-1}(\mu) = \varphi_{2n-1}^{-1}(\boldsymbol{p}_{2n-1})$ of the spectral measure are in the interior of the moment space $\mathcal{M}_{2n-1}([0,1])$. The Jacobian of the transformation φ_{2n-1}^{-1} is given by $\prod_{k=1}^{n}(p_k(1-p_k))^{2n-1-k}$, which gives for the density of the random moments $\boldsymbol{m}_{2n-1}(\mu)$

$$\begin{aligned} f_{\boldsymbol{m}}(\boldsymbol{m}_{2n-1}) =& c\prod_{k=1}^{n} p_{2k-1}(\boldsymbol{m}_{2n-1})^{(n-k)\beta/2-(2n-2k)+\gamma_0}(1-p_{2k-1}(\boldsymbol{m}_{2n-1}))^{(n-k)\beta/2-(2n-2k)+\delta_0} \\ &\times \prod_{k=1}^{n-1} p_{2k}(\boldsymbol{m}_{2n-1})^{(n-k)\beta/2-(2n-2k-1)-1}(1-p_{2k}(\boldsymbol{m}_{2n-1}))^{(n-k-1)\beta/2-(2n-2k-1)+\gamma_0+\delta_0+1} \\ =& c\prod_{k=1}^{n} p_{2k-1}(\boldsymbol{m}_{2n-1})^{(\beta/2-2)(n-k)+\gamma_0}(1-p_{2k-1}(\boldsymbol{m}_{2n-1}))^{(\beta/2-2)(n-k)+\delta_0} \\ &\times \prod_{k=1}^{n-1} p_{2k}(\boldsymbol{m}_{2n-1})^{(\beta/2-2)(n-k)}(1-p_{2k}(\boldsymbol{m}_{2n-1}))^{(\beta/2-2)(n-k-1)+\gamma_0+\delta_0}. \end{aligned}$$

This is a density as in (2.16) with the asserted parameters. □

An interesting case is obtained by the choice $\beta = 4$. Here the parameters γ_k, δ_k of the density $f_{2n-1}^{(\gamma,\delta)}$ do not depend on k. If additionally $\gamma_0 = \delta_0 = 0$, the vector of moments $\boldsymbol{m}_{2n-1}$ is uniformly distributed on $\mathcal{M}_{2n-1}([0,1])$. In other words, starting from a moment vector $\boldsymbol{m}_{2n-1}$ drawn uniformly from the moment space, we obtain a random measure

$$\mu = \sum_{k=1}^{n} w_k \delta_{\lambda_k}$$

with first moments given by $\boldsymbol{m}_{2n-1}$ and with support points distributed according to the density

(3.7) $$f_\lambda(\lambda) = |\Delta(\lambda)|^4 \mathbb{1}_{\{0 \le \lambda_k \le 1 \ \forall k\}}.$$

The roots of the n-th orthogonal polynomial $P_n(x)$ with respect to the measure μ are precisely the support points $\lambda_1, \ldots, \lambda_n$. Consequently the roots of the n-th random orthogonal polynomial $P_n(x)$ associated with $\boldsymbol{m}_{2n-1}$. are distributed according to the density (3.7) [see also Birke and Dette (2009)].

The following results show that the spectral measures of the Laguerre ensemble and Hermite ensemble have moments distributed according to the density obtained in Theorem 2.2 and Theorem 2.4, respectively. The proofs are analogous to the proof of Theorem 3.2 and use the tridiagonal matrix models for the Laguerre and Hermite ensemble provided by Dumitriu and Edelman (2002).

Theorem 3.3. *Let $\mu \sim L\beta E_n(\gamma_0)$ denote a random spectral measure of the Laguerre ensemble, then the distribution of the corresponding random moment vector $\boldsymbol{m}_{2n-1}(\mu)$ on $\mathcal{M}_{2n-1}([0,\infty))$ is absolute continuous with density $g_n^{(\gamma,\delta)}$ defined by (2.19), where the parameters of the density are given by $\delta_k = -1$ for $1 \le k \le 2n-1$ and*

$$\begin{aligned} \gamma_{2k-1} &= (\tfrac{\beta}{2} - 2)(n-k) + \gamma_0 && 1 \le k \le n, \\ \gamma_{2k} &= (\tfrac{\beta}{2} - 2)(n-k) && 1 \le k \le n-1. \end{aligned}$$

Theorem 3.4. *Suppose $\mu \sim G\beta E_n$ is a random spectral measure of the Gaussian ensemble, then the distribution of the corresponding random moment vector $\boldsymbol{m}_{2n-1}(\mu)$ on the moment space $\mathcal{M}_{2n-1}(\mathbb{R})$ is absolute continuous with density $h_{2n-1}^{(\gamma,\delta)}$ defined by (2.23) where the parameters of the density are given by*

$$\delta_{2k-1} = \tfrac{1}{2} \quad (1 \le k \le n), \qquad \delta_{2k} = 1 \quad (1 \le k \le n-1),$$

and $\gamma_k = (\frac{\beta}{2} - 2)(n-k) - 1$ $(1 \le k \le n)$.

4 Weak convergence of random moments

In this section we study the probabilistic properties of random vector on the moment spaces $\mathcal{M}_n([a,b])$, $\mathcal{M}_n([0,\infty))$ and $\mathcal{M}_n(\mathbb{R})$ distributed according to the measures introduced in Section 2. We begin with random moments defined on the moment space corresponding to probability measures on a compact interval. Chang et al. (1993) and Gamboa and Lozada-Chang (2004) investigated the uniform distribution on $\mathcal{M}_n([a,b])$, and we first demonstrate that weak convergence of random moment vectors can be established for a rather broad class of distributions on $\mathcal{M}_n([a,b])$. An important role in the discussion of moment spaces corresponding to probability measures with bounded support $[a,b]$ plays the arcsine distribution ν with density

$$d\nu(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}} \mathbb{1}_{\{a<x<b\}} dx.$$

The canonical moments of the arcsine distribution are given by 1/2 [see Dette and Studden (1997)] and therefore its sequence of moments could be considered as the center of the moment space $\mathcal{M}([a,b])$. The following statements establish the asymptotic properties of the (random) canonical moments corresponding to distributions on the moment space $\mathcal{M}_n([a,b])$ defined in (2.14). The only assumption necessary to obtain almost sure convergence is that for all $\varepsilon > 0$

$$\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_k(x)dx > 0. \tag{4.1}$$

Throughout this paper the symbol $\xrightarrow{\mathcal{D}}$ stands for weak convergence.

Theorem 4.1. *Suppose that the distribution of the random moment vector* $\boldsymbol{m}_n \in \mathcal{M}_n([a,b])$ *is absolute continuous with density* f_n *defined in* (2.14)*, where the functions* f_k *satisfy for all* $\epsilon > 0$ *condition* (4.1)*, and denote by* $p_k^{(n)}$ *the* k*-th canonical moment of* $\boldsymbol{m}_n$ $(k = 1, \ldots, n)$*.*

(a) If $n \to \infty$*, then almost surely for any* $k \geq 1$

$$p_k^{(n)} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

(b) If additionally the function f_k *in the density* (2.14) *is continous at* $\frac{1}{2}$ *and* $f_k(\frac{1}{2}) > 0$*, then the* k*-th canonical moment corresponding to* $\boldsymbol{m}_n$ *satisfies*

$$\sqrt{8n}(p_k^{(n)} - \tfrac{1}{2}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

Proof: For notational convenience, we consider only the case $[a,b] = [0,1]$. We introduce the random variable $q_k^{(n)} = p_k^{(n)} - \frac{1}{2}$ with density

$$c_{k,n} f_k(x + \tfrac{1}{2})(\tfrac{1}{4} - x^2)^{n-k} \mathbb{1}_{\{-1/2<x<1/2\}}$$

and show $q_k^{(n)} \to 0$, which proves the first assertion of the Theorem. For $0 < \varepsilon < \frac{1}{4}$, let $U(\varepsilon) = [-\varepsilon, \varepsilon]$, then clearly,

$$1 = c_{k,n} \int_{U(\varepsilon)} f_k(x + \tfrac{1}{2})(\tfrac{1}{4} - x^2)^{n-k} dx + c_{k,n} \int_{U(\varepsilon)^C} f_k(x + \tfrac{1}{2})(\tfrac{1}{4} - x^2)^{n-k} dx,$$

and we obtain

$$\begin{aligned}
\Big(c_{k,n} \int_{U(\varepsilon)^C} f_k(x + \tfrac{1}{2})(\tfrac{1}{4} - x^2)^{n-k} dx\Big)^{-1} &= 1 + \frac{\int_{U(\varepsilon)} f_k(x + \frac{1}{2})(\frac{1}{4} - x^2)^{n-k} dx}{\int_{U(\varepsilon)^C} f_k(x + \frac{1}{2})(\frac{1}{4} - x^2)^{n-k} dx} \\
&\geq \frac{\int_{U(\frac{\varepsilon}{2})} f_k(x + \frac{1}{2})(\frac{1}{4} - x^2)^{n-k} dx}{\int_{U(\varepsilon)^C} f_k(x + \frac{1}{2})(\frac{1}{4} - x^2)^{n-k} dx} \\
&\geq \frac{(\frac{1}{4} - \frac{\varepsilon^2}{4})^{n-k} \int_{U(\frac{\varepsilon}{2})} f_k(x + \frac{1}{2}) dx}{(\frac{1}{4} - \varepsilon^2)^{n-k} \int_{U(\varepsilon)^C} f_k(x + \frac{1}{2}) dx} \geq \Big(\frac{1 - \varepsilon^2}{1 - 4\varepsilon^2}\Big)^{n-k} c_{\varepsilon,f},
\end{aligned}$$

where $c_{\varepsilon,f}$ is a positive constant by condition (4.1) and independent of n. This yields

$$P\left(q_k^{(n)} \in U(\varepsilon)^C\right) = c_{k,n} \int_{U(\varepsilon)^C} f_k(x+\tfrac{1}{2})(\tfrac{1}{4}-x^2)^{n-k}dx \le \left(\frac{1-\varepsilon^2}{1-4\varepsilon^2}\right)^{k-n} c_{\varepsilon,f}^{-1},$$

which implies

$$\sum_{n=k}^{\infty} P\left(q_k^{(n)} \in U(\varepsilon)^C\right) \le c_{\varepsilon,f}^{-1} \sum_{n=k}^{\infty} \left(\frac{1-\varepsilon^2}{1-4\varepsilon^2}\right)^{k-n} < \infty.$$

By the Borel-Cantelli Lemma, $q_k^{(n)}$ converges to 0 almost surely, which proves the first part of the Theorem.

For a proof of part (b) we note that the rescaled canonical moment $\sqrt{8n}(p_k^{(n)} - \frac{1}{2})$ has the density

$$c_{k,n}\frac{1}{\sqrt{8n}} 2^{-2(n-k)} f_k(\tfrac{1}{2}+\tfrac{x}{\sqrt{8n}}) \left(1-\frac{x^2}{2n}\right)^{n-k} \mathbb{1}_{\{-\sqrt{2n}<x<\sqrt{2n}\}}, \tag{4.2}$$

and the assertion follows from the convergence of the density (4.2) to the density of the standard normal distribution [see the convergence Theorem by Scheffé (1947)]. Clearly,

$$f_k(\tfrac{1}{2}+\tfrac{x}{\sqrt{8n}}) \left(1-\frac{x^2}{2n}\right)^{n-k} \mathbb{1}_{\{-\sqrt{2n}<x<\sqrt{2n}\}} \xrightarrow[n\to\infty]{} f(\tfrac{1}{2})e^{-x^2/2},$$

while the factors in (4.2) not depending on x have the integral representation

$$\left(c_{k,n}\frac{1}{\sqrt{8n}} 2^{-2(n-k)}\right)^{-1} = \int_{-\sqrt{2n}}^{\sqrt{2n}} f_k(\tfrac{1}{2}+\tfrac{x}{\sqrt{8n}}) \left(1-\frac{x^2}{2n}\right)^{n-k} dx =: I_n.$$

With the definitions

$$I := \int_{\mathbb{R}} f(\tfrac{1}{2})e^{-x^2/2}dx, \qquad I_n' := \int_{-\sqrt{2n}}^{\sqrt{2n}} f_k(\tfrac{1}{2}) \left(1-\frac{x^2}{2n}\right)^{n-k} dx,$$

we have the inequality

$$|I_n - I| \le |I_n - I_n'| + |I_n' - I|, \tag{4.3}$$

and we need to show that the right hand side converges to 0. Obviously $|I_n' - I|$ converges to 0 by theorem of dominated convergence. For the first term on the right hand side of (4.3) we have for n sufficiently large

$$|I_n - I_n'| \le 2\int_{-\sqrt{2n}}^{\sqrt{2n}} |f_k(\tfrac{1}{2}) - f_k(\tfrac{1}{2}+\tfrac{x}{\sqrt{8n}})| e^{-x^2/2}dx = \int_{-1}^{1} |f_k(\tfrac{1}{2}) - f_k(\tfrac{1}{2}+\tfrac{x}{2})| \sqrt{2n}e^{-nx^2}dx.$$

For given $\varepsilon > 0$ we choose $\delta > 0$, such that $|f_k(\frac{1}{2}) - f_k(\frac{1}{2} + \frac{x}{2})| < \varepsilon$ for all $x \in (-\delta, \delta)$ then

$$\begin{aligned} |I_n - I_n'| \leq & \varepsilon \int_{(-\delta,\delta)} \sqrt{2n} e^{-nx^2} dx + \int_{(-\delta,\delta)^C} |f_k(\tfrac{1}{2}) - f_k(\tfrac{1}{2} + \tfrac{x}{2})| \sqrt{2n} e^{-nx^2} dx \\ \leq & \varepsilon \int e^{-x^2/2} dx + \int_{(-\delta,\delta)^C} |f_k(\tfrac{1}{2}) - f_k(\tfrac{1}{2} + \tfrac{x}{2})| \sqrt{2n} e^{-nx^2} dx \end{aligned}$$

and the second integral converges to 0 by the theorem of monotone convergence. □

The weak convergence of the canonical moments was shown by Chang et al. (1993) for a uniform distribution on $\mathcal{M}_n([0,1])$ (i.e., $f_k \equiv 1$). These authors showed weak convergence of the vector $\boldsymbol{m}_k^{(n)}$ of the first k components of a uniformly distributed moment vector $\boldsymbol{m}_n = (m_1, \ldots, m_n)$ on the moment space $\mathcal{M}_n([0,1])$, that is

$$\sqrt{8n} A^{-1} (\boldsymbol{m}_k^{(n)} - \boldsymbol{m}_k(\nu)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, I_k), \tag{4.4}$$

where $\boldsymbol{m}_k(\nu)$ denotes the vector of the first k moments of the arcsine distribution and A is a $k \times k$ lower triangular matrix with entries

$$a_{i,j} = 2^{-2i+2} \binom{2i}{i-j} \qquad j \leq i.$$

By part (b) of Theorem 4.1, it is easy to see that the weak convergence in (4.4) holds for the more general densities f_n on $\mathcal{M}_n([0,1])$. We conclude this paper with a discussion of corresponding results for distributions on the non compact moment spaces $\mathcal{M}_n([0,\infty))$ and $\mathcal{M}_n(\mathbb{R})$. In this case the analogs of the arcsine distribution in this context are the Marchenko-Pastur distribution defined by

$$d\eta(x) = \frac{\sqrt{x(4-x)}}{2\pi x} \mathbb{1}_{\{0<x<4\}} dx$$

and Wigners semicircle distribution on the interval $[-2, 2]$, that is

$$d\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{\{-2<x<2\}} dx \tag{4.5}$$

(see Nica and Speicher (2006)). The moments of the Marchenko-Pastur law η are the Catalan numbers c_n defined by

$$m_n(\eta) = c_n = \frac{1}{n+1} \binom{2n}{n} \; n \in \mathbb{N}, \tag{4.6}$$

and the moments of the semicircle law ρ are given by

$$m_n(\rho) = \begin{cases} \frac{1}{m+1}\binom{2m}{m} & \text{if } n = 2m \\ 0 & \text{if } n = 2m-1. \end{cases}$$

Our next results establish the asymptotic properties of the quantities z_k corresponding to a random vector on the moment space $\mathcal{M}_n([0,\infty))$ with density $g_n^{(\gamma,\delta)}$ defined in (2.19). The following result is a well known consequence of the asymptotic behavior of the density of the Gamma distribution and the proof therefore omitted.

Theorem 4.2. *Suppose $\boldsymbol{m}_n$ is a random vector of moments on the moment space $\mathcal{M}_n([0,\infty))$ with density $g_n^{(\gamma,\delta)}$, where the γ_k are fixed, $\delta_1 = \cdots = \delta_n = n$, and let $z_k^{(n)}$ denote the k-th component of the vector $\boldsymbol{z}_n = (z_1^{(n)}, \ldots, z_n^{(n)})$. Then the standardized random variable $z_k^{(n)}$ is asymptotically normal distributed, that is*

$$\sqrt{n}(z_k^{(n)} - 1) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

By Theorem 4.2 the vector

$$\sqrt{n}(\boldsymbol{z}_k^{(n)} - \boldsymbol{1}) = \sqrt{n}((z_1^{(n)}, \ldots, z_k^{(n)})^T - (1, \ldots, 1)^T)$$

is asymptotically multivariate normal distributed. In order to derive a corresponding statement of the random vector $\boldsymbol{m}_k^{(n)} = \psi_k^{-1}(\boldsymbol{z}_k^{(n)})$ we will use the Delta method and study first the image of the vector $(1, \ldots, 1)^T$ under the mapping ψ_k^{-1}.

Lemma 4.3. *Let $(c_n)_{n\geq 1}$ denote the sequence of Catalan numbers defined in* (4.6)*, then*

$$\psi_n(c_1, \ldots, c_n) = (1, \ldots, 1)^T.$$

Proof: The proof presented here relies on the combinatorical interpretation of the Catalan numbers and a recursive algorithm given in Skibinsky (1968) to calculate the moments in terms of the variables z_k. The k-th Catalan number counts the paths in $\mathbb{N} \times \mathbb{N}$ starting in $(0,0)$ and ending in $(2k, 0)$, where one is only allowed to make steps in the direction $(1,1)$ or $(1,-1)$. Skibinsky (1968) defines the triangular array $\{g_{i,j}\}_{i,j\geq 0}$ by $g_{i,j} = 0$ for $i > j$, $g_{0,j} = 1$ and the recursion

$$g_{i,j} = g_{i,j-1} + z_{j-i+1} g_{i-1,j}, \qquad 1 \leq i \leq j. \tag{4.7}$$

He showed that $g_{k,k} = m_k$. Consequently, if $z_i = 1$ $(i = 1, 2, \ldots)$ the quantity $g_{k,k}$ is the number of paths through the lattice $\{(i,j)\}_{i,j\geq 0}$, starting in (k,k) and ending in $(0,0)$, where in each vertex we can only make steps upward or to the left and where we are not allowed to cross the diagonal $\{(i,i)\}$. This number is exactly the k-th Catalan number c_k. □

Theorem 4.4. *If the vector of random moments $\boldsymbol{m}_n \in \mathcal{M}_n([0,\infty))$ is governed by a law with density $g_n^{(\gamma,\delta)}$, where $\delta_1 = \cdots = \delta_n = n$ and the γ_k are fixed, then the projection $\boldsymbol{m}_k^{(n)} = \Pi_k^n(\boldsymbol{m}_n)$ of $\boldsymbol{m}_n$ onto the first k coordinates satisfies*

$$\sqrt{n}C^{-1}(\boldsymbol{m}_k^{(n)} - \boldsymbol{m}_k(\eta)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k),$$

where the vector $\boldsymbol{m}_k(\eta) = (c_1, \ldots, c_k)^T$ *contains the first* k *moments of the Marchenko-Pastur distribution and* C *is a lower triangular matrix with entries* $c_{1,1} = \cdots = c_{k,k} = 1$*, and*

$$c_{i,j} = \binom{2i}{i-j} - \binom{2i}{i-j-1} \qquad j < i.$$

Proof: It suffices to calculate the Jacobi matrix

$$C = \frac{\partial \psi_k^{-1}}{\partial \boldsymbol{z}_k}(\boldsymbol{z}_k^0)$$

of the mapping ψ_k^{-1} at $\boldsymbol{z}_k^0 = (1, \ldots, 1)^T$, then the independence of the recursion variables z_k and Theorem 4.2 yield the convergence

$$\sqrt{n}(\boldsymbol{m}_k^{(n)} - \boldsymbol{m}_k(\eta)) = \sqrt{n}C(\boldsymbol{z}_k^{(n)} - \boldsymbol{z}_k^0) + o_P(1) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, CC^T).$$

Note that the moment m_i depends only on $z_1, \ldots, z_i$ and consequently C is a lower triangular matrix. To identify the entries of the matrix C we consider the triangular array $\{g_{i,j}\}_{i,j\geq 0}$ defined in (4.7). For a fixed r with $1 \leq r \leq k$ we introduce the notation

$$u_{i,j} = \frac{\partial g_{i,j}}{\partial z_r}(\boldsymbol{z}_k^0),$$

and obtain a new triangular array $\{u_{i,j}\}_{i,j\geq 0}$. Obviously we have $u_{i,j} = 0$ for $i > j$ and the other values of $u_{i,j}$ are determined by the initial condition $u_{0,j} = 0$ and the recursion

$$u_{i,j} = u_{i,j-1} + u_{i-1,j} + \delta_{r,j-i+1} g^0_{i-1,j}, \qquad 1 \leq i \leq j, \tag{4.8}$$

were $\delta_{i,j}$ denotes the Kronecker symbol and $g^0_{i,j}$ is the coefficient in the recursion (4.8), if all z_i are equal to 1, that is

$$g^0_{i,j} = \binom{i+j}{i} - \binom{i+j}{i-1}.$$

The numbers $g^0_{i,j}$ are sometimes called generalized Catalan numbers [see Finucan (1976)]. At the end of this proof we will show that the entries in the new triangular array are given by

$$u_{i,j} = \begin{cases} \binom{i+j}{i-1} - \binom{i+j}{i-r-1} & \text{if } j - i \geq r, \\ \binom{i+j}{j-r} - \binom{i+j}{i-r-1} & \text{if } 0 \leq j - i < r. \end{cases} \tag{4.9}$$

With this identity we obtain for the entries of the matrix C

$$c_{i,r} = \frac{\partial m_i}{\partial z_r}(\boldsymbol{z}_k^0) = u_{i,i} = \binom{2i}{i-r} - \binom{2i}{i-r-1}$$

for $1 \leq r \leq i$, which proves the assertion of the Theorem.

The proof of (4.9) follows by an induction argument for the row number i. Obviously we have $u_{0,j} = 0$ and therefore $u_{1,j} = 0$ as long as $j < r$. The value of $u_{1,r}$ is equal to $g^0_{0,r} = 1$ and for $j > r$ we have $u_{1,j} = 1$. These values are also given by the formula (4.9). In the induction step $i-1 \to i$ we perform an induction for j and have to distinguish between five cases with different recursion for $u_{i,j}$. The scheme in (4.10) illustrates these cases in the i-th row.

$$(4.10) \qquad \ddots \underbrace{(i,i)}_{(1)} \ddots \underbrace{(i,i+1)\ldots(i,i+r-2)}_{(2)} \ddots \underbrace{(i,i+r-1)}_{(3)} \ddots \underbrace{(i,i+r)}_{(4)} \ddots \underbrace{(i,i+r+1)\ldots}_{(5)}$$

(1) $j = i$: First consider $r = 1$, Then case (1) is identical to case (3) and

$$\begin{aligned} u_{i,i} =& u_{i-1,i} + g^0_{i-1,i} = \tbinom{2i-1}{i-2} - \tbinom{2i-1}{i-3} + \tbinom{2i-1}{i-1} - \tbinom{2i-1}{i-2} \\ =& \tbinom{2i}{i-1} - \tbinom{2i-1}{i-2} - \tbinom{2i-1}{i-3} = \tbinom{2i}{i-1} - \tbinom{2i}{i-2}. \end{aligned}$$

For $r > 1$ it is

$$u_{i,i} = u_{i-1,i} = \tbinom{2i-1}{i-r} - \tbinom{2i-1}{i-r-2} = \tbinom{2i}{i-r} - \tbinom{2i-1}{i-r-1} - \tbinom{2i-1}{i-r-2} = \tbinom{2i}{i-r} - \tbinom{2i}{i-r-1}.$$

(2) $i < j < i + r - 1$:

$$u_{i,j} = u_{i,j-1} + u_{i-1,j} = \tbinom{i+j-1}{j-1-r} - \tbinom{i+j-1}{i-r-1} + \tbinom{i+j-1}{j-r} - \tbinom{i+j-1}{i-r-2} = \tbinom{i+j}{j-r} - \tbinom{i+j}{i-r-1}$$

(3) $j = i + r - 1$: The case $r = 1$ was considered in (1), for $r > 1$ we have

$$\begin{aligned} u_{i,j} &= u_{i,j-1} + u_{i-1,j} + g^0_{i-1,j} \\ &= \tbinom{i+j-1}{j-1-r} - \tbinom{i+j-1}{i-r-1} + \tbinom{i+j-1}{i-2} - \tbinom{i+j-1}{i-r-2} + \tbinom{i+j-1}{i-1} - \tbinom{i+j-1}{i-2} \\ &= \tbinom{i+j-1}{j-1-r} - \tbinom{i+j}{i-r-1} + \tbinom{i+j-1}{j-r} = \tbinom{i+j}{j-r} - \tbinom{i+j}{i-r-1}. \end{aligned}$$

(4) $j = i + r$:

$$\begin{aligned} u_{i,j} =& u_{i,j-1} + u_{i-1,j} = \tbinom{i+j-1}{j-1-r} - \tbinom{i+j-1}{i-r-1} + \tbinom{i+j-1}{i-2} - \tbinom{i+j-1}{i-r-2} = \tbinom{i+j-1}{j-r-1} + \tbinom{i+j-1}{j-r-2} - \tbinom{i+j}{i-r-1} \\ =& \tbinom{i+j}{j-r-1} - \tbinom{i+j}{i-r-1} = \tbinom{i+j}{i-1} - \tbinom{i+j}{i-r-1} \end{aligned}$$

(5) $j > i + r$:

$$u_{i,j} = u_{i,j-1} + u_{i-1,j} = \tbinom{i+j-1}{i-1} - \tbinom{i+j-1}{i-r-1} + \tbinom{i+j-1}{i-2} - \tbinom{i+j-1}{i-r-2} = \tbinom{i+j}{i-1} - \tbinom{i+j}{i-r-1}$$

□

We conclude the discussion of random moments considering the moment space $\mathcal{M}_{2n-1}(\mathbb{R})$. Recall the bijective mapping (2.11) from the interior of the moment space $\mathcal{M}_{2n-1}(\mathbb{R})$ onto

$(\mathbb{R} \times \mathbb{R}^+)^{n-1} \times \mathbb{R}$ corresponding to the range for coefficients in the recursive relation of the orthogonal polynomials (2.3). The following results give the weak asymptotics of random recursion coefficients and moments and correspond to Theorem 4.2 and 4.4. The proof of Theorem 4.6 follows by similar arguments as presented in the proof of Theorem 4.4 and is therefore omitted.

Theorem 4.5. *Let the random vector $\boldsymbol{m}_{2n-1} \in \mathcal{M}_{2n-1}(\mathbb{R})$ be governed by a law with density $h_{2n-1}^{(\gamma,\delta)}$ where $\gamma_k > -1$ is fixed and $\delta_k = n$ $(k = 1, \dots, n)$. For fixed k denote by $b_k^{(n)}$ and by $a_k^{(n)}$ the $(2k-1)$-th component of the vector $\xi_{2n-1}(\boldsymbol{m}_{2n-1})$ and the $2k$-th component, respectively. Then*

$$\sqrt{2n}b_k^{(n)} \sim \mathcal{N}(0,1),$$
$$\sqrt{2n}(a_k^{(n)} - 1) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

Theorem 4.6. *Let the vector of random moments $\boldsymbol{m}_{2n-1} \in \mathcal{M}_{2n-1}(\mathbb{R})$ be governed by a law with density $h_{2n-1}^{(\gamma,\delta)}$ where $\delta_k = n$ $(k = 1, \dots, n)$. For $k \in \mathbb{N}$ denote by $\boldsymbol{m}_k^{(n)} = \Pi_k^n(\boldsymbol{m}_{2n-1})$ the projection onto the first k coordinates and by $\boldsymbol{m}_k(\rho) = \Pi_k(0, c_1, 0, c_2, \dots)$ the vector of the first k moments of the semicircle law defined in (4.5). Then*

$$\sqrt{2n}D^{-1}(\boldsymbol{m}_k^{(n)} - \boldsymbol{m}_k(\rho)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k),$$

where D is a $k \times k$ lower triangular matrix with $d_{i,j} = 0$ if $i + j$ is odd and the remaining entries are given by

$$d_{i,j} = \binom{i}{\frac{i-j}{2}} - \binom{i}{\frac{i-j}{2} - 1}.$$

By the results in Section 3, the moment density of the three classical ensembles is the moment density investigated asymptotically in this Section. Although for the random matrix ensembles the parameters γ_k, δ_k depend on n, only minor changes are necessary to obtain a weak convergence result for the first k moments. In the case of the Jacobi ensemble, the canonical moment $p_k^{(n)}$ follows a Beta distribution with parameters behaving like $\frac{\beta}{2}n$ and therefore,

$$\sqrt{4\beta n}(p_k^{(n)} - \tfrac{1}{2}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1)$$

and we obtain easily the right scaling for the ordinary moments in the following Corollary.

Corollary 4.7. *Let $\mu_n \sim J\beta E_n(\gamma_0, \delta_0)$ be a spectral measure of the Jacobi ensemble, then the first k moments $\boldsymbol{m}_k(\mu_n)$ of μ_n satisfy*

$$\sqrt{4\beta n}A^{-1}(\boldsymbol{m}_k(\mu_n) - \boldsymbol{m}_k(\nu)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k),$$

where $\boldsymbol{m}_k(\nu)$ is the moment vector of the arcsine measure and A is the $k \times k$ matrix in (4.4).

In particular, the moment convergence implies the weak convergence of the spectral measure to the arcsine measure. This is also a consequence of the well-known convergence of the emprical eigenvalue distribution to the arcsine measure, since the (unscaled) moments of the spectral measure have the same asymptotic behaviour as the moments of the empirical eigenvalue distribution. By Corollary 4.7, the fluctuations around this limit in terms of the moments are Gaussian. A corresponding result holds for the Laguerre ensemble with rescaled eigenvalue density

$$f_L(\lambda) = c_L^{\gamma_0} |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^{\gamma_0} e^{-\beta n \lambda_i/2} \mathbb{1}_{\{\lambda_i > 0\}}.$$

For the first k moments of a spectral measure μ_n with this eigenvalue density we obtain from Theorem 3.3 together with Theorem 2.3 the asymptotic

$$\sqrt{\beta n/2}\, C^{-1}(\boldsymbol{m}_k(\mu_n) - \boldsymbol{m}_k(\eta)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k).$$

From Theorem 3.4 and Theorem 2.5 we can deduce for a spectral measure μ_n of the rescaled Gaussian Ensemble with eigenvalue density

$$f_G(\lambda) = c_G |\Delta(\lambda)|^\beta \prod_{i=1}^n e^{-\beta n \lambda_i^2/4}.$$

the weak convergence of the first k moments

$$\sqrt{\beta n/2}\, D^{-1}(\boldsymbol{m}_k(\mu_n) - \boldsymbol{m}_k(\rho)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k).$$

The different scaling in these two cases comes from the scaling necessary to obtain weak convergence of the recursion parameters z_k and the recursion coefficients a_k, b_k, respectively. Again, this convergence results can be related to the convergence of the empirical eigenvalue density to the Marchenko-Pastur law and the semicircle distribution and give the fluctuations around the limit distribution.

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