Quantile Spectral Analysis for Locally Stationary Time Series

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Abstract

Classical spectral methods are subject to two fundamental limitations: they only can account for covariance-related serial dependencies, and they require second-order stationarity. Much attention has been devoted recently to quantile-based spectral methods that go beyond covariance-based serial dependence features. At the same time, methods relaxing stationarity into much weaker *local stationarity* conditions have been developed for a variety of time-series models. Here, we are combining those two approaches by proposing quantile-based spectral methods for locally stationary processes. We therefore introduce time-varying versions of the copula spectra and periodograms that have been recently proposed in the literature, along with a new definition of *strict* local stationarity that allows us to handle completely general non-linear processes without any moment assumptions, thus accommodating our quantile-based concepts and methods. We establish the consistency of our methods, and illustrate their power by means of simulations and an empirical study of the Standard & Poor's 500 series. This empirical study brings evidence of important variations in serial dependence structures both across time (crises and quiet periods exhibit quite different dependence structures) and across quantiles (dependencies between extreme quantiles are not the same as in the "median" range of the series). Such variations remain completely undetected, and are actually undetectable, via classical covariance-based spectral methods.

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1 Introduction

For more than a century, spectral methods have been among the favorite tools of time-series analysis. The concept of periodogram was proposed and discussed as early as 1898 by Schuster, who coined the term in a study (Schuster (1898)) of meteorological series. The modern mathematical foundations of the approach were laid between 1930 and 1950 by such big names as Wiener, Cramér, Kolmogorov, Bartlett, and Tukey. The main reason for the unwavering success of spectral methods

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is that they are entirely *model-free*, hence fully nonparametric; as such, they can be considered a precursor to the subsequent development of nonparametric techniques in the area and, despite their age, they still are part of the leading group of methods in the field.

The classical spectral approach to time series analysis, however, remains deeply marked by two major restrictions:

(i) as a second-order theory, it is essentially limited to modeling first- and second-order dynamics: being entirely covariance-based, it cannot accommodate heavy tails and infinite variances, and cannot account for any dynamics in conditional skewness, kurtosis, or tail behavior;

(ii) the assumption of second-order stationarity is pervasive: except for processes that, after some adequate transformation such as differencing or cointegration, are second-order stationary, observations exhibiting time-varying distributional features are ruled out.

The first of these two limitations recently has attracted much attention, and new quantile-related spectral analysis tools have been proposed, which do not require second-order moments, and are able to capture serial features that cannot be accounted for by the classical second-order approach. Pioneering contributions in that direction are Hong (1999) and Li (2008), who coined the names of *Laplace spectrum* and *Laplace periodogram*. The Laplace spectrum concept was further studied by Hagemann (2011), and extended into *cross-spectrum* and spectral *kernel* concepts by Dette et al. (2014), who also introduced *copula-based* versions of the same. Those cross-spectral quantities are indexed by couples (τ_1, τ_2) of quantile levels, and their collections (for $(\tau_1, \tau_2) \in [0, 1]^2$) account for any features of the joint distributions of pairs (X_t, X_{t-k}) in a strictly stationary process $\{X_t\}$ without requiring any distributional assumptions such as the existence of finite moments.

That thread of literature also includes Li (2012, 2014), Kley et al. (2014), and Lee and Subba Rao (2012). Somewhat different approaches were taken by Hong (2000), Davis et al. (2013), and several others; in the time domain, Linton and Whang (2007), Davis and Mikosch (2009), and Han et al. (2014) introduced the related concepts of *quantilograms* and *extremograms*. Strict stationarity, however, is essential in all those contributions.

The pictures in Figure 1 show that the copula-based spectral methods developed in Dette et al. (2014) (where we refer to for details) indeed successfully account for serial features that remain out of reach in the traditional approach. The series considered in Figure 1 is the classical S&P500 index series, with $T = 13092$ observations from 1962 through 2014; more precisely, that series contains the differences of logarithms of daily opening and closing prices for about 52 years. That series is generally accepted to be white noise, yielding perfectly flat periodograms. Three rank-based copula periodograms are provided, for the quantile levels 0.1, 0.5 and 0.9, respectively. The central one, corresponding to the central part of the marginal distribution, is compatible with the assumption of white noise. But the more extreme ones (associated with quantile levels 0.1 and 0.9) yield a peak at the origin, pointing at a long-memory-like behavior in the tails which is definitely not present in the median part of the (marginal) distribution.

Now, the periodograms in Figure 1 were computed from the whole series ($1 \leq t \leq 13092$), under the presumption of stationarity (more precisely, stationarity in distribution, for all k, of the couples (X_t, X_{t-k})). Is that assumption likely to hold true? Traditional periodograms computed from the four disjoint subseries corresponding to the periods 1962-1974, 1974-1987, 1987-2000, and 2000-2014 are shown in Figure 2, and suggest an evolution in time, by which the descending spectral density of the 1962-1974 period evolves into the ascending one of the more recent 2000-2014 years.

This brings us to questioning the second limitation of traditional spectral methods, second-order stationarity, and motivated the development of a rich strand of literature, mainly along four (largely overlapping) lines:

(a) *models with time-dependent parameters*: inherently parametric, those models are mimicking the traditional ones, but with parameters varying over time—see Subba Rao (1970) for a

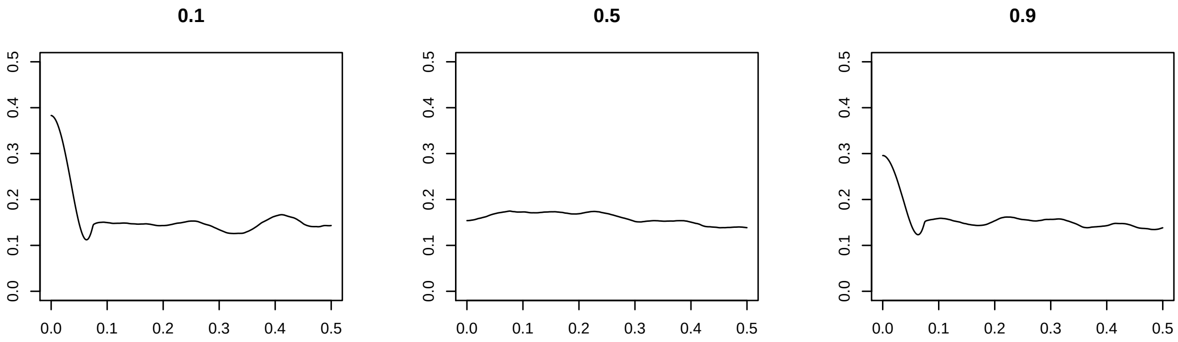


Figure 1: S&P500, 1962-2014: the smoothed rank-based copula periodograms for $\tau_1 = \tau_2 = \tau = 0.1$, 0.5 and 0.9, respectively. All curves are plotted against $\omega/2\pi$.

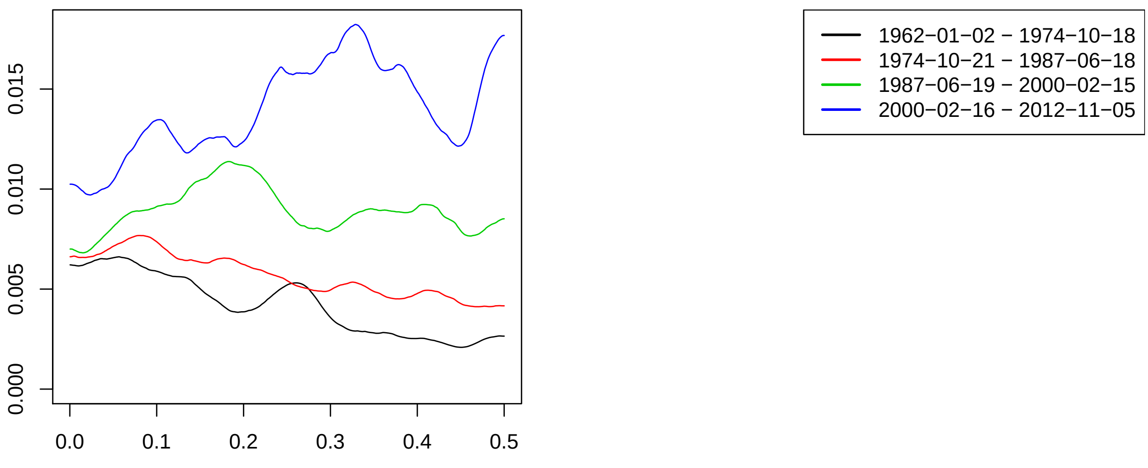


Figure 2: S&P500, 1962-2014: the traditional smoothed periodograms for the periods 1962-1974, 1974-1987, 1987-2000, and 2000-2014, respectively. All curves are plotted against $\omega/2\pi$.

prototypical contribution, Azrak and Mélard (2006) for an in-depth study of the time-varying ARMA case;

(b) the *evolutionary spectral methods*, initiated by Priestley (1965), where the process under study admits a spectral representation with *time-varying transfer function*—a second-order characterization, thus, but wavelet-based versions also have been considered, as in Nason et al. (2000);

(c) *piecewise stationary processes*, in relation with change-point analysis: see, e.g., Davis et al. (2005);

(d) the *locally stationary process* approach initiated by Dahlhaus (1997, 2000) based on the assumption that, over a short period of time (that is, locally in time), the process under study behaves approximately as a stationary one. We refer to Dahlhaus (2012) for a survey of this approach and mention that related concepts have been recently developed by Zhou and Wu (2009); Zhao and Wu (2009) and Vogt (2012).

Those four approaches, as already mentioned, are not without overlaps: Dahlhaus (1996) actually is about varying-parameter autoregressive models; so is Dahlhaus et al. (1999); Dahlhaus (1997) is based on time-varying (second-order) spectral representations, turned into time-domain linear MA(∞) ones by Dahlhaus and Polonik (2009); Dahlhaus and Subba Rao (2006) and Fryzlewicz et al. (2008) deal with locally stationary ARCH models, hence also resort to (a); most references require moment assumptions, either by nature (because they are based on a spectral representation), or by the nature of the stationary approximation they are considering.

In this paper, we are trying to address the two limitations (i) and (ii) of traditional spectral analysis simultaneously by developing a local stationary version of the quantile-related spectral analysis proposed in Dette et al. (2014). While adopting the local stationary ideas of (d), however, we turn them into a fully non-parametric and moment-free approach, adapted to the nature of quantile- and copula-based spectral concepts (see Harvey (2010) for a related, time-domain, attempt). The definitions of local stationarity existing in the literature indeed are not general enough to accommodate quantile spectra, and we therefore formulate a new concept of *strict local stationarity.* Contrary to Dahlhaus (1996), which deals with time-varying autoregressions, to Dahlhaus (1997), which is based on time-varying second-order spectra, or to Vogt (2012) where the approximation is in terms of stochastic variables and requires finite moments of order $\rho > 0$, our approximation is directly based on joint distribution functions and does not involve any moments. This very general concept of local stationarity allows us to handle completely general non-linear processes without moment assumptions, and to extend to the quantile context the definitions of a local spectrum and a local periodogram. The *time-varying copula spectrum* and its estimators are introduced in Section 2 and Section 3, respectively. In Section 4 we illustrate the application of the new methodology by means of a small simulation study and a data example, while the theoretical properties of time-varying copula spectra are investigated in Section 5. In particular, consistency of the corresponding smoothed local periodograms is established. The main ideas and arguments of the proofs are collected in an appendix in Section 6, while additional technical results and explanations are deferred to an online supplement.

When applied to the S&P500 series of Figures 1 and 2, our local periodograms yield the estimated copula-based spectra shown in Figure 3 (to be compared with those in Figure 2). Time-varying periodogram values in those figures, are represented by a color, ranging from cyan and light blue ("small" values) to orange and red ("large values"), in such a way that dark blue regions correspond to those where the periodogram does not significantly differ from that of a white noise process; see Section 4.1 for details. Whereas the central periodograms ($\tau_1 = \tau_2 = 0.5$) are pretty flat (dark blue) with the exception of some long-memory-like behavior limited to the early seventies, the more extreme ones ($\tau_1 = \tau_2 = 0.1$ and 0.9) suggest an alternance of high low-frequency

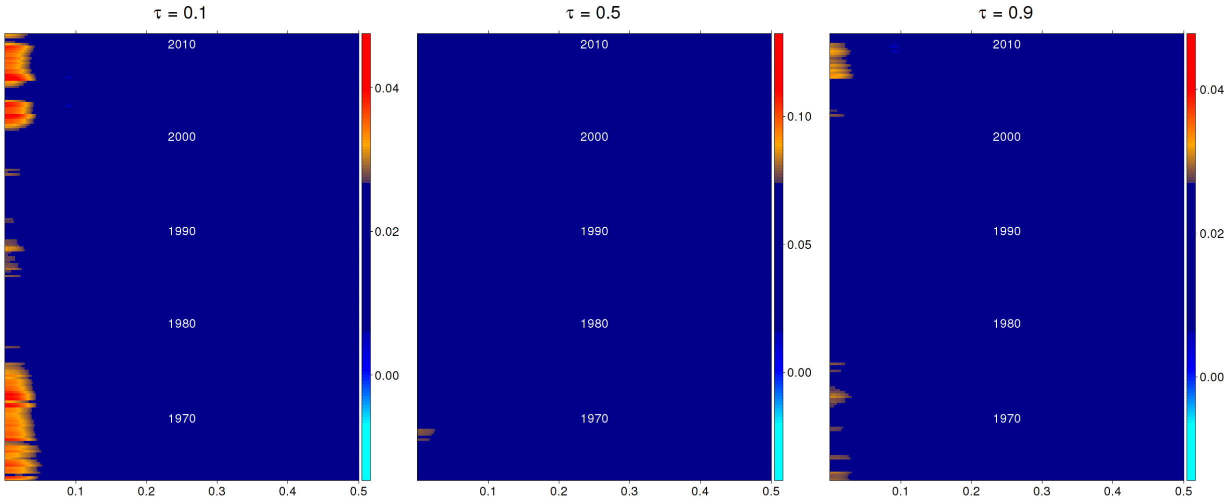


Figure 3: *Levelplots of the time-varying smoothed rank-based copula periodograms for* $\tau_1 = \tau_2 = \tau = 0.1$, 0.5, *and* 0.9, *respectively. The horizontal axis represents the frequencies* $\omega/2\pi$ *from* 0 *to* 0.5, *the vertical axis is time (1963-2014;* $1 \le t \le 13092$*); for each value of* t*, a periodogram is plotted against frequencies via the color code provided along the right-hand side of each figure.*

spectral densities (yellow and red) and perfectly "flat" (dark blue) periods. A closer analysis of this S&P500 series is provided in Section 4.3, and reveals that those periods of "long memory regime" correspond to well identified crises and booms. Another interesting observation is the asymmetry between the time-varying spectra associated with the left ($\tau = 0.1$) and right ($\tau = 0.9$) tails. That asymmetry is amply confirmed by comparing the periodograms associated with $\tau = 0.2$ and $\tau = 0.8$ shown in Figure 4. Inspection of local stationary periodograms thus suggests that the S&P500 series, perhaps, is not as close to white noise as claimed. However, it takes a combination of quantile-related and local stationarity tools to bring some evidence for that fact.

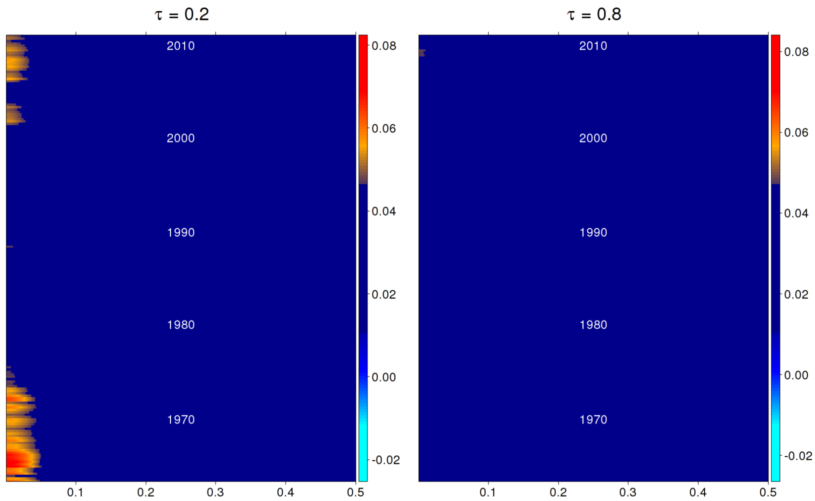


Figure 4: *Levelplot of the time-varying periodograms for* $\tau_1 = \tau_2$ *and* $\tau_1 \in \{0.2, 0.8\}$*. The horizontal axis represents the frequencies* $\omega/2\pi$ *from* 0 *to* 0.5*, the vertical axis is time (1963-2014;* $1 \le t \le 13092$*); for each value of* t*, a periodogram is plotted against frequencies via the color code provided along the right-hand side of each figure.*

2 Strict local stationarity and local copula spectra

2.1 Strictly locally stationary processes

Consider an observed series $(X_1, \ldots, X_T)$ of length T as being part of a triangular array $(X_{t,T},\ 1 \leq t \leq T)$, $T \in \mathbb{N}$, of finite-length realizations of nonstationary processes $\{X_{t,T},\ t \in \mathbb{Z}\}$, $T \in \mathbb{N}$. The intuitive idea behind the definitions of local stationarity by Dahlhaus (1996, 2000), Zhou and Wu (2009) and Vogt (2012) (to quote only a few) is the assumption that those processes have an approximately stationary behavior over a short period of time. More formally, all those authors assume the existence of a collection, indexed by $\vartheta \in (0,1)$, of stationary processes $\{X_t^{\vartheta},\ t \in \mathbb{Z}\}$ such that the nonstationary process $\{X_{t,T},\ t \in \mathbb{Z}\}$ can be approximated (in a suitable way), in the vicinity of time t, by the stationary process $\{X_t^{\vartheta},\ t \in \mathbb{Z}\}$ associated with $\vartheta = t/T$.

The exact nature of this approximation has to be adapted to the specific problem under study. If the objective is an extension of classical spectral analysis, only the autocovariances $\mathrm{Cov}(X_{t,T}, X_{s,T})$ have to be approximated. In the quantile-related context considered here, the joint distributions of $X_{t,T}$ and $X_{s,T}$ is the feature of interest, and traditional autocovariances are to be replaced with autocovariances of indicators, of the form $\mathrm{Cov}(\mathbb{I}_{\{X_{t,T} \leq q_{t,T}(\tau_1)\}}, \mathbb{I}_{\{X_{s,T} \leq q_{s,T}(\tau_2)\}})$, where $q_{t,T}(\tau_1)$ stands for $X_{t,T}$'s quantile of order τ_1 and $q_{s,T}(\tau_2)$ for $X_{s,T}$'s quantile of order τ_2, with $\tau_1, \tau_2 \in (0,1)$ (see Li (2008, 2012), Hagemann (2011), or Dette et al. (2014)). Such covariances only depend on the bivariate copulas of $X_{t,T}$ and $X_{s,T}$.

In the strictly stationary context, this leads to the so-called *Laplace spectrum*, first considered by Li (2008) for a strictly stationary process $\{Y_t, t \in \mathbb{Z}\}$ with marginal median zero. The Laplace spectrum is defined as

$$\mathcal{C}_{0,0}(\omega) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-i\omega k} \mathrm{Cov}(\mathbb{I}_{\{Y_0 \leq 0\}}, \mathbb{I}_{\{Y_{-k} \leq 0\}}), \qquad \omega \in (-\pi, \pi].$$

That concept was extended by Hagemann (2011), Dette et al. (2014), and Li (2012) to general quantile levels. The most general version, which also takes into account cross-covariances of indicators, was introduced by Dette et al. (2014). Denoting by $q(\tau)$ the marginal quantile function of Y_t, they define the *copula spectral density kernel* as

$$\mathcal{C}_{\tau_1,\tau_2}(\omega) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-i\omega k} \mathrm{Cov}(\mathbb{I}_{\{Y_0 \leq q(\tau_1)\}}, \mathbb{I}_{\{Y_{-k} \leq q(\tau_2)\}}), \quad \tau_1, \tau_2 \in (0,1), \qquad \omega \in (-\pi, \pi].$$

Those definitions heavily rely on the strict stationarity of the underlying time series; without strict stationarity, actually, they do not make much sense. It seems natural, thus, to ask whether some adequate notion of local stationarity can be employed to characterize the notion of a local copula-based spectrum. However, the definitions of local stationarity previously considered in the literature are placing unnecessarily strong restrictions on the classes of processes that can be considered. In particular, Dahlhaus (1996)) and Vogt (2012) rely on moment assumptions that are not natural in a quantile context, and are not required for the definition of copula spectra. We therefore introduce a new concept of *strict local stationarity* which completely avoids moment assumptions while allowing us to define and estimate local versions of the copula spectral density kernel. Our concept, however, is not totally unrelated to the existing ones, and we also show that, under adequate conditions, processes that are locally stationary in the sense of Dahlhaus (1996) are strictly locally stationary in the new sense, see Section 5.1 for details.

The Laplace and Copula spectral density kernels of a stationary process $\{Y_t\}$ are defined in terms of its bivariate marginal distribution functions. Therefore, it is natural to use bivariate marginal distribution functions when evaluating, in the definition of local stationarity, the distance between the non-stationary process $\{X_{t,T}\}$ under study and its stationary approximation $\{X_t^{\vartheta}\}$.

Definition 2.1. *A triangular array* $\{(X_{t,T})_{t\in\mathbb{Z}}\}_{T\in\mathbb{N}}$ *of processes is called* locally strictly stationary (of order two) *if there exists a constant* $L > 0$ *and, for every* $\vartheta \in (0,1)$*, a strictly stationary process* $\{X_t^\vartheta, t \in \mathbb{Z}\}$ *such that for every* $1 \le r, s \le T$

$$\left\| F_{r,s;T}(\cdot,\cdot) - G^\vartheta_{r-s}(\cdot,\cdot) \right\|_\infty \le L\Big(\max(|r/T - \vartheta|, |s/T - \vartheta|) + 1/T \Big) \tag{2.1}$$

where $\|\cdot\|_\infty$ *stands for the supremum norm, while* $F_{r,s;T}(\cdot,\cdot)$ *and* $G^\vartheta_k(\cdot,\cdot)$ *denote the joint distribution functions of* $(X_{r,T}, X_{s,T})$ *and* $(X_0^\vartheta, X_{-k}^\vartheta)$*, respectively.*

Here, "of order two" refers to the fact that (2.1) is based on bivariate distributions only. Letting y tend to infinity in $F_{r,s;T}(x,y)$ and $G^\vartheta_k(x,y)$, we get an analogous condition for the marginal distributions $F_{t;T}$ and G^ϑ of $X_{t,T}$ and X_0^ϑ, namely

$$\left\| F_{t;T}(\cdot) - G^\vartheta(\cdot) \right\|_\infty \le L\big|t/T - \vartheta\big| + L/T. \tag{2.2}$$

Intuitively, (2.1) and (2.2) imply that the univariate and bivariate distribution functions $F_{t;T}$ and $F_{r,s;T}$ of the process $\{X_{t,T}\}$ are allowed to change smoothly over time. One advantage of this definition is its nonparametric character, as it does not depend on any specific data-generating mechanism.

2.2 Local copula spectral density kernels

Turning to the definition of a localized version of copula spectral density kernels, first consider the copula cross-covariance kernels associated with the strictly stationary $\{X_t^\vartheta, t \in \mathbb{Z}\}$, $\vartheta \in (0,1)$. The lag-h-copula cross-covariance kernel of $\{X_t^\vartheta\}$, as defined in Dette et al. (2014), is

$$\gamma_h^\vartheta(\tau_1,\tau_2) := \mathrm{Cov}(\mathbb{I}_{\{X_t^\vartheta \le q^\vartheta(\tau_1)\}}, \mathbb{I}_{\{X_{t-h}^\vartheta \le q^\vartheta(\tau_2)\}}), \quad \tau_1,\tau_2 \in (0,1),$$

where $q^\vartheta(\tau)$ denotes X_t^ϑ's marginal quantile of order τ.

These cross-covariances always exist; their collection (for $\tau_1,\tau_2 \in (0,1)$) provides a canonical characterization of the joint copula of $(X_t^\vartheta, X_{t-h}^\vartheta)$, hence, an approximate (in the sense of (2.1)) description of the joint copula of $(X_{t,T}, X_{t-h,T})$. Therefore we also call $\gamma_h^\vartheta(\tau_1,\tau_2)$ the *time-varying lag h copula cross-covariance kernel* of $\{X_{t,T}\}$. If we assume that the lag-h-covariance kernels $\gamma_h^\vartheta(\tau_1,\tau_2)$ are absolutely summable for all $\tau_1,\tau_2 \in (0,1)$, we moreover can define the *local* or *time-varying Laplace spectral density kernel* of $\{X_{t,T}\}$ as

$$\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2) := \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} \gamma_h^\vartheta(\tau_1,\tau_2)e^{-ih\omega}, \quad \tau_1,\tau_2 \in (0,1), \quad \omega \in (-\pi,\pi]. \tag{2.3}$$

The time-varying covariance kernel then admits the representation

$$\gamma_h^\vartheta(\tau_1,\tau_2) = \int_{-\pi}^{\pi} e^{ih\omega}\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)d\omega, \qquad \omega \in (-\pi,\pi], \quad \tau_1,\tau_2 \in (0,1).$$

In Section 5.3, we provide an additional theoretical justification for considering time-varying Laplace spectral density kernels demonstrating that these kernels can be considered as approximations of indicator versions

$$\mathfrak{W}_{t_0,T}(\omega,\tau_1,\tau_2) := \sum_{s=-\infty}^{\infty} \mathrm{Cov}\Big(\mathbb{I}_{\{X_{\lfloor t_0+s/2\rfloor,T} \le F^{-1}_{\lfloor t_0+s/2\rfloor,T}(\tau_1)\}}, \mathbb{I}_{\{X_{\lfloor t_0-s/2\rfloor,T} \le F^{-1}_{\lfloor t_0-s/2\rfloor,T}(\tau_2)\}}\Big)\frac{e^{-i\omega s}}{2\pi} \tag{2.4}$$

of the so-called *Wigner-Ville spectrum* of $\{X_{t,T}\}$ (see Martin and Flandrin (1985)). Additional evidence for the usefulness of the concepts discussed here for data analysis is provided in Section 4, where we discuss both simulation evidence and a data analysis of the S&P 500 time series.

3 Estimation of local copula spectra

Given observations $X_{1,T}, \ldots, X_{T,T}$, the classical approach to the estimation of the time-varying spectral density of a locally stationary time series consists in considering a subset of n data points centered around a time point t_0. To formalize ideas, let m_T be a sequence of positive integers that converges to infinity as $T \to \infty$. Define the discrete neighborhood

$$\mathcal{N}_{t_0,T} := \Big\{ t \in \mathbb{Z} : |t_0 - t| < m_T \Big\},$$

denote by $n = n(m_T, T)$ the cardinality of the set $\mathcal{N}_{t_0,T}$, and let $t_{\min} := \min\{t \in \mathcal{N}_{t_0,T}\}$. Define the local rank of $X_{t,T}$ as its rank $R_{t_0,T}(X_{t,T})$ within the n-tuple $\{X_{t,T} \mid t \in \mathcal{N}_{t_0,T}\}$. Denote by $\omega_{j,n} = 2\pi j/n$, $1 \le j \le \lfloor \frac{n+1}{2} \rfloor$ the positive Fourier frequencies, by $x \mapsto \rho_\tau(x) := x(\tau - \mathbb{I}_{\{x \le 0\}})$ the so-called *check function* (see Koenker (2005)), let $\mathbf{c}_t'(\omega) := (1, \cos(\omega(t - t_{\min} + 1)), \sin(\omega(t - t_{\min} + 1)))$, and introduce the piecewise constant function φ_n defined on the interval $(0, \pi)$ by

$$\varphi_n(\omega) := \omega_{j,n}, \tag{3.1}$$

where $\omega_{j,n}$ is the Fourier frequency closest to ω—more precisely, $\omega_{j,n}$ is such that ω belongs to the interval $(\omega_{j,n} - \frac{2\pi}{n}, \omega_{j,n} + \frac{2\pi}{n}]$. Following Dette et al. (2014), the *local rank-based Laplace periodogram* is defined as

$$\hat{L}_{t_0,T}(\omega, \tau_1, \tau_2) := \frac{n}{4} \hat{\boldsymbol{b}}_{t_0,T}(\varphi_n(\omega), \tau_1)' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \hat{\boldsymbol{b}}_{t_0,T}(\varphi_n(\omega), \tau_2), \quad \omega \in (0, \pi), \ \tau_1, \tau_2 \in (0, 1), \tag{3.2}$$

with

$$(\hat{a}_{t_0,T}(\omega_{j,n}, \tau), \hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n}, \tau)) := \operatorname*{argmin}_{(a,b) \in \mathbb{R}^3} \sum_{t \in \mathcal{N}_{t_0,T}} \rho_\tau(n^{-1} R_{t_0,T}(X_{t,T}; \vartheta) - (a, \boldsymbol{b})\mathbf{c}_t(\omega_{j,n})). \tag{3.3}$$

In Theorem 5.1, we show that the local estimators $\hat{L}_{t_0,T}$ defined in (3.2)-(3.3) converge in distribution to non-degenerate complex random variables with expected values $\mathfrak{f}^\vartheta(\omega, \tau_1, \tau_2)$. Thus, local periodograms, just as the traditional ones, yield inconsistent estimators of the corresponding spectral densities—here, the local Laplace spectra. In the stationary case, a smoothed version of the estimator is used to circumvent this problem. We will show that this technique also works in a local stationary context. For this purpose, we introduce a *smoothed* version

$$\hat{f}_{t_0,T}(\omega_{j,n}, \tau_1, \tau_2) := \sum_{|k| \le K_n} W_{t_0,T}(k) \hat{L}_{t_0,T}(\omega_{j+k,n}, \tau_1, \tau_2), \tag{3.4}$$

of time-varying periodograms at the Fourier frequencies $\omega_{j,n} = 2\pi j/n$, where $K_n \to \infty$ as $n \to \infty$ and $\{W_{t_0,T}(k) : |k| \le K_n\}$ is a sequence of positive weights satisfying

$$W_{t_0,T}(k) = W_{t_0,T}(-k) \quad \text{and} \quad \sum_{|k| \le K_n} W_{t_0,T}(k) = 1.$$

The function $\hat{f}_{t_0,T}(\cdot, \tau_1, \tau_2)$ is extended to the interval $(0, \pi)$ by letting

$$\hat{f}_{t_0,T}(\omega, \tau_1, \tau_2) := \hat{f}_{t_0,T}(\varphi_n(\omega), \tau_1, \tau_2).$$

In Section 5.2, we prove that, under mild conditions on the weights and bandwidth parameters, the smoothed time-varying periodograms defined in (3.4) provide consistent estimates of the copula spectral density $\mathfrak{f}^\vartheta(\omega, \tau_1, \tau_2)$. In Section 4, we illustrate the properties of the proposed estimators by means of simulated and real data.

4 Simulations and an empirical study

4.1 Calibrating the color scale

As in Section 1, plots of time-varying spectral densities and their estimations are provided in the form of *heat diagrams*: the horizontal axis represents frequencies ($0 \leq \omega/2\pi \leq 0.5$), the vertical axis a subset of the span of time $1, \ldots, T$ over which the time-varying spectral quantities are estimated. The spectral values themselves (for $\tau_1 = \tau_2 = \tau$), or their real and imaginary parts (for $\tau_1 \neq \tau_2$) are represented via a continuous color code, ranging from cyan and light blue (for small values) to dark blue, yellow, orange, and red (for large values). As we shall explain below, this color code also has an interpretation in terms of significance of certain p-values. This requires a preliminary calibration step, though. Indeed, being "small", for a $(\tau_1 = \tau_2 = \tau)-$periodogram value (which by nature is nonnegative real) cannot have the same meaning as being "small" for the imaginary or the real part of some $(\tau_1', \tau_2')-$cross-periodogram (for which negative values are possible): a meaningful color code therefore should be (τ_1, τ_2)-specific. We therefore introduce a distribution-free simulation-based calibration that fully exploits the properties of copula-based quantities.

To explain the idea behind this calibration step, consider plotting, for some $\mathcal{T}_0 \subset \{1, ..., T\}$ and $\Omega \subset (0, \pi)$, a collection $\big(\Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2)\big)_{t_0 \in \mathcal{T}_0, \omega \in \Omega}$ of the real parts (the imaginary parts are dealt with in exactly the same way) of estimators computed from the realization $X_1, ..., X_T$ of some time series of interest. A color is then attributed to each value of $\Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2)$ along the following steps:

(i) simulate $M = 1000$ independent realizations $(U_{1,m}, \ldots, U_{T,m})$, $m = 1, ..., M$ of an i.i.d. sequence of random variables of length T (one can assume them to be uniform over $[0, 1]$, but, in view of the distribution-freeness of our rank-based method, this is not required);

(ii) for each of those M realizations, compute the estimator $\hat{f}^{U,m}_{t_0,T}(\omega, \tau_1, \tau_2)$ of the local spectral density;

(iii) define, for each $m = 1, ..., 1000$, the quantities

$$Q^m_{\max} := \max_{\omega, t_0} \Re \hat{f}^{U,m}_{t_0,T}(\omega, \tau_1, \tau_2) \quad \text{and} \quad Q^m_{\min} := \min_{\omega, t_0} \Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2),$$

and obtain the empirical 99.5% quantiles $q_{\max}$ of $(Q^{U,m}_{\max})_{m=1,...,M}$ and the 0.5% quantile $q_{\min}$ of $(Q^m_{\min})_{m=1,...,M}$, respectively.

The color palette then is set as follows: all points $(t_0, \omega) \in \mathcal{T}_0 \times \Omega$ with $\Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2)$ value in $[q_{\min}, q_{\max}]$ receive dark blue color. Next, letting

$$\begin{aligned} v_{\min} &:= \min(\min_{t_0,\omega} \Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2), q_{\min} - (q_{\max} - q_{\min})), \\ v_{\max} &:= \max(\max_{t_0,\omega} \Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2), q_{\max} + (q_{\max} - q_{\min})), \end{aligned}$$

all points (t_0, ω) for which $\Re \hat{f}_{t_0,T}(\omega, \tau_1, \tau_2)$ lies in the interval $[v_{\min}, q_{\min}]$ receive a color ranging, according to a linear scale, from cyan to light and dark blue, while the colors for the interval $[q_{\max}, v_{\max}]$ similarly range from dark blue to yellow and red. The correspondence between the actual size of the estimate and the colors used is provided by the numerical scale on the right-hand side of each diagram.

All our heat diagrams thus have the following interpretation. For each given choice of (τ_1, τ_2), the probability, under the hypothesis of white noise, that the real (resp., the imaginary) part of the smoothed (τ_1, τ_2)-time-varying periodogram lies entirely in the dark blue area is approximately 0.01.

Hence, the presence of light blue, cyan or orange-red zones in a diagram indicates a significant (at probability level 1%) deviation from white noise behavior. The location of those zones moreover tells us where in the spectrum, and when in the period of observation, those significant deviations take place, along with an evaluation of their magnitude.

This calibration method yields a universal distribution-free and model-free color scaling which at the same time provides (as far as dark blue regions are concerned) a hypothesis testing interpretation of the results. The same color code was used for the SP500 data in Sections 1, and 4.3, as well as for the simulations in Section 4.2. Currently, an R-package containing the programs which were used in the simulations and data analysis is in preparation. A preliminary version is available from the authors upon request.

4.2 Simulations

This section provides a numerical illustration of the performances of our estimator of the time-varying quantile spectral density in several time-varying models that have been considered elsewhere in the literature. For each of those models, two arrays of time-varying copula cross-spectral densities are provided, side by side, under the form of heat diagrams, for each combination of the quantile levels 0.1, 0.5, and 0.9, using the color code described in Section 4.1:

(a) the smoothed rank-based periodogram estimators of the copula-based spectral densities, and

(b) the "actual" time-varying copula-based spectral densities (of which (a) provides an estimator).

The estimators in (a) are computed from one realization, of length $T = 2^{13}$, of the (nonstationary) process under consideration. For the smoothing weights, we use

$$W_{t_0,T}(k) := \frac{\tilde{W}_{t_0,T}(k)}{\sum_{|m|\le nb_n} \tilde{W}_{t_0,T}(m)} \quad \text{and} \quad \tilde{W}_{t_0,T}(k) := b_n^{-1} \sum_{j=-\infty}^{\infty} W(b_n^{-1}[2\pi k/n + 2\pi j]), \tag{4.1}$$

where

$$W(u) := \frac{15}{32\pi}\Big(7(u/\pi)^4 - 10(u/\pi)^2 + 3\Big) I\{|u| \le \pi\}$$

is a kernel (chosen in accordance with the recommendations in Gasser et al. (1985)), b_n a bandwidth given in Table 1, and $K_n := \lceil nb_n \rceil$ (see Kley et al. (2014) for a similar approach). In each case, $\mathcal{T}_0$ was given by $\{64k + n/2 | k = 0, \dots, \lfloor (T-n)/64 \rfloor\}$, and we used $\Omega := \{2\pi j/n | j = 1, ..., (n-2)/2\}$. Table 1 provides the bandwidths b_n and window lengths n which were used for each specific model.

The actual (cross-)spectral densities in (b) were obtained by simulating, for each $t_0 \in \mathcal{T}_0$, $R = 1000$ independent replications, all of length 2^{11}, of the strictly stationary approximation $(X_t^{t_0/T})_{t=1,\dots,2^{11}}$, computing the corresponding rank-based Laplace periodograms $\hat{L}^r_{t_0,T}(\omega, \tau_1, \tau_2)$, say, $r = 1, ..., R$, and averaging them, for each fixed $(t_0, \omega) \in \mathcal{T}_0 \times \Omega$, over $r = 1, ..., R$.

The following models were considered.

(1) In Figure 5, we display the results for a classical Gaussian time-varying AR(2) process, taken from Dahlhaus (2012), with equation

$$X_{t,T} = 1.8\cos(1.5 - \cos(2\pi t/T))X_{t-1} - 0.81X_{t-2} + Z_t \tag{4.2}$$

and $Z_t \sim \mathcal{N}(0, 1)$. Its strictly stationary approximation at $t_0 = \vartheta T$, $0 \le \vartheta \le 1$, is given by

$$X_t^{\vartheta} = 1.8\cos(1.5 - \cos(2\pi\vartheta))X_{t-1}^{\vartheta} - 0.81X_{t-2}^{\vartheta} + Z_t. \tag{4.3}$$

This process exhibits a time-varying periodicity which is clearly visible in the heat diagram for the real parts of its time-varying copula-based spectral (cross-)densities, which are displayed

Model	Bandwidth b_n	window length n
(1) time-varying Gaussian AR(2)	0.075	512
(2) time-varying Cauchy AR(2)	0.075	512
(3) time-varying Gaussian ARCH(1)	0.1	2048
(4) time-varying QAR(1)	0.125	2048

Table 1: The weights, bandwidths, and window lengths used in the estimation of the copula-based spectral densities in parts (a) of Figures 5-8.

in the lower triangular part of Figure 5(b). The uniformly dark blue imaginary parts in the upper triangular part are a consequence of the fact that those imaginary parts actually are zero, since Gaussian processes are time-reversible [see Proposition 2.1 in Dette et al. (2014)]. Those spectral densities are quite well recovered by our estimator (Figure 5(a)). As expected, no additional information can be gained from observing different quantiles, since conditional distributions, hence all conditional quantiles, in Gaussian processes, get shifted by the same quantity.

(2) In Figure 6, we show heat diagrams for the same time-varying AR(2) process, now driven by independent Cauchy innovations. This model violates the moment assumptions of classical spectral analysis. The imaginary parts of the spectra are shown in the upper triangular part of Figure 6(b); note that, due to time-irreversibility, the actual spectral density (b) exhibits significant yellow parts which, however, are too narrow to be picked up by our estimator (a). Also note the significant peak around zero appearing in the diagrams associated with extreme quantiles ($\tau_1, \tau_2 = 0.1$ and 0.9); they indicate long-memory-like persistence in tail events—a phenomenon that totally escapes traditional analyses.

(3) In Figure 7, results for a time-varying ARCH(1) model of the form

$$X_{t,T} = \sqrt{1/2 + (0.9t/T)X_{t-1}^2}Z_t$$

with $Z_t \sim \mathcal{N}(0,1)$ are displayed. Here, the strictly stationary approximation at $t_0 = \vartheta T$, $0 \leq \vartheta \leq 1$, takes the form

$$X_t^\vartheta = \sqrt{1/2 + 0.9\vartheta (X_{t-1}^\vartheta)^2}Z_t.$$

In these stationary approximations, the influence of X_{t-1}^ϑ on the variance of X_t^ϑ gradually increases over time. This is reflected in the diagrams associated with extreme quantiles, but is not visible in the median ones.

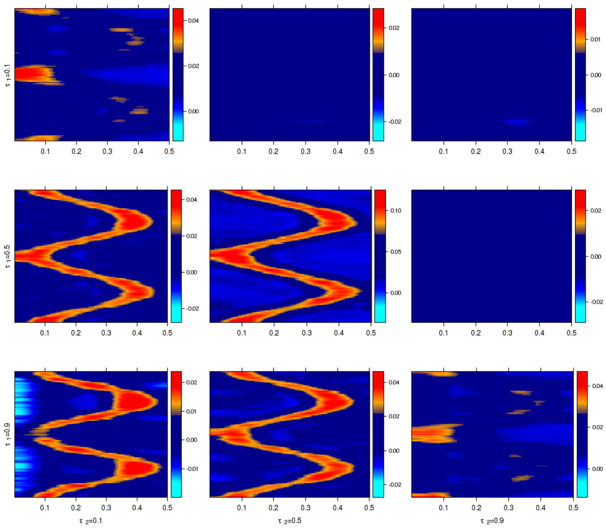
(4) Finally, we show in Figure 8 the heat diagram for the QAR(1) (Quantile Autoregression) model of order one

$$X_{t,T} = [(1.9U_t - 0.95)(t/T) + (-1.9U_t + 0.95)(1 - (t/T))]X_{t-1} + (U_t - 1/2),$$

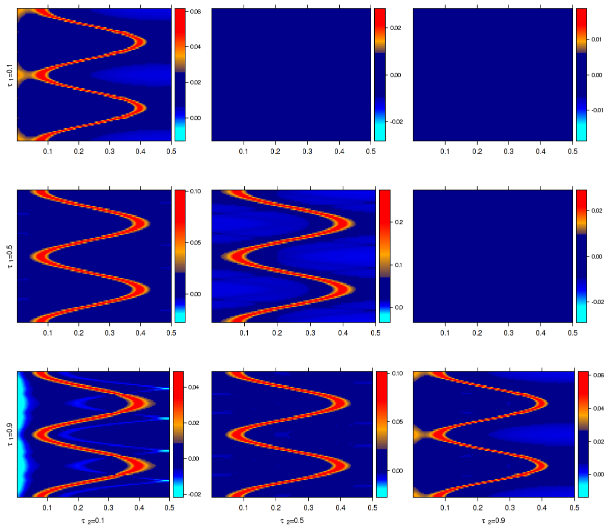
where the U_t's are are i.i.d. uniform over $[0,1]$ (see Koenker and Xiao (2006)). The corresponding strictly stationary approximation at $t_0 = \vartheta T$, $0 \leq \vartheta \leq 1$, is of the form

$$X_t^\vartheta = [(1.9U_t - 0.95)\vartheta + (-1.9U_t + 0.95)(1 - \vartheta)]X_{t-1}^\vartheta + (U_t - 1/2).$$

The gradient of the coefficient function changes slowly from $1.9\,U_t - 0.95$ to $-1.9\,U_t + 0.95$, so that the spectral density of the lower quantiles for small values of t_0/T is the same as as the spectral density for the upper quantiles for $1 - t_0/T$ and vice versa.

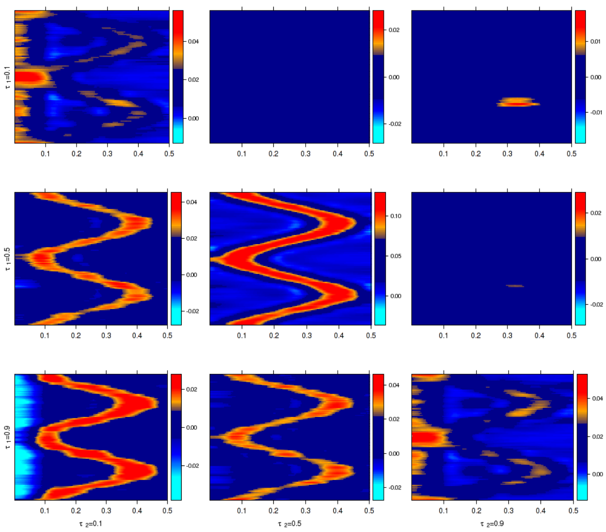


(a) Estimated cross-spectral densities

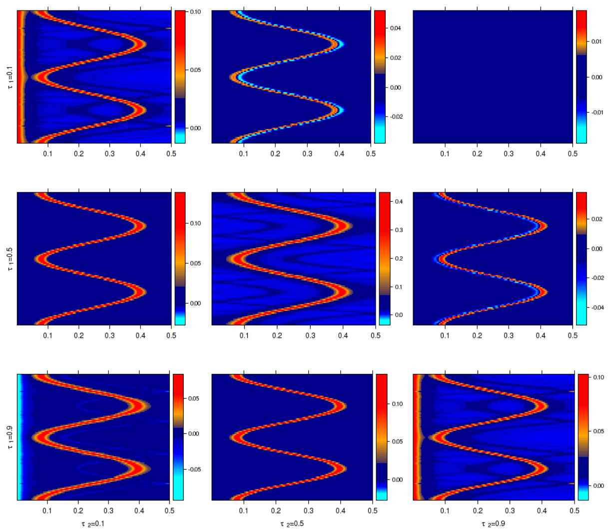


(b) Actual cross-spectral densities (simulated)

Figure 5: The Gaussian locally stationary AR(2) process described in (1).



(a) Estimated cross-spectral densities

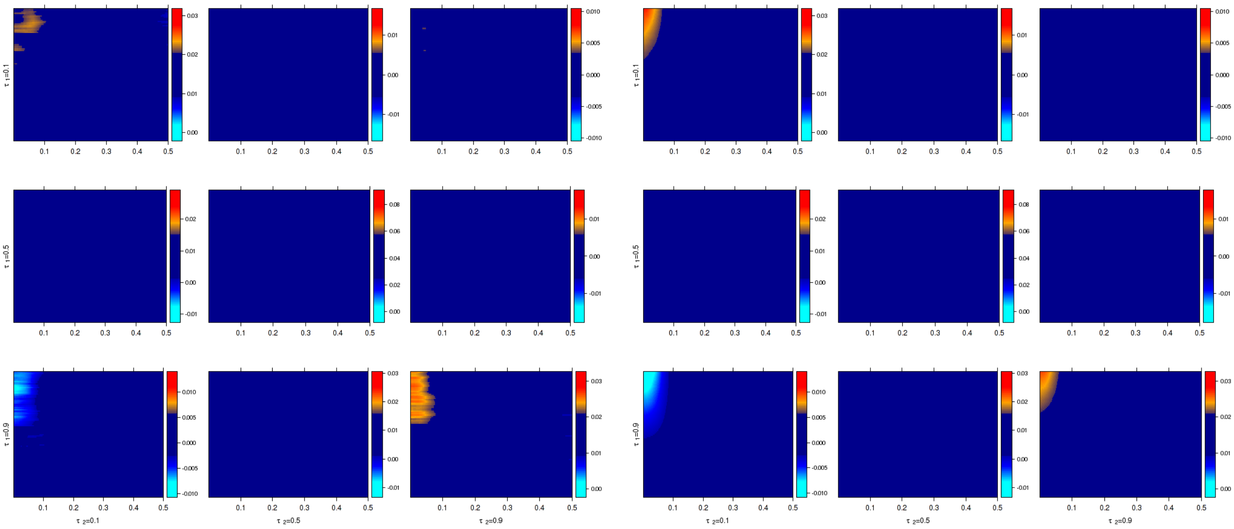


(b) Actual cross-spectral densities (simulated)

Figure 6: The Cauchy-driven locally stationary AR(2) process described in (2).

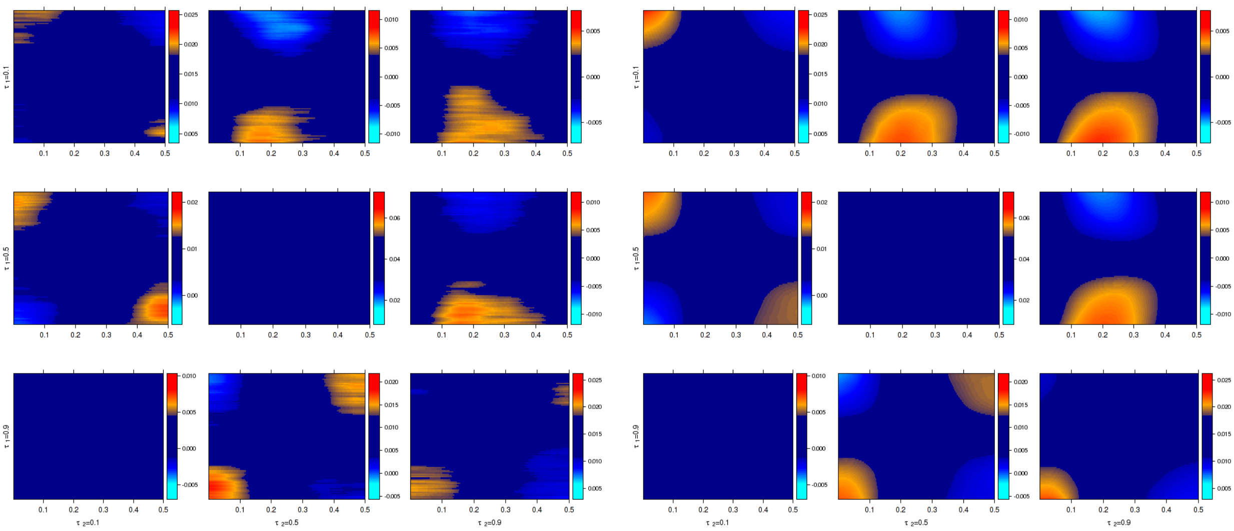
4.3 Standard & Poor's 500

We now turn back to the S&P500 index series already considered in the introduction, with $T = 13092$ daily observations from 1962 through 2014 (differences of the logarithms of daily opening and closing prices for about 52 years). We applied the same estimation method as above: smoothing



(a) Estimated cross-spectral densities (b) Actual cross-spectral densities (simulated)

Figure 7: The locally stationary ARCH(1) process described in (3).



(a) Estimated cross-spectral densities (b) Actual cross-spectral densities (simulated)

Figure 8: The locally stationary QAR(1) process described in (4).

was performed by using the same weights as described in Section 4.2 (see equation (4.1)), with bandwidth $b_n = 0.075$, a window length $n = 1024$ and we considered the sets

$$\mathcal{T}_0 = \{256 + 64j | 0 \le j \le 200\} \quad \text{and} \quad \Omega = \{2\pi j/n | j = 1, ..., (n-2)/2\}.$$

The calibration for the heat plots was performed as described in Section 4.1.

The presence of yellow and red areas in more than 1% of of the t_0 values seriously challenges the general opinion that this series is white noise, yielding perfectly flat periodograms. Deviations from white noise behavior are particularly visible in the diagrams associated with tail quantile levels. Concentrating on the $\tau_1 = \tau_2 = 0.1$ case, closer inspection of the diagram reveals a relation between low-frequency spectral peaks and financial crisis events: in Figure 10, horizontal white lines are identifying the Oil Crisis of 1973, the Black Monday (19.10.1987) which took place during the Savings and Loan Crisis in the USA, bursting of the dot-com bubble in 2001 (followed by the early 2000s recession) and the financial crisis from 2007-2012. Those episodes seem to match the low-frequency peaks quite well, indicating an association between crises and a local, long-memory-like, persistence of low returns.

This apparent relation of low-frequency peaks to crises is confirmed when focusing on the periods of crises. In Figures 11-12, we provide plots of the $\tau_1 = \tau_2 = 0.1$ periodograms before and after two of those four crises, the 2001 bursting of the dot-com bubble and the 2007 financial crisis. More precisely, for each of them, we calculated periodograms using only observations before the critical date, and compared them to periodograms using only observations taken after it. None of the pre-crisis periodograms indicates a significant deviation from white noise, whereas both of the post-crisis ones do. The interpretation is that crises, locally but quite suddenly, produce long-memory-like persistence in low returns. As shown by Figure 10, that persistence eventually fades away—more slowly, though, than it has appeared. The atypical spectra in the late sixties are probably an indication that the market, at that time, was much smaller, and less efficient, than nowadays.

5 Theoretical properties of time-varying copula spectra and local rank-based Laplace periodograms

5.1 A brief comparative discussion of some concepts of local stationarity

In this section, we provide a brief comparison of our concept of local stationarity with some other notions that have been previously discussed in the literature. Lemma 5.1 below shows that, under relatively mild assumptions (which are required for the comparison to make sense), processes that are locally stationary in the sense of Dahlhaus (1996) are also locally strictly stationary in the sense of our definition. More precisely, consider a process with time-varying MA(∞) representation of the form

$$X_{t,T} = \mu(t/T) + \sum_{j=0}^{\infty} a_{t,T}(j)\xi_{t-j}, \tag{5.1}$$

where $\{\xi_t\}$ is i.i.d. white noise. Under assumptions similar to those used by Dahlhaus and Polonik (2006), that process is locally strictly stationary in the sense of Definition 2.1. The following is proved in the online appendix (see Section 7.5).

Lemma 5.1. *If the processes $\{X_{t,T}, t \in \mathbb{Z}\}$, $T \in \mathbb{N}$, admit MA(∞) representations of the form* (5.1) *such that conditions (7.24)-(7.27) (see Appendix 7.5) are satisfied, then the triangular array $\{X_{t,T}\}_{T\in\mathbb{N}}$ is locally strictly stationary in the sense of Definition 2.1.*

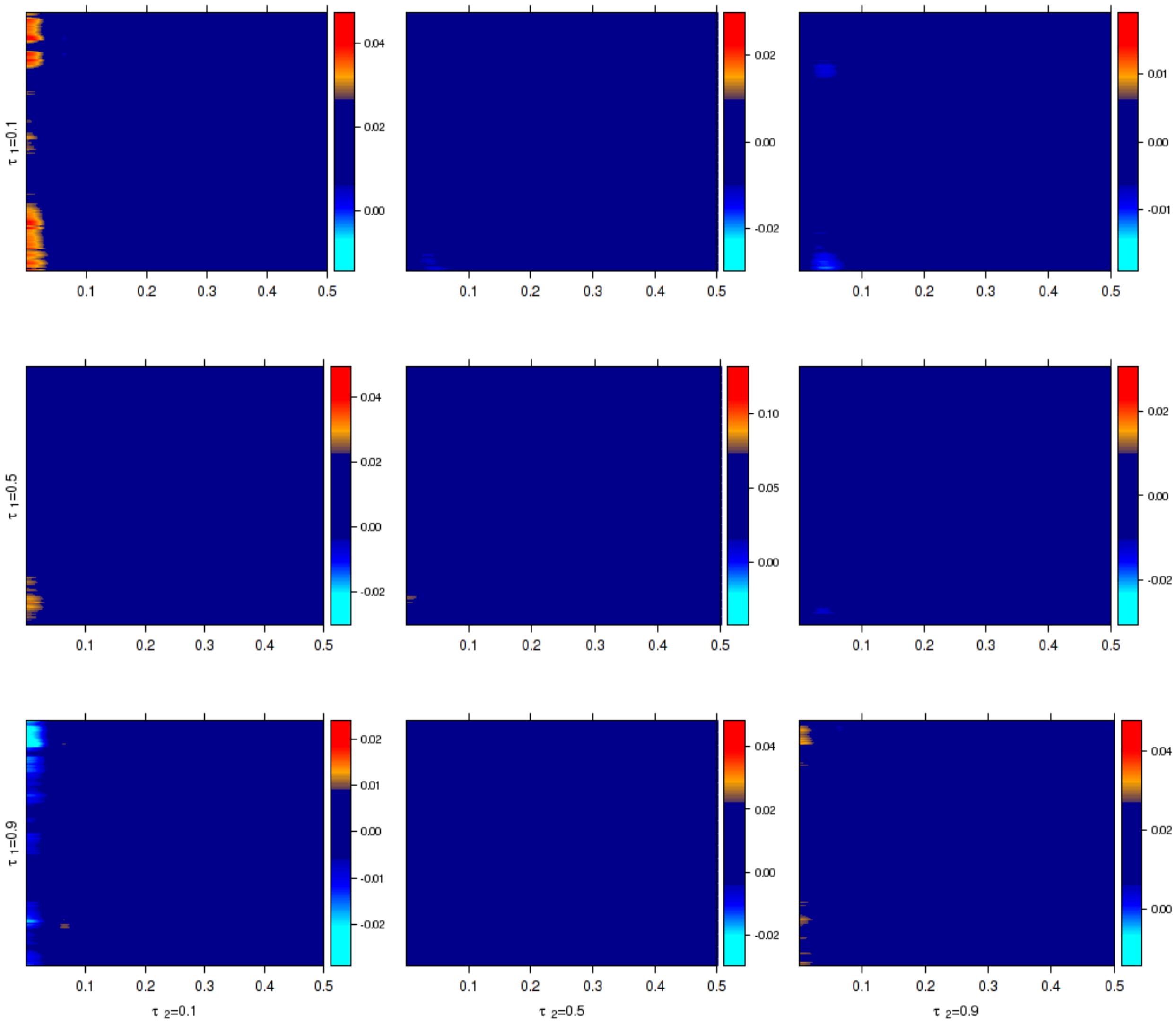


Figure 9: The S&P500 index series from 1962 through 2014. Estimated cross-spectral densities.

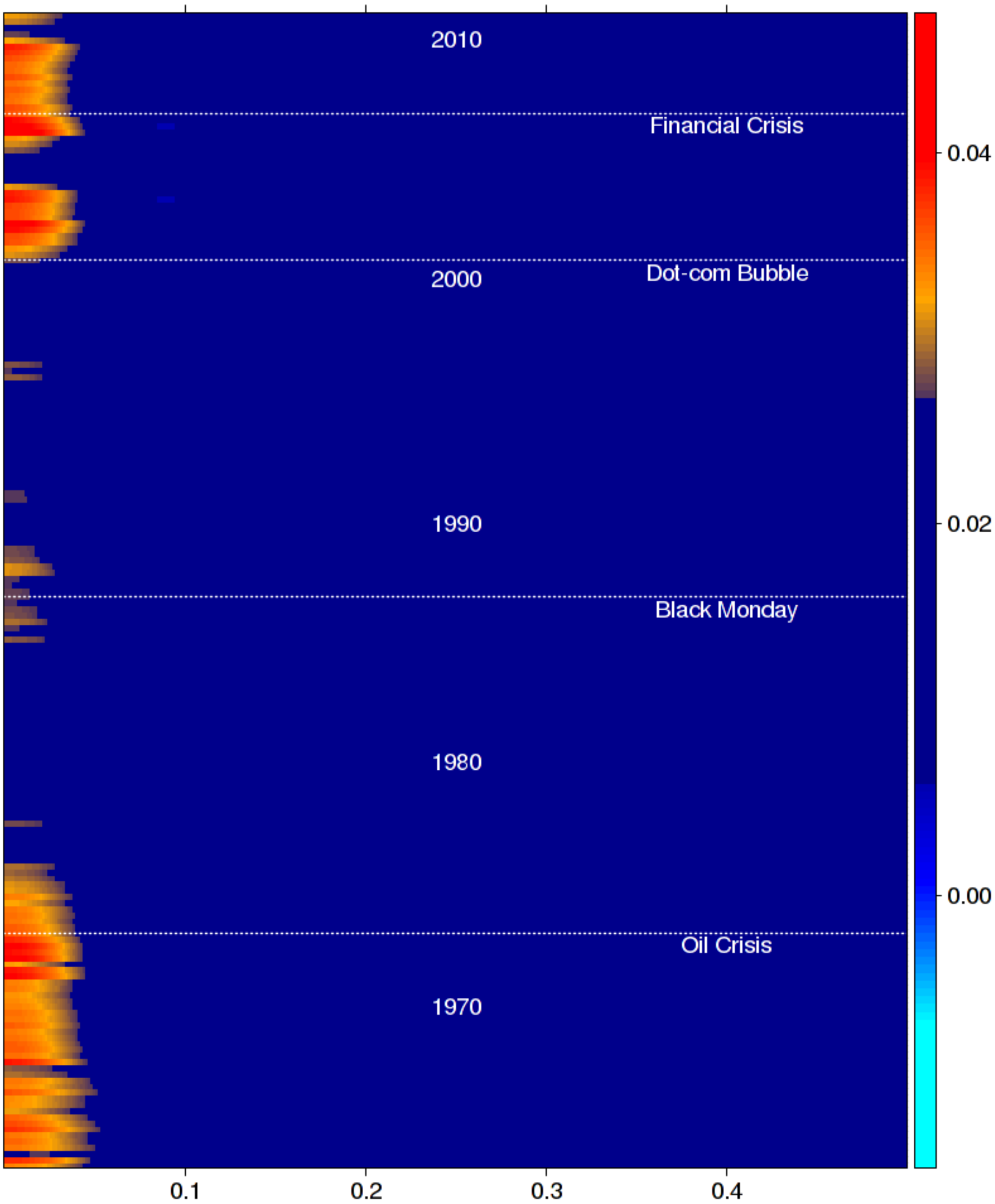


Figure 10: The $\tau_1 = \tau_2 = 0.1$ periodogram of Figure 9; horizontal lines indicate historical financial crises, namely the Oil Crisis of 1973, the Black Monday (19.10.1987) which took place during the Savings and Loan Crisis in the USA, the bursting of the dot-com bubble in 2001 (followed by the early 2000s recession), and the 2007-2012 financial crisis.

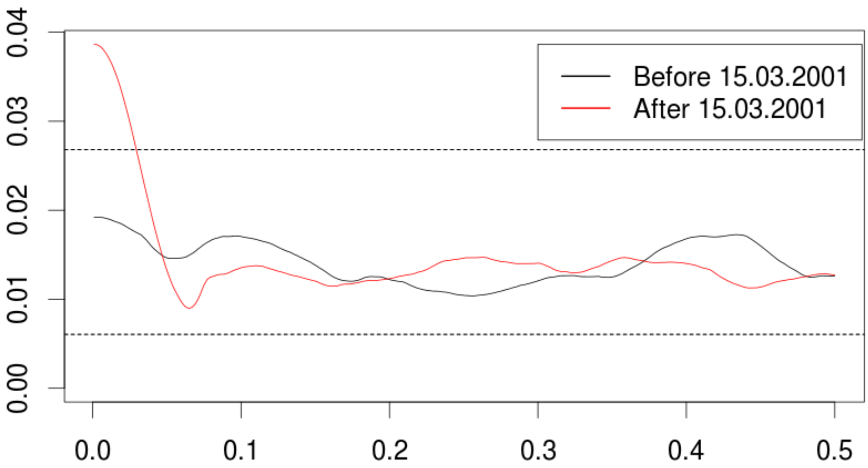


Figure 11: Single periodograms calculated before (blue) and after (red) the bursting of the dot-com bubble in 2001; the dashed horizontal lines represent the values of $q_{\min}$ and $q_{\max}$ from Section 4.1(*iii*); smoothing and bandwidth choices as in Figure 9.

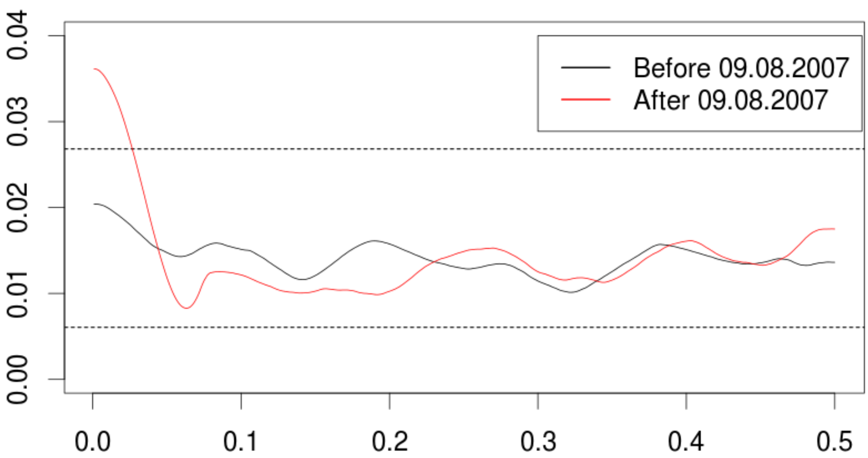


Figure 12: Single periodograms calculated before (blue) and during (red) the Financial Crisis (2007-2012); the dashed horizontal lines represent the values of $q_{\min}$ and $q_{\max}$ from Section 4.1(*iii*); smoothing and bandwidth choices as in Figure 9.

The definition proposed by Vogt (2012) avoids the parametric flavor of Dahlhaus (1996). Rather than bounding a distance between distribution functions (which are nonrandom objects), it places a bound on the difference between $X_{t,T}$ and its stationary approximation. Whether this approach is suitable for defining a sensible local notion of copula spectral density is unclear. Another approach, which can be considered as a generalization of the ideas of Dahlhaus (1996) to nonlinear processes, is developed by Zhou and Wu (2009) and Zhao and Wu (2009), who consider processes of the form $X_{t,T} = G(t/T; \xi_t, \xi_{t-1}, \ldots)$ where $\{\xi_t\}$ is i.i.d. white noise and G some measurable function. This is considerably more general than Dahlhaus (1996); whether it can be used in the context of a local notion of copula spectral density again is unclear.

5.2 Asymptotic theory

Before we proceed with the derivation of the asymptotic properties of the rank-based estimators of Laplace spectral density kernels, we collect here some necessary technical assumptions. First, let us recall the definition of a $\beta-$mixing array. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and let $\mathscr{B}$ and $\mathscr{C}$ be subfields of $\mathscr{A}$. Define

$$\beta(\mathscr{B}, \mathscr{C}) = \mathbb{E} \sup_{C \in \mathscr{C}} |\mathbb{P}(C) - \mathbb{P}(C|\mathscr{B})|$$

and, for an array $\{Z_{t,T} : 1 \leq t \in \mathbb{Z}\}$, $T \in \mathbb{N}$,

$$\beta(k) = \sup_T \sup_{t \in \mathbb{Z}} \beta(\sigma(\{Z_{s,T}, s \leq t\}), \sigma(\{Z_{s,T}, t+k \leq s\})),$$

where $\sigma(\{Z\})$ is the $\sigma-$field generated by the set $\{Z\}$ of random variables. An array is called *$\beta-$mixing* or *uniformly mixing* if $\beta(k) \to 0$ as $k \to \infty$. The following assumptions will be considered in the sequel.

(A1) The triangular array $\{X_{t,T} : 1 \leq t \leq T\}_{T \in \mathbb{N}}$ is $\beta-$mixing with $\beta(k) = o(k^{-\delta})$ where $\delta > 1$ and locally strictly stationary with approximating processes $\{X_t^{\vartheta}\}_{t \in \mathbb{Z}}$.

(A2) For all T, the distribution functions $F_{t;T}$ of $X_{t,T}$ and, for any $1 \leq t_1,\ t_2 \leq T$, the joint distribution functions $F_{t_1,t_2;T}(\cdot,\cdot)$ of $(X_{t_1,T}, X_{t_2,T})$ are twice continuously differentiable, with uniformly bounded derivatives (with respect to t_1, t_2, T and all their arguments). Moreover, there exist constants $d_\tau > 0, f_{\min} > 0$ and $T_0 < \infty$ such that for all $T \geq T_0$

$$\inf_t \inf_{|x - q_{t,T}(\tau)| \leq d_\tau} f_{t,T}(x) \geq f_{\min} > 0,$$

where $f_{t,T}$ and $q_{t,T}(\tau) := F_{t;T}^{-1}(\tau)$ denote the density and τ-quantile corresponding to the distribution function $F_{t;T}$.

(A3) For all ϑ the process $\{X_t^{\vartheta}\}_{t \in \mathbb{Z}}$, with marginal distribution function $G^{\vartheta}(\cdot)$, joint distribution functions $G_h^{\vartheta}(\cdot,\cdot)$, τ-quantiles $q^{\vartheta}(\tau)$, and marginal density $g^{\vartheta}(\cdot)$, satisfies (A2) (with $g_{\min}$ instead of $f_{\min}$).

We now are ready to state our first result which concerns the joint asymptotic distribution of a finite collection of local rank-based Laplace periodograms. Denote by $\mathcal{F}_n := \{2\pi j/n | 1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor\}$ a set of Fourier frequencies.

Theorem 5.1. *Let $\Omega := \{\omega_1, \ldots, \omega_\nu\} \subset (0, \pi)$ denote a $\nu-$tuple of distinct frequencies and let Assumptions (A1)-(A3) be satisfied. Assume that $n \to \infty$, $nT^{-1/2} \to 0$ and $|t_0/T - \vartheta| = o(T^{-1/2})$ as $T \to \infty$. Then*

$$(\hat{L}_{t_0,T}(\omega_1, \tau_1, \tau_2), \ldots, \hat{L}_{t_0,T}(\omega_\nu, \tau_1, \tau_2)) \xrightarrow[T \to \infty]{\mathcal{L}} (L^{\vartheta}(\omega_1, \tau_1, \tau_2), \ldots, L^{\vartheta}(\omega_\nu, \tau_1, \tau_2)), \qquad (5.2)$$

where the random variables $L^\vartheta(\omega,\tau_1,\tau_2)$ *associated with distinct frequencies are mutually independent and*

$$L^\vartheta(\omega,\tau_1,\tau_2) \overset{D}{=} \begin{cases} \pi\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)\chi_2^2 & \text{if} \quad \tau_1=\tau_2 \\ \frac{1}{4}(Z_{11},Z_{12})\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}Z_{21}\\ Z_{22}\end{pmatrix} & \text{if} \quad \tau_1\neq\tau_2, \end{cases}$$

where $(Z_{11},Z_{12},Z_{21},Z_{22})$ *is multivariate normal, with mean* $(0,\ldots,0)$ *and covariance matrix*

$$\mathbf{\Sigma}^\vartheta(\omega) := 4\pi\begin{pmatrix} \mathfrak{f}^\vartheta(\omega,\tau_1,\tau_1) & 0 & \Re(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) & \Im(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) \\ 0 & \mathfrak{f}^\vartheta(\omega,\tau_1,\tau_1) & -\Im(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) & \Re(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) \\ \Re(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) & -\Im(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) & \mathfrak{f}^\vartheta(\omega,\tau_2,\tau_2) & 0 \\ \Im(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) & \Re(\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)) & 0 & \mathfrak{f}^\vartheta(\omega,\tau_2,\tau_2) \end{pmatrix}.$$

To prove consistency for the smoothed versions of the local rank-based Laplace periodograms defined in (3.4), we additionally need the following assumptions.

(A4) As $n\to\infty$, $K_n/n\to 0$ and $\sum_{|k|\le K_n}(W_{t_0,T})^2(k)=o(1)$.

(A5) The functions $\omega\to\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)$ are continuously differentiable for all $\tau_1,\tau_2,\vartheta\in(0,1)$.

(A6) The arrays $\{X_{t,T}|0<t\le T\}$, $T\in\mathbb{N}$ are beta-mixing with rate $\beta(k)=o(k^{-\delta})$ for some $\delta\ge 2$.

Proposition 5.1. *Let (A1)-(A6) hold and assume that* $n\to\infty$, $nT^{-1/2}\to 0$ *and* $|t_0/T-\vartheta| = o(T^{-1/2})$ *as* $T\to\infty$*. Then the estimator* $\hat{f}^\vartheta_{t_0,T}(\omega,\tau_1,\tau_2)$ *defined in* (3.4) *is consistent for the Laplace spectral density* $\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)$*. More precisely, we have*

$$\hat{f}_{t_0,T}(\omega,\tau_1,\tau_2) = 2\pi\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)+o_{\mathbb{P}}(1).$$

Remark 5.1. A direct generalization to the locally stationary context of the ideas from Li (2008) and Li (2012) would be a periodogram of the form

$$\mathring{L}_{t_0,T}(\omega,\tau_1,\tau_2) := \frac{n}{4}\mathring{\boldsymbol{b}}_{t_0,T}(\varphi_n(\omega),\tau_1)'\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\mathring{\boldsymbol{b}}_{t_0,T}(\varphi_n(\omega),\tau_2), \qquad \omega\in(0,\pi),\ \tau_1,\tau_2\in(0,1), \quad (5.3)$$

where

$$(\mathring{a}_{t_0,T}(\omega_{j,n},\tau),\mathring{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau)) := \underset{(a,b)\in\mathbb{R}^3}{\operatorname{argmin}} \sum_{t\in\mathcal{N}_{t_0,T}} \rho_\tau(X_{t,T}-(a,\boldsymbol{b})c_t(\omega_{j,n})).$$

The crucial difference between (5.3) and (3.2) is that the ranks appearing in (3.2) have been replaced, in (5.3), by the original time series values. For this version of the periodograms, results similar to Theorem 5.1 and Proposition 5.1 are established in the online appendix. Informally, the statements of Theorem 5.1 and Proposition 5.1 remain true if all occurrences of $\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)$ are replaced by the weighted versions

$$\mathring{\mathfrak{f}}^\vartheta(\omega,\tau_1,\tau_2) := \frac{\mathfrak{f}^\vartheta(\omega,\tau_1,\tau_2)}{g^\vartheta(q^\vartheta(\tau_1))g^\vartheta(q^\vartheta(\tau_2))}.$$

In a locally stationary setting, the scaling with the marginal densities g^ϑ has the significant disadvantage that a change in the marginal distribution cannot be distinguished from a change in the dependence structure.

5.3 Relation to the Wigner-Ville spectra

In this section, we provide a theoretical justification for considering the time-varying Laplace spectral density kernel by establishing a connection to a classical concept from the analysis of locally stationary time series. In particular, we show that the time-varying Laplace spectral density $\mathfrak{f}^{\vartheta}(\omega,\tau_1,\tau_2)$, as defined in (2.3), is unique and provides a local spectral description of the non-stationary process under study, which justifies the terminology *time-varying Laplace spectral density* of $\{X_{t,T}\}$.

Lemma 5.2. *Let $\{X_{t,T}\}$ be locally strictly stationary with approximating processes $\{X_t^{\vartheta}\}$, and assume that conditions (A1) - (A3) hold. If moreover $\gamma_h^{\vartheta}(\tau_1,\tau_2)$ are absolutely summable for any $\vartheta,\tau_1,\tau_2 \in (0,1)$, then, for any fixed $\vartheta,\tau_1,\tau_2 \in (0,1)$ and any sequence $t_0 = t_0(T)$ such that $t_0/T \to \vartheta$,*

$$\sup_{\omega\in(-\pi,\pi]}\Big|\mathfrak{f}^{\vartheta}(\omega,\tau_1,\tau_2)-\mathfrak{W}_{t_0,T}(\omega,\tau_1,\tau_2)\Big| = o(1).$$

where $\mathfrak{W}_{t_0,T}$ denotes the Wigner-Ville spectrum (defined in (2.4)) of the indicators.

6 Appendix: proofs and technical details

6.1 Proof of Lemma 5.2

It follows from the absolute summability of $\gamma_h^{\vartheta}(\tau_1,\tau_2)$ that

$$\mathfrak{f}^{\vartheta}(\omega,\tau_1,\tau_2) = \frac{1}{2\pi}\sum_{h=-T^{1/4}}^{T^{1/4}}\gamma_h^{\vartheta}(\tau_1,\tau_2)e^{-i\omega h} + o(1),$$

while assumption (A3) yields

$$\mathfrak{W}_{t_0,T}(\omega,\tau_1,\tau_2) = \frac{1}{2\pi}\sum_{h=-T^{1/4}}^{T^{1/4}}\Big(F_{\lfloor t_0-h/2\rfloor,\lfloor t_0+h/2\rfloor;T}(F^{-1}_{\lfloor t_0-h/2\rfloor;T}(\tau_1),F^{-1}_{\lfloor t_0+h/2\rfloor;T}(\tau_2))-\tau_1\tau_2\Big)e^{-i\omega h}+o(1).$$

Hence, up to $o(1)$ quantities, the difference $|\mathfrak{W}_{t_0,T}(\omega,\tau_1,\tau_2)-\mathfrak{f}^{\vartheta}(\omega,\tau_1,\tau_2)|$ is bounded by

$$\begin{aligned}&\frac{1}{2\pi}\sum_{h=-T^{1/4}}^{T^{1/4}}\big|F_{\lfloor t_0-h/2\rfloor,\lfloor t_0+h/2\rfloor;T}(F^{-1}_{\lfloor t_0-h/2\rfloor;T}(\tau_1),F^{-1}_{\lfloor t_0+h/2\rfloor;T}(\tau_2))-G_h^{\vartheta}(q^{\vartheta}(\tau_1),q^{\vartheta}(\tau_2))\big|\\ &\qquad\le\frac{1}{\pi}\sum_{h=-T^{1/4}}^{T^{1/4}}\frac{L}{g_{\min}}\Big|\frac{h}{T}+\frac{1}{T}\Big|,\end{aligned}$$

a quantity which, in view of Equation (7.4) in the online supplement, is $o(1)$ as $T\to\infty$. □

6.2 Proof of Theorem 5.1 and Proposition 5.1

The proofs of both results are based on a uniform linearization of $\hat{\boldsymbol{b}}_{t_0,T}(\omega,\tau)$ which takes the following form

$$\sup_{\omega\in\mathcal{F}_n,\tau\in\mathcal{T}}\Big\|\sqrt{n}\hat{\boldsymbol{b}}_{t_0,T}(\omega,\tau)-2n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}}\begin{pmatrix}\cos(\omega(t-t_{\min}+1))\\ \sin(\omega(t-t_{\min}+1))\end{pmatrix}(\tau-\mathbb{I}_{\{U_{t,T}\le\tau\}})\Big\| = o_{\mathbb{P}}(1), \quad (6.1)$$

where $t_{\min} := \min\{t \in \mathcal{N}_{t_0,T}\}$ and $U_{t,T} := F_{t;T}(X_{t,T})$. In what follows, we briefly sketch the main arguments which are needed to establish (6.1), while most technical details are deferred to the online supplement. The proofs of Theorem 5.1 and Proposition 5.1 are provided in Sections 6.2.1 and 6.2.2, respectively.

Let $\hat{F}_{t_0,T}(x)$ denote the empirical distribution function of $\{X_{t,T}|t \in \mathcal{N}_{t_0,T}\}$, namely,

$$\hat{F}_{t_0,T}(x) = n^{-1} \sum_{t\in\mathcal{N}_{t_0,T}} \mathbb{I}_{\{X_{t,T}\le x\}},$$

and introduce the functions

$$\hat{Z}^R_{t_0,T}(\boldsymbol{\delta}) := \sum_{t\in\mathcal{N}_{t_0,T}} \rho_\tau(\hat{F}_{t_0,T}(X_{t,T}) - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(\hat{F}_{t_0,T}(X_{t,T}) - \tau),$$

$$\hat{Z}^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau) := \sum_{t\in\mathcal{N}_{t_0,T}} \rho_\tau(U_{t,T} - \tau - n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}) - \rho_\tau(U_{t,T} - \tau) - \delta_1\sqrt{n}\Big(n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}} F_{t,T}(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau\Big),$$

and

$$\hat{Z}^{\vartheta,U}_{t_0,T}(\boldsymbol{\delta},\omega,\tau) := -\boldsymbol{\delta}'(\boldsymbol{\zeta}^U_{t_0,T}(\omega,\tau) + \boldsymbol{e}_1\sqrt{n}(G^\vartheta(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau)) + \frac{1}{2}\boldsymbol{\delta}'\boldsymbol{Q}^U(\omega)\boldsymbol{\delta},$$

where $\boldsymbol{e}_1 := (1,0,0)'$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\delta_3)$,

$$\boldsymbol{Q}^U(\omega) := \frac{1}{n}\sum_{t=1}^n \mathbf{c}_t(\omega)\mathbf{c}_t'(\omega) = \begin{pmatrix} 1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2}\end{pmatrix}, \quad \text{and} \quad \boldsymbol{\zeta}^U_{t_0,T}(\omega,\tau) := n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}} \mathbf{c}_t(\omega)(\tau - \mathbb{I}_{\{U_{t,T}\le\tau\}}).$$

Furthermore, set

$$\boldsymbol{\delta}_{t_0,T}(\omega,\tau) := \operatorname*{argmin}_{\boldsymbol{\delta}\in\mathbb{R}^3} \hat{Z}^R_{t_0,T}(\boldsymbol{\delta}) \qquad \text{and} \qquad \boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau) := \operatorname*{argmin}_{\boldsymbol{\delta}\in\mathbb{R}^3} \hat{Z}^{\vartheta,U}_{t_0,T}(\boldsymbol{\delta},\omega,\tau).$$

Observe that the last two components of $\boldsymbol{\delta}_{t_0,T}(\omega,\tau)$ coincide with the components of $\hat{\boldsymbol{b}}_{t_0,T}(\omega,\tau)$, admitting the representation

$$\boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau) = (\boldsymbol{Q}^U(\omega))^{-1}(\boldsymbol{\zeta}^U_{t_0,T}(\omega,\tau) + \boldsymbol{e}_1\sqrt{n}(G^\vartheta(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau)).$$

Therefore, it suffices to show that $\|\boldsymbol{\delta}_{t_0,T}(\omega,\tau) - \boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau)\|_\infty$ is uniformly small in probability. To prove this, we need a couple of intermediate results which are established in the online supplement. More precisely, we show (Section 7.3.1) that there exists a constant $A > 0$ with

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{\omega\in\mathcal{F}_n} \|\boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau)\|_\infty > \frac{A}{2}\log n\Big) = 0 \tag{6.2}$$

and that for this constant A we have (Section 7.3.2)

$$\sup_{\omega\in\mathcal{F}_n}\ \sup_{\|\boldsymbol{\delta}\|_\infty < A\log n} |\hat{Z}^R_{t_0,T}(\boldsymbol{\delta}) - \hat{Z}^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau)| = O_{\mathbb{P}}(n^{-\frac{1}{4}\frac{\delta-1}{\delta+1}}(\log n)^3) \tag{6.3}$$

and (Section 7.3.3)

$$\sup_{\omega\in\mathcal{F}_n}\ \sup_{\|\boldsymbol{\delta}\|_\infty < A\log n} |\hat{Z}^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau) - \hat{Z}^{\vartheta,U}_{t_0,T}(\boldsymbol{\delta},\omega,\tau)| = O_{\mathbb{P}}(n^{-\frac{1}{4}\frac{\delta-1}{\delta+1}}(\log n)^3), \tag{6.4}$$

where $\delta > 1$ is the constant from assumption (A1). Combining (6.2)-(6.4), we find that

$$\sup_{\omega\in\mathcal{F}_n}\ \sup_{\|\boldsymbol{\delta}-\boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau)\|_\infty < \epsilon} |\hat{Z}^R_{t_0,T}(\boldsymbol{\delta}) - \hat{Z}^{\vartheta,U}_{t_0,T}(\boldsymbol{\delta},\omega,\tau)| = O_{\mathbb{P}}(n^{-\frac{1}{4}\frac{\delta-1}{\delta+1}}(\log n)^3). \tag{6.5}$$

Finally, similar arguments as those in the proof of Lemma 6.1 in Dette et al. (2014) yield

$$\sup_{\omega\in\mathcal{F}_n} \|\boldsymbol{\delta}_{t_0,T}(\omega,\tau) - \boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau)\|_\infty = O_{\mathbb{P}}(n^{-\frac{1}{8}\frac{\delta-1}{\delta+1}}(\log n)^{3/2}) = o_{\mathbb{P}}(1),$$

which establishes the desired result (6.1). □

6.2.1 Proof of Theorem 5.1

The result clearly follows if we can show that

$$\sqrt{n}\left(\hat{\boldsymbol{b}}_{t_0,T}(\varphi_n(\omega),\tau)\right)_{\tau\in\mathcal{T},\,\omega\in\Omega} \xrightarrow[T\to\infty]{\mathcal{L}} (\mathbf{N}^{\vartheta}(\omega,\tau))_{\tau\in\mathcal{T},\,\omega\in\Omega},$$

where the $\mathbf{N}^{\vartheta}(\omega,\tau)$'s are Gaussian random vectors with mean $\mathbf{0}$ and covariances

$$\mathrm{Cov}(\mathbf{N}^{\vartheta}(\omega_1,\tau_{k_1}),\mathbf{N}^{\vartheta}(\omega_2,\tau_{k_2})) = \mathbf{M}^{\vartheta}(\tau_{k_1},\tau_{k_2},\omega_1,\omega_2),$$

where

$$\mathbf{M}^{\vartheta}(\tau_{k_1},\tau_{k_2},\omega_1,\omega_2) := \begin{cases} 4\pi\begin{pmatrix} \Re(\mathfrak{f}^{\vartheta}(\omega,\tau_{k_1},\tau_{k_2})) & \Im(\mathfrak{f}^{\vartheta}(\omega,\tau_{k_1},\tau_{k_2})) \\ -\Im(\mathfrak{f}^{\vartheta}(\omega,\tau_{k_1},\tau_{k_2})) & \Re(\mathfrak{f}^{\vartheta}(\omega,\tau_{k_1},\tau_{k_2})) \end{pmatrix} & \text{if} \quad \omega_1=\omega_2=:\omega \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if} \quad \omega_1\neq\omega_2. \end{cases} \tag{6.6}$$

By (6.1), it is sufficient to prove the weak convergence

$$\left(2n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}}\begin{pmatrix}\cos(\varphi_n(\omega)(t-t_{\min}+1)) \\ \sin(\varphi_n(\omega)(t-t_{\min}+1))\end{pmatrix}(\tau-\mathbb{I}_{\{U_{t,T}\le\tau\}})\right)_{\tau\in\mathcal{T},\,\omega\in\Omega} \xrightarrow[T\to\infty]{\mathcal{L}} (\mathbf{N}^{\vartheta}(\omega,\tau))_{\tau\in\mathcal{T},\,\omega\in\Omega}. \tag{6.7}$$

The latter follows from a routine application of the application of the Cramér-Wold device. Define arbitrary coefficients $\boldsymbol{\lambda}_{ik}\in\mathbb{R}^2$, $i=1,\dots,v$, $k=1,\dots,p$, and let

$$\tilde{\mathbf{c}}_t(\omega) := (\cos(\varphi_n(\omega)(t-t_{\min}+1)),\sin(\varphi_n(\omega)(t-t_{\min}+1)))' \quad \text{with} \quad t_{\min} := \min\{t\in\mathcal{N}_{t_0,T}\}.$$

We need to show that

$$2\sum_{k=1}^{p}\sum_{i=1}^{\nu}\sum_{u\in\mathcal{N}_{t_0,T}}\boldsymbol{\lambda}'_{ik}\frac{\tilde{\mathbf{c}}_u(\omega_i)}{\sqrt{n}}\big(\tau_k-\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau_k)\}}\big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0,\mathrm{Var}\Big[\sum_{k=1}^{p}\sum_{i=1}^{\nu}\boldsymbol{\lambda}'_{ik}\mathbf{N}^{\vartheta}(\omega_i,\tau_k)\Big]\Big) \tag{6.8}$$

where the $\mathbf{N}^{\vartheta}(\omega_i,\tau_k)$'s are centered normal random variables with covariances $\mathrm{Cov}(\mathbf{N}^{\vartheta}(\omega_i,\tau_k),\mathbf{N}^{\vartheta}(\omega_j,\tau_l))$ of the form (6.6). To prove this claim, consider the covariances

$$\begin{aligned}
&\mathrm{Cov}\Big(\sum_{u\in\mathcal{N}_{t_0,T}}\frac{\tilde{\mathbf{c}}_u(\omega_i)}{\sqrt{n}}(\tau_k-\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau_k)\}}),\sum_{u\in\mathcal{N}_{t_0,T}}\frac{\tilde{\mathbf{c}}_u(\omega_j)}{\sqrt{n}}(\tau_l-\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau_l)\}})\Big) \\
&\quad=\frac{1}{n}\sum_{u\in\mathcal{N}_{t_0,T}}\sum_{v\in\mathcal{N}_{t_0,T}}\tilde{\mathbf{c}}_u(\omega_i)\tilde{\mathbf{c}}'_v(\omega_j)\mathrm{Cov}\big(\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau_k)\}},\mathbb{I}_{\{X_{v,n}\le q_{v,T}(\tau_l)\}}\big) \\
&\quad=\frac{1}{n}\sum_{u\in\mathcal{N}_{t_0,T}}\sum_{v\in\mathcal{N}_{t_0,T}}\tilde{\mathbf{c}}_u(\omega_i)\tilde{\mathbf{c}}'_v(\omega_j)\mathrm{Cov}\big(\mathbb{I}_{\{X_u^{\vartheta}\le q^{\vartheta}(\tau_k)\}},\mathbb{I}_{\{X_v^{\vartheta}\le q^{\vartheta}(\tau_l)\}}\big)+o(1),
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
&\sup_{u\in\mathcal{N}_{t_0,T}}\Big|\sum_{v\in\mathcal{N}_{t_0,T}}\mathrm{Cov}\big(\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau_k)\}},\mathbb{I}_{\{X_{v,n}\le q_{v,T}(\tau_l)\}}\big)-\mathrm{Cov}\big(\mathbb{I}_{\{X_u^{\vartheta}\le q^{\vartheta}(\tau_k)\}},\mathbb{I}_{\{X_v^{\vartheta}\le q^{\vartheta}(\tau_l)\}}\big)\Big| \\
&\quad\le\sup_{u\in\mathcal{N}_{t_0,T}}\sum_{v\in\mathcal{N}_{t_0,T}}|F_{u,v;T}(q_{u,T}(\tau_k),q_{v,T}(\tau_l))-G^{\vartheta}_{u-v}(q^{\vartheta}_u(\tau_k),q^{\vartheta}_v(\tau_l))| \xrightarrow[T\to\infty]{} 0,
\end{aligned}$$

itself a consequence of equation (7.4). Along the same lines as in the proof of Theorem 2 in Li (2008), we obtain

$$\lim_{T\to\infty}\frac{4}{n}\sum_{u\in\mathcal{N}_{t_0,T}}\sum_{v\in\mathcal{N}_{t_0,T}}\tilde{\mathbf{c}}_u(\omega_i)\tilde{\mathbf{c}}_v'(\omega_j)\mathrm{Cov}\big(\mathbb{I}_{\{X_u^\vartheta\le q^\vartheta(\tau_k)\}},\mathbb{I}_{\{X_v^\vartheta\le q^\vartheta(\tau_l)\}}\big)=\mathbf{M}^\vartheta(\tau_k,\tau_l,\omega_i,\omega_j).$$

Hence, we have

$$\mathrm{Var}\Big(2\sum_{k=1}^{p}\sum_{i=1}^{\nu}\sum_{u\in\mathcal{N}_{t_0,T}}\boldsymbol{\lambda}_{ik}'\frac{\tilde{\mathbf{c}}_u(\omega)}{\sqrt{n}}(\tau_k-\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau_k)\}})\Big)\xrightarrow[T\to\infty]{}\mathrm{Var}\Big(\sum_{k=1}^{p}\sum_{i=1}^{\nu}\boldsymbol{\lambda}_{ik}'\mathbf{N}^\vartheta(\omega_i,\tau_k)\Big).$$

To conclude, we apply a central limit theorem from Francq and Zakoïan (2005) with $\kappa=0$, $T_n=0$, $r*=(\delta-1)/(2+4\delta)$ and $v*=3/(\delta-1)$, and obtain (6.8). The claim follows. □

6.2.2 Proof of Proposition 5.1

First define

$$\mathcal{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2):=\frac{1}{n}d_{t_0,T}^\vartheta(-\omega_{j,n},\tau_1)d_{t_0,T}^\vartheta(-\omega_{j,n},\tau_2)\tag{6.9}$$

with $d_{t_0,T}^\vartheta(\omega_{j,n},\tau):=\frac{n}{2}(1,i)\boldsymbol{b}_{t_0,T}^\vartheta(\omega_{j,n},\tau)$ and

$$\boldsymbol{b}_{t_0,T}^\vartheta(\omega_{j,n},\tau):=2n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}}\begin{pmatrix}\cos(\omega_{j,n}\tilde{t})\\ \sin(\omega_{j,n}\tilde{t})\end{pmatrix}(\tau-\mathbb{I}_{\{X_{t,T}\le q_{t,T}(\tau)\}}),$$

where $\tilde{t}=t-t_{\min}+1$, $t_{\min}=\min\{t\in\mathcal{N}_{t_0,T}\}$. In Section 7.3.4, we show that, uniformly in ω,

$$\hat{L}_{t_0,T}(\varphi_n(\omega),\tau_1,\tau_2)=\mathcal{L}_{t_0,T}(\varphi_n(\omega),\tau_1,\tau_2)+o_{\mathbb{P}}(1).\tag{6.10}$$

Together with $\sum_{|k|\le K_n}W_{t_0,T}(k)=1$ and $W_{t_0,T}(k)\ge0$, this implies that we can write the estimator (3.4) as (assuming that $\varphi_n(\omega)=\omega_{j_n,n}$)

$$\hat{f}_{t_0,T}(\omega,\tau_1,\tau_2)=\sum_{|k|\le K_n}W_{t_0,T}(k)\mathcal{L}_{t_0,T}(\omega_{j_n+k,n},\tau_1,\tau_2)+o_{\mathbb{P}}(1).\tag{6.11}$$

In Section 7.3.5 we show that, for any deterministic sequence j_n in $\{1,...,n-1\}$,

$$\sum_{|k|\le K_n}W_{t_0,T}(k)\Big[\mathcal{L}_{t_0,T}(\omega_{j+k,n},\tau_1,\tau_2)-2\pi\mathfrak{f}^\vartheta(\omega_{j+k,n},\tau_1,\tau_2)\Big]=o_{\mathbb{P}}(1).\tag{6.12}$$

Now, for any $\omega\in(0,\pi)$, observe that the point $\omega_{j_n,n}:=\varphi_n(\omega)$ is such that $|\omega_{j_n,n}-\omega|=O(K_n/n)$, and that, for $\mathfrak{f}=Re(\mathring{\mathfrak{f}}^\vartheta)$ and $\mathfrak{f}=Im(\mathring{\mathfrak{f}}^\vartheta)$,

$$\begin{aligned}\Big|\sum_{|k|\le K_n}W_{t_0,T}(k)(\mathfrak{f}(\omega_{j_n+k,n})-\mathfrak{f}(\omega))\Big| &\le \sum_{|k|\le K_n}W_{t_0,T}(k)|\mathfrak{f}'(\xi_{j_n+k,n})||\omega_{j_n+k,n}-\omega|\\ &\le C_n\sum_{|k|\le K_n}W_{t_0,T}(k)|2\pi k/n+\omega_{j_n,n}-\omega|\\ &\le C_n\sum_{|k|\le K_n}W_{t_0,T}(k)|2\pi k/n|+C_n\sum_{|k|\le K_n}W_{t_0,T}(k)|\omega_{j_n,n}-\omega|\\ &\le C_n(2\pi K_n/n+|\omega_{j_n,n}-\omega|)\sum_{|k|\le K_n}W_{t_0,T}(k)=O(K_n/n),\end{aligned}$$

where $|\xi_{j_n+k,n} - \omega| \leq |\omega_{j_n+k,n} - \omega|$ and $C_n := \sup_{\xi \in \Xi} |\mathfrak{f}(\xi)|$ is the supremum of the first derivative of $\mathfrak{f}$ in the interval $\Xi := \big[\omega - |\omega - \omega_{j_n+k,n}| - \omega_{K_n,n},\, \omega + |\omega - \omega_{j_n+k,n}| + \omega_{K_n,n}\big]$. Note that C_n is a bounded sequence since $|\omega - \omega_{j_n+k,n}| \pm \omega_{K_n,n} \to 0$ and, by assumption (A5), $C_n \to |\mathfrak{f}'(\omega)|$.

This implies that

$$\Big| \sum_{|k| \leq K_n} W_{t_0,T}(k)(\mathfrak{f}^{\vartheta}(\omega_{j_n+k,n}, \tau_1, \tau_2) - \mathfrak{f}^{\vartheta}(\omega, \tau_1, \tau_2))\Big| = O(K_n/n)$$

which, together with (6.11) and (6.12), completes the proof. □

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7 Online Appendix

This online appendix collects the technical results for the main body of the paper. Sections 7.1 and 7.2 contain basic probabilistic and technical details that are used throughout the proofs. Sections 7.3 and 7.4 provide the missing steps in the proofs of Theorem 5.1, Proposition 5.1, and additional statements for Remark 5.1, respectively. Section 7.5 contains a proof of Lemma 5.1.

7.1 Probabilistic details

7.1.1 A lemma on cumulants

Let $(X_t)_{t\in\mathbb{Z}}$ be an arbitrary real-valued stochastic process; for all $p \in \mathbb{N}$, $(t_1, \dots, t_p) \in \mathbb{Z}^p$, and all p-tuple $A_1, \dots, A_p$ of Borel sets, consider the *cumulant*

$$
\begin{aligned}
&|\mathrm{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \dots, \mathbb{I}\{X_{t_p} \in A_p\})| \\
&\quad := \Big| \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)!\mathbb{P}\Big(\bigcap_{i\in\nu_1}\{X_{t_i} \in A_i\}\Big)\cdots\mathbb{P}\Big(\bigcap_{i\in\nu_R}\{X_{t_i} \in A_i\}\Big)\Big|, \qquad (7.1)
\end{aligned}
$$

where the sum $\sum_{\{\nu_1,\dots,\nu_R\}}$ runs over all partitions $\{\nu_1, \dots, \nu_R\}$ of the set $\{1, \dots, p\}$ (see Brillinger (1975) p.19). Define

$$
\alpha(n) := \sup_{t\in\mathbb{Z}} \sup_{A\in\sigma(\dots,X_{t-1},X_t),\, B\in\sigma(X_{t+n},X_{t+n+1},\dots)} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.
$$

Lemma 7.1. *There exists a constant K_p depending on p only such that, for any $(t_1, ..., t_p) \in \mathbb{Z}^p$ and any p-tuple $A_1, \dots, A_p$ of Borel sets,*

$$
\Big|\mathrm{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \dots, \mathbb{I}\{X_{t_p} \in A_p\})\Big| \le K_p\alpha\Big(\big\lfloor p^{-1}\max_{i,j}|t_i - t_j|\big\rfloor\Big).
$$

Proof The lemma trivially holds for $t_1 = ... = t_p$. If at least two t_i's in $(t_1, \dots, t_p)$ are distinct, choose j such that $\max_{i=1,\dots,p-1}(t_{i+1}-t_i) = t_{j+1}-t_j > 0$ and let $(Y_{t_{j+1}}, \dots, Y_{t_p})$ be a random vector that is independent of $(X_{t_1}, \dots, X_{t_j})$ and possesses the same joint distribution as $(X_{t_{j+1}}, \dots, X_{t_p})$. By an elementary property of the cumulants (cf. Theorem 2.3.1 (iii) in Brillinger (1975)), we have

$$
\mathrm{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \dots, \mathbb{I}\{X_{t_j} \in A_j\}, \mathbb{I}\{Y_{t_{j+1}} \in A_{j+1}\}, \dots, \mathbb{I}\{Y_{t_p} \in A_p\}) = 0.
$$

Therefore,

$$
\begin{aligned}
&\Big|\mathrm{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \dots, \mathbb{I}\{X_{t_p} \in A_p\}) \\
&\quad - \mathrm{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \dots, \mathbb{I}\{X_{t_j} \in A_j\}, \mathbb{I}\{Y_{t_{j+1}} \in A_{j+1}\}, \dots, \mathbb{I}\{Y_{t_p} \in A_p\})\Big| \\
&= \Big| \sum_{\{\nu_1,\dots,\nu_R\}} (-1)^{R-1}(R-1)![P_{\nu_1}\cdots P_{\nu_R} - Q_{\nu_1}\cdots Q_{\nu_R}]\Big|,
\end{aligned}
$$

where

$$
P_{\nu_r} := \mathbb{P}\Big(\bigcap_{i\in\nu_r}\{X_{t_i} \in A_i\}\Big) \quad \text{and} \quad Q_{\nu_r} := \mathbb{P}\Big(\bigcap_{\substack{i\in\nu_r\\ i\le j}}\{X_{t_i} \in A_i\}\Big)\mathbb{P}\Big(\bigcap_{\substack{i\in\nu_r\\ i>j}}\{X_{t_i} \in A_i\}\Big),
$$

$r = 1, \dots, R$, with $\mathbb{P}(\bigcap_{i\in\emptyset}\{X_{t_i} \in A_i\}) := 1$ by convention. By the definition of $\alpha(n)$, it follows that, for any partition $\{\nu_1, ..., \nu_R\}$ and any $r = 1, ..., R$, we have $|P_{\nu_r} - Q_{\nu_r}| \le \alpha(t_{j+1} - t_j)$. Thus, for every partition $\{\nu_1, ..., \nu_R\}$,

$$
|P_{\nu_1}\cdots P_{\nu_R} - Q_{\nu_1}\cdots Q_{\nu_R}| \le \sum_{r=1}^{R} |P_{\nu_r} - Q_{\nu_r}| \le R\alpha(t_{j+1} - t_j).
$$

All together, this yields

$$\left|\text{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \ldots, \mathbb{I}\{X_{t_p} \in A_p\})\right| \leq \alpha(t_{j+1} - t_j) \sum_{\{\nu_1,\ldots,\nu_R\}} R!\ .$$

Noting that $p(t_{j+1} - t_j) \geq \max_{i_1,i_2} |t_{i_1} - t_{i_2}|$ and observing that $\alpha(n)$ is monotone in n, we obtain

$$\left|\text{cum}(\mathbb{I}\{X_{t_1} \in A_1\}, \ldots, \mathbb{I}\{X_{t_p} \in A_p\})\right| \leq K_p \alpha(\max |t_i - t_j|).$$

□

7.1.2 A blocking technique for nonstationary $\beta-$mixing processes

In her paper, Yu (1994) constructs an *independent block sequence* to transfer classical tools used in the analysis of i.i.d. data to the case of $\beta-$mixing stationary time series. We are applying her technique here to derive an exponential inequality for sums of $\beta-$mixing local stationary variables, which will be used on multiple occasions in the sequel. For this purpose, let $\{X_{t,n}\}$ be a $\beta-$mixing triangular array with mixing coefficient β_n. For each fixed n, divide the process $X_{t,n}$ into $2\mu_n$ blocks of length $a_n = \lfloor n/2\mu_n \rfloor$, with a remainder block of length $n - 2\mu_n a_n$. Define

$$\begin{aligned}
\Gamma_j &= \{i : 2(j-1)a_n + 1 \leq i \leq (2j-1)a_n\}, \\
\Delta_j &= \{i : (2j-1)a_n + 1 \leq i \leq (2j)a_n\}, \\
R &= \{i : 2\mu_n a_n + 1 \leq i \leq n\},
\end{aligned}$$

and introduce the notation

$$X(\Gamma_j) = \{X_{i,n}, i \in \Gamma_j\}, \quad X(\Delta_j) = \{X_{i,n}, i \in \Delta_j\}, \quad X(R) = \{X_{i,n}, i \in R\},$$

where the dependence on n is omitted for the sake of brevity. We now have a sequence of alternating Γ and Δ blocks

$$X = X(\Gamma_1), X(\Delta_1), X(\Gamma_2), \ldots, X(\Gamma_{\mu_n}), X(\Delta_{\mu_n}), X(R).$$

To use the concept of *coupling* we take a one-dependent block sequence

$$Y = Y(\Gamma_1), Y(\Delta_1), Y(\Gamma_2), \ldots, Y(\Gamma_{\mu_n}), Y(\Delta_{\mu_n}),$$

where $Y(\Gamma_j) = \{\xi_i : i \in \Gamma_j\}$ and $Y(\Delta_j) = \{Y_i : i \in \Delta_j\}$ such that the sequence is independent of X and each block of Y has the same distribution as a block in X. That is,

$$Y(\Gamma_i) \overset{D}{=} X(\Gamma_i) \quad \text{and} \quad Y(\Delta_i) \overset{D}{=} X(\Delta_i).$$

The existence of such a sequence and the measurability issues that arise are addressed in Yu (1994). The block sequences that belong to the Γ blocks are denoted by X_Γ and Y_Γ and those belonging to the Δ blocks are denoted by X_Δ and Y_Δ, e.g

$$X_\Gamma = X(\Gamma_1), X(\Gamma_2), \ldots, X(\Gamma_{\mu_n}).$$

We obtain X_Γ by leaving out every other block in the original sequence, which is $\beta-$mixing, so that the dependence between the blocks in X_Γ becomes weaker as block sizes increase. Denote by Q and $\tilde{Q}$ the distributions of X_Γ and Y_Γ, respectively. The following Lemma from Yu (1994) establishes an upper bound for the difference between expectations computed from the Γ block sequences from the original and the independent block sequences, respectively.

Lemma 7.2. *For any measurable function h on $\mathbb{R}^{\mu_n a_n}$ with $\|h\|_\infty \leq M$,*

$$\big|\mathbb{E}_Q[h(X_\Gamma)] - \mathbb{E}_{\tilde{Q}}[h(Y_\Gamma)]\big| \leq M(\mu_n - 1)\beta_{a_n}.$$

The same bound is valid for X_Δ and Y_Δ. We can now consider a sum of $\beta-$mixing random variables, namely $\sum_{t=1}^n f(X_{t,n})$, and link its probabilistic behavior to that of the sum of independent blocks $\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n})$ where f is a function contained in some appropriate class F_n of functions from $\mathbb{R}$ to $\mathbb{R}$, which will be specified later on. For simplicity, we assume that $\mathbb{E}(f(X_{i,n})) = 0$ for all $f \in F_n$. The following Lemma is a slight adjustment of Lemma 4.2 from Yu (1994).

Lemma 7.3. *Let F_n be a sequence of permissible classes of functions bounded by a constant M_n. Let $(r_n)_{n\in\mathbb{N}}$ be such that, for n large enough, $2r_n\mu_n \geq nM_n$. Then*

$$\mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{t=1}^n f(X_{t,n})\Big| > 4r_n\Big) \leq \mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n})\Big| > r_n\Big) + \mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Delta_j} f(Y_{i,n})\Big| > r_n\Big) + 2\mu_n\beta_{a_n}.$$

Proof We can split the sum $\sum_{t=1}^n f(X_{t,n})$ into three parts, yielding

$$\begin{aligned}\mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{t=1}^n f(X_{t,n})\Big| > 4r_n\Big) \leq &\mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(X_{i,n})\Big| > r_n\Big)\\ &+ \mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Delta_j} f(X_{i,n})\Big| > r_n\Big) + \mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{i\in R} f(X_{i,n})\Big| > 2r_n\Big).\end{aligned}$$

The last part, which deals with the remainder term, is bounded by $M_n(2a_n) \leq M_n n/\mu_n$. Since $2r_n\mu_n \geq nM_n$, the probability associated with that remainder term is zero. The second term can be treated by the same arguments. Therefore, we just have to deal with the first term. Applying Lemma 7.2 with

$$h = \mathbb{I}\Big\{\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(X_{i,n})\Big| > r_n\Big\},$$

we get that

$$\mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(X_{i,n})\Big| > r_n\Big) \leq \mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n})\Big| > r_n\Big) + \mu_n\beta_{a_n},$$

which concludes the proof. □

The upper bound in Lemma 7.3 only involves i.i.d. blocks, which allows us to use classical techniques. In particular, we will apply the Benett inequality to obtain further bounds. For this purpose, assume that F_n contains at most a finite number $m_f(n)$ of functions, so that

$$\mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n})\Big| > r_n\Big) \leq m_f(n)\sup_{f\in F_n}\mathbb{P}\Big(\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n})\Big| > r_n\Big).$$

If we furthermore assume that the variance $\mathrm{Var}(\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n}))$ of the blocks is bounded by V_n, the Benett inequality yields

$$\mathbb{P}\Big(\Big|\sum_{j=1}^{\mu_n}\sum_{i\in\Gamma_j} f(Y_{i,n})\Big| > r_n\Big) \leq \exp\Big(-\frac{\mu_n V_n}{a_n^2 M_n^2}h\Big(\frac{r_n a_n M_n}{2\mu_n V_n}\Big)\Big), \tag{7.2}$$

where $h(x) = (1+x)log(1+x) - x$. Straightforward calculations finally provide, for (7.2), the bound

$$\exp\Big(-\frac{\log 2}{2}\big(\frac{r_n^2}{4\mu_n V_n} \wedge \frac{r_n}{2a_n M_n}\big)\Big).$$

Summing up, we just have proven the following lemma.

Lemma 7.4. *Let $\{X_{t,n}\}$ be a $\beta-$mixing triangular array, and denote by F_n a sequence of classes of functions from $\mathbb{R}$ to $\mathbb{R}$, with cardinality $\#F_n$, satisfying*

(i) $\#F_n \leq m_f(n)$; *(ii)* $\sup_{f\in F_n} |f(X_{t,n})| \leq M_n$; *(iii)* $\mathbb{E}(f(X)) = 0$.

Consider a blocking structure induced by the sequence (μ_n, a_n) of pairs of integers, where $n/2 - a_n \leq \mu_n a_n \leq n/2$, $a_n \to \infty$, and $\mu_n \to \infty$, satisfying

(a) $\mu_n \beta_{a_n} \xrightarrow{n\to\infty} 0$; *(b)* $2r_n\mu_n \geq nM_n$;

(c) $\mathrm{Var}\big(\sum_{i\in\Gamma_j} f(X_{i,n})\big) \vee \mathrm{Var}\big(\sum_{i\in\Delta_j} f(X_{i,n})\big) \leq V_n$ *for all* $1 \leq j \leq \mu_n$.

Then,

$$\mathbb{P}\Big(\sup_{f\in F_n}\Big|\sum_{t=1}^{n} f(X_{t,n})\Big| > 4r_n\Big) \leq 2m_f(n)\exp\Big(-\frac{\log 2}{2}\Big(\frac{r_n^2}{4\mu_n V_n} \wedge \frac{r_n}{2a_n M_n}\Big)\Big) + o(1).$$

7.2 Auxiliary technical results

Throughout this section, let $\{X_{t,T}\}$ denote a triangular array of locally strictly stationary (in the sense of Definition 2.1) time series satisfying Assumptions (A1)-(A3). The notation introduced in Sections 2 and 3 is used throughout. We start with a simple auxiliary result.

Lemma 7.5. *Let F and G denote functions from $\mathbb{R}$ to $\mathbb{R}$, with $|G(x) - G(y)| > c|x - y|$ for $x, y \in [a, b]$ where c is some positive constant. Let $x_1, x_2 \in (a, b)$ be such that $F(x_1) = G(x_2)$: if $\|F(\cdot) - G(\cdot)\|_\infty \leq \epsilon$, then $|x_1 - x_2| \leq \epsilon/c$.*

Proof The claim readily follows from $c|x_1 - x_2| < |G(x_1) - G(x_2)| = |G(x_1) - F(x_2)| \leq \epsilon$. □

Lemma 7.5 can be used to bound distances between the quantiles of two distribution functions: in view of Assumption (A3), it applies to $F = F_{u;T}$ and $G = G_\vartheta^\vartheta(\cdot)$, yielding

$$|q_{u,T}(\tau) - q_\tau^\vartheta| \leq \frac{L}{g_{\min}}\Big|\frac{u - \vartheta T}{T} + \frac{1}{T}\Big|. \tag{7.3}$$

Furthermore if $n/T = o(n^{-1})$ and $|t_0 T^{-1} - \vartheta| = o(T^{-1/2})$, a Taylor expansion yields

$$\sup_{s,t\in\mathcal{N}_{t_0,T}} \|F_{s,t;T}(q_{s,T}(\tau_1), q_{t,T}(\tau_2)) - G_{s-t}^\vartheta(q^\vartheta(\tau_1), q^\vartheta(\tau_2); \vartheta)\|_\infty = o(n^{-1}). \tag{7.4}$$

Next, define

$$H_{u,T}(\boldsymbol{\delta}, \omega, \tau) := \int_0^{b(\omega)} \big(\mathbb{I}\{X_{u,T} \leq s + q_{u,T}(\tau)\} - \mathbb{I}\{X_{u,T} \leq q_{u,T}(\tau)\}\big)ds$$

where $b(\omega) = n^{-1/2}\mathbf{c}_u'(\omega)\boldsymbol{\delta}$, and

$$W_{u,T}(\boldsymbol{\delta}, \omega, \tau) := H_{u,T}(\boldsymbol{\delta}, \omega, \tau) - g^\vartheta(q_\tau^\vartheta)(\mathbf{c}_u'(\omega)\boldsymbol{\delta})^2/2n. \tag{7.5}$$

Denote by $\mathcal{F}_n$ the set of Fourier frequencies $\omega_{j,n}$.

Lemma 7.6. *There exists a finite constant C such that, for any τ, $\boldsymbol{\delta}$, u_1, u_2, n, and T large enough,*

$$\sup_{\omega\in\mathcal{F}_n}\sup_{u\in\mathcal{N}_{t_0,T}}\left|\mathbb{E}[(W_{u,T}(\boldsymbol{\delta},\omega,\tau))]\right|\le C\|\boldsymbol{\delta}\|^3 n^{-3/2}, \tag{7.6}$$

$$\sup_{\omega\in\mathcal{F}_n}\sup_{u\in\mathcal{N}_{t_0,T}}\left|(W_{u,T}(\boldsymbol{\delta},\omega,\tau))\right|\le C(\|\boldsymbol{\delta}\|^2\vee 1)n^{-1/2}\quad a.s.$$

and

$$\sup_{\omega\in\mathcal{F}_n}\left|\mathbb{E}[W_{u_1,T}(\boldsymbol{\delta},\omega,\tau)W_{u_2,T}(\boldsymbol{\delta},\omega,\tau)]\right|\le C(\|\boldsymbol{\delta}\|^4\vee 1)(n^{-3/2}\mathbb{I}_{\{u_1=u_2\}}+n^{-2}\mathbb{I}_{\{u_1\neq u_2\}}). \tag{7.7}$$

Proof Let h denote a function from $\mathbb{R}$ to $\mathbb{R}$: then,

$$\begin{aligned}\Big|\int_0^{b(\omega)}F_{u;T}(h(s))-G^{\vartheta}(h(s))ds\Big| &\le Cn^{-1/2}\|\boldsymbol{\delta}\|\|F_{u;T}(x)-G^{\vartheta}(x)\|_\infty\\ &\le Cn^{-1/2}\|\boldsymbol{\delta}\|\frac{|u-\vartheta T|}{T}\le Cn^{1/2}T^{-1}\|\boldsymbol{\delta}\|\le C\|\boldsymbol{\delta}\|n^{-3/2};\end{aligned}$$

similarly, and using the same arguments, for a function (h_1,h_2) from $\mathbb{R}^2$ to $\mathbb{R}^2$,

$$\Big|\int_0^{b(\omega)}\int_0^{b(\omega)}F_{u;T}(h_1(s,t))-G^{\vartheta}(h_1(s,t))dsdt\Big|\le C\|\boldsymbol{\delta}\|^2n^{-2}$$

and

$$\Big|\int_0^{b(\omega)}\int_0^{b(\omega)}F_{u,v;T}(h_1(s,t),h_2(s,t))-G^{\vartheta}_{u-v}(h_1(s,t),h_2(s,t);\vartheta)dsdt\Big|\le C\|\boldsymbol{\delta}\|^2n^{-2}.$$

Along with a Taylor expansion, these inequalities yield, for $\omega\in\mathcal{F}_n$,

$$\begin{aligned}\mathbb{E}[H_{u,T}(\boldsymbol{\delta},\omega,\tau)] &= \int_0^{b(\omega)}F_{u;T}(s+q_{u,T}(\tau))-F_{u;T}(q_{u,T}(\tau))ds\\ &= \int_0^{b(\omega)}G^{\vartheta}(s+q_{u,T}(\tau))-G^{\vartheta}(q_{u,T}(\tau))ds+O(\|\boldsymbol{\delta}\|n^{-3/2})\\ &= \int_0^{b(\omega)}sg^{\vartheta}(q_{u,T}(\tau))+r_1(s,\tau)ds+O(\|\boldsymbol{\delta}\|n^{-3/2})\\ &= \frac{1}{2n}g^{\vartheta}(q_{u,T}(\tau))(\mathbf{c}'_u(\omega)\boldsymbol{\delta})^2+r_2(\tau,\omega),\end{aligned}$$

where $|r_1(s,\tau)|\le Cs^2$, hence $|r_2(\tau,\omega_{j,n})|\le C(\|\boldsymbol{\delta}\|^3\vee\|\boldsymbol{\delta}\|)n^{-3/2}$. With equation (7.3), Assumption (A2) and a Taylor expansion, we obtain

$$g^{\vartheta}(q_{u,T}(\tau))-g^{\vartheta}(q^{\vartheta}(\tau))|=|q_{u,T}(\tau)-q^{\vartheta}(\tau)||(g^{\vartheta})'(\xi)|\le C\frac{|u-\vartheta T|}{T}=o(n^{-1}). \tag{7.8}$$

The first part of (7.6) follows. The second part is obtained by bounding each term of the difference in the definition (7.5) of $W_{u,T}$. To prove (7.7) we consider the cases $u_1=u_2$ and $u_1\neq u_2$ separately.

First, observe that in case $u_1 = u_2 = u$, we have

$$
\begin{aligned}
&\mathbb{E}\big[H_{u,T}^2(\boldsymbol{\delta}, \omega, \tau)\big] \\
&\quad = \mathbb{E}\Big[\int_0^{b(\omega)}\int_0^{b(\omega)} (\mathbb{I}_{\{X_{u,T} \le s + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_{u,T} \le q_{u,T}(\tau)\}})(\mathbb{I}_{\{X_{u,T} \le t + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_{u,T} \le q_{u,T}(\tau)\}}) dsdt\Big] \\
&\quad = \mathbb{E}\Big[\int_0^{b(\omega)}\int_0^{b(\omega)} (\mathbb{I}_{\{X_{u,T} \le (s\wedge t) + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_{u,T} \le (s\wedge 0) + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_{u,T} \le (0\wedge t) + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_{u,T} \le q_{u,T}(\tau)\}} dsdt\Big] \\
&\quad = \mathbb{E}\Big[\int_0^{b(\omega)}\int_0^{b(\omega)} (\mathbb{I}_{\{X_u^{\vartheta} \le (s\wedge t) + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_u^{\vartheta}\vartheta \le (s\wedge 0) + q_{u,T}(\tau)\}} \\
&\qquad\qquad - \mathbb{I}_{\{X_u^{\vartheta}\vartheta \le (0\wedge t) + q_{u,T}(\tau)\}} - \mathbb{I}_{\{X_u^{\vartheta} \le q_{u,T}(\tau)\}} dsdt\Big] + O(\|\boldsymbol{\delta}\|^2 n^{-2}) \\
&\quad = \int_0^{b(\omega)}\int_0^{b(\omega)} (s\wedge t - s\wedge 0 - 0\wedge t) g^{\vartheta}(q_{u,T}(\tau)) + r_3(s,t,\tau) dsdt + O(\|\boldsymbol{\delta}\|^2 n^{-2}),
\end{aligned}
$$

where $|r_3(s,t,\tau)| \le C(s^2+t^2)$, which can be calculated from the remainder of the Taylor expansion. Now, $\int_0^x\int_0^x (s\wedge t - s\wedge 0 - 0\wedge t) dsdt = |x|^3/3$, and $b = n^{-1/2}\mathbf{c}_u'(\omega)\boldsymbol{\delta}$ implies

$$
\mathbb{E}\big[(H_{u,T}(\boldsymbol{\delta}, \omega, \tau))^2\big] = \frac{1}{3} n^{-3/2} f_{u,T} |\mathbf{c}_u'(\omega)\boldsymbol{\delta}|^3 + r_4(\omega_{j,n}, \tau), \tag{7.9}
$$

where $r_4(\omega_{j,n}, \tau) \le C\|\boldsymbol{\delta}\|^4 n^{-2}$. Similarly, if $u_1 \ne u_2$ we obtain

$$
\begin{aligned}
&g^{\vartheta}(q^{\vartheta}(\tau))(2n)^{-1}(\mathbf{c}_u'(\omega)\boldsymbol{\delta})^2 \mathbb{E}\big[H_{u_1,T}(\boldsymbol{\delta}, \omega, \tau) H_{u_2,T}(\boldsymbol{\delta}, \omega, \tau)\big] \\
&\quad = \mathbb{E}\Big[\int_0^{b(\omega)}\int_0^{b(\omega)} (\mathbb{I}_{\{X_{u_1,T} \le s + q_{u_1,T}(\tau)\}} - \mathbb{I}_{\{X_{u_1,T} \le q_{u_1,T}(\tau)\}})(\mathbb{I}_{\{X_{u_2,T} \le t + q_{u_2,T}(\tau)\}} - \mathbb{I}_{\{X_{u_2,T} \le q_{u_2,T}(\tau)\}}) dsdt\Big] \\
&\quad = \int_0^{b(\omega)}\int_0^{b(\omega)} F_{u_1,u_2;T}(s + q_{u_1,T}(\tau) + q_{u_2,T}(\tau)) - F_{u_1,u_2;T}(q_{u_1,T}(\tau), t + q_{u_2,T}(\tau)) \\
&\qquad\qquad - F_{u_1,u_2}(q_{u_1,T}(\tau), t + q_{u_2,T}(\tau)) + F_{u_1,u_2;T}(q_{u_1,T}(\tau), q_{u_2,T}(\tau)) dsdt \\
&\quad = \int_0^{b(\omega)}\int_0^{b(\omega)} r_5(s,t,\tau) dsdt + O(\|\boldsymbol{\delta}\|^2 n^{-2}),
\end{aligned}
$$

where the last equality follows from a two-dimensional Taylor expansion that leads to $|r_5(s,t,\tau)| \le C(s^2+t^2)$. Hence,

$$
\big|\mathbb{E}\big[H_{u_1,T}(\boldsymbol{\delta}, \omega, \tau) H_{u_2,T}(\boldsymbol{\delta}, \omega, \tau)\big]\big| \le C(\|\boldsymbol{\delta}\|^4 \vee 1) n^{-2},
$$

which completes the proof of (7.7). □

Lemma 7.7. *For any bounded set $S \subset \mathbb{R}$ and positive sequence $b_n = o(1)$,*

(i) $\sup_{x\in S} |\hat{F}_{t_0,T}(x) - G^{\vartheta}(x)| = O_{\mathbb{P}}(n^{-1/2}\sqrt{\log(n)})$ *and*

(ii) $\sup_{x\in S}\sup_{|y|\le b_n} |\hat{F}_{t_0,T}(x+y) - \hat{F}_{t_0,T}(x) - G^{\vartheta}(x+y) + G^{\vartheta}(x)| = O_{\mathbb{P}}(\rho_n(b_n, \delta))$

where

$$
\rho_n(b_n, \delta) := \Big(\frac{b_n + n^{1/(1+\delta)} b_n^2 \log n}{n} \log n\Big)^{1/2} \vee (n^{-\delta/(1+\delta)} \log n) \tag{7.10}
$$

and δ is the exponent in the $\beta-$mixing rate of Assumption (A1).

Proof To prove (i), let us show that

$$
\sup_{x\in S}\Big|\hat{F}_{t_0,T}(x) - n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}} F_{t;T}(x)\Big| = O_{\mathbb{P}}(n^{-1/2}\sqrt{\log(n)}). \tag{7.11}
$$

The claim then follows from Definition 2.1 and the triangle inequality, that is,

$$\sup_{x\in S}\left|\hat{F}_{t_0,T}(x)-G^{\vartheta}(x)\right| \le \sup_{x\in S}\left|\hat{F}_{t_0,T}(x)-n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}}F_{t;T}(x)\right| + \left|n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}}F_{t;T}(x)-G^{\vartheta}(x)\right|$$
$$= O_{\mathbb{P}}(n^{-1/2}\sqrt{\log(n)}) + O(n^{-1}).$$

Set

$$\hat{F}_{t_0,T}(x)-n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}}F_{t;T}(x) = n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}}(\mathbb{I}_{\{X_{t,T}\le x\}}-F_{t;T}(x)) =: n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}}W_{t,T}(x).$$

It is possible to cover the set S with $N=O(n)$ spheres of radius n^{-1} and centers $z_j, j=1,\dots,N$. If we restrict the function $W_{t,T}$ to this finite subset, we get

$$\sup_{|z_j-x|<n^{-1}}|W_{t,T}(x)-W_{t,T}(z_j)| = \sup_{|z_j-x|<n^{-1}}|\mathbb{I}_{\{X_{t,T}\le x\}}-\mathbb{I}_{\{X_{t,T}\le z_j\}}+F_{t;T}(x)-F_{t;T}(z_j)|$$
$$\le \mathbb{I}_{\{|X_{t,T}-z_j|\le n^{-1}\}}+Cn^{-1} =: V_{t,T}(j)$$

and therefore, for some C_D to be chosen later on,

$$\mathbb{P}\Big(\sup_{x\in S}\Big|n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}}W_{t,T}(x)\Big|\ge C_D n^{-1/2}\sqrt{\log(n)}\Big) \tag{7.12}$$
$$\le \mathbb{P}\Big(\max_{1\le j\le N}\Big|\sum_{t\in\mathcal{N}_{t_0,T}}W_{t,T}(z_j)\Big|\ge\frac{C_D}{2}n^{1/2}\sqrt{\log(n)}\Big)+\mathbb{P}\Big(\max_{1\le j\le N}\Big|\sum_{t\in\mathcal{N}_{t_0,T}}V_{t,T}(j)\Big|\ge\frac{C_D}{2}n^{1/2}\sqrt{\log(n)}\Big)$$
$$= P_T^1+P_T^2, \text{ say.}$$

We will now use the blocking technique from Lemma 7.4 to show that both probabilities in the right-hand side of (7.12) tend to zero. Observe that $W_{t,T}(z_j)$ and $\mathring{V}_{t,T}(j) := V_{t,T}(j)-\mathbb{E}(V_{t,T}(j))$ are centered $\beta-$mixing random variables with

$$\sup_{1\le j\le N}\sup_{t,T}\left|W_{t,T}(z_j)\right|\le M \quad\text{and}\quad \sup_{1\le j\le N}\sup_{t,T}\left|\mathring{V}_{t,T}(j)\right|\le M$$

for some constant M independent of j,t and T, so that conditions (i)-(iii) in Lemma 7.4 are satisfied. Also set

$$a_n=\lceil n^{\frac{1}{\delta+1}}\rceil,\quad \mu_n=\lfloor\frac{n}{2a_n}\rfloor\quad\text{and}\quad r_n=n^{1/2}\sqrt{\log(n)},$$

so that conditions (a) and (b) are satisfied as well for n large enough. To bound the variances, observe that, for $|t_1-t_2|\le a_n$,

$$\mathrm{Var}\Big(\sum_{t=t_1}^{t_2}W_{t,T}(x)\Big) = \sum_{u=t_1}^{t_2}\sum_{v=t_1}^{t_2}\mathbb{E}\big(W_{u,T}(x)W_{v,T}(x)\big)$$
$$\le \sum_{u=t_1}^{t_2}\big(F_{u;T}(x)-F_{u;T}^2(x)\big)+\sum_{u=t_1}^{t_2}\sum_{v=t_1}^{t_2}\big(F_{u,v;T}(x,x)-F_{u;T}(x)F_{v;T}(x)\big)$$
$$\le \frac{1}{4}a_n+Ca_n=O(a_n),$$

where the inequality for the second sum follows from the $\beta-$mixing properties in Assumption (A1). Therefore, the inequality $\mathbb{E}(V_{t,T}(x)) \leq n^{-1}$ yields

$$\begin{aligned}
\mathrm{Var}\Big(\sum_{t=t_1}^{t_2} V_{t,T}(x)\Big) &= \sum_{u=t_1}^{t_2}\sum_{v=t_1}^{t_2} \mathbb{E}\big(V_{u,T}(x)V_{v,T}(x)\big) \\
&= \sum_{u=t_1}^{t_2} \big(\mathbb{P}(|X_{u,T}-z_j|<n^{-1}) + O(n^{-1})\big) + \sum_{u=t_1}^{t_2}\sum_{v=t_1}^{t_2} \big(\mathbb{P}(|X_{u,T}-z_j|<n^{-1},\ |X_{v,T}-z_j|<n)^{-1}\big) \\
&\leq O\big(\frac{a_n}{n}\big) + O\big(\frac{a_n^2}{n}\big) = O(a_n).
\end{aligned}$$

It thus follows from Lemma 7.4 lthat $P_T^1 \leq Nn^{-D} + o(1)$, since

$$\frac{C_D^2 n\log(n)}{4n} \wedge \frac{C_D\sqrt{n\log(n)}}{2Mn^{1/(1+\delta)}} \geq D\log(n)$$

for an appropriate constant C_D and sufficiently large n. The same conclusion holds for P_T^2, which deals with $V_{t,T}(j)$; (7.11) follows. Part (ii) of he lemma follows along the same lines; see Lemma 6.9 in Dette et al. (2014) for a proof in the stationary case. □

Lemma 7.8. *Let F^{-1} denotes the generalized inverse of a non-decreasing function F.*

(i) Fix $\vartheta \in (0,1)$ and assume that, for some $\gamma > 0$ such that $[a-\gamma, b+\gamma] \subset (0,1)$,

$$\liminf_{T\to\infty} \inf_{u\in[a-\gamma,b+\gamma]} \min_{t\in\mathcal{N}_{t_0,T}} f_{t,T}((G^\vartheta)^{-1}(u)) > 0.$$

Then,

$$\sup_{u\in[a,b]} |G^\vartheta(\hat{F}_{t_0,T}^{-1}(u)) - u| = O_{\mathbb{P}}(n^{-1/2}\sqrt{\log n}).$$

(ii) If, moreover, $\rho_n(b_n,\delta) = o(b_n)$ with $0 < b_n = o(1)$, where $\rho_n(b_n,\delta)$ is defined in (7.10), then

$$\sup_{\substack{u,v\in[a,b]\\|u-v|\leq b_n}} \Big|G^\vartheta(\hat{F}_{t_0,T}^{-1}(u)) - G^\vartheta(\hat{F}_{t_0,T}^{-1}(v)) - (u-v)\Big| = O_{\mathbb{P}}(\rho_n(2b_n,\delta)).$$

Proof Let $h:\mathbb{R}\mapsto\mathbb{R}$ be a nondecreasing function. Then $\sup_{w\in[u,v]}|h(w)-w| \leq \gamma_n$ implies $\sup_{w\in[u+2\gamma_n,v-2\gamma_n]}|h^{-1}(w)-w| \leq \gamma_n$. Setting $h(w) = \hat{F}_{t_0,T}((G^\vartheta)^{-1}(w))$ yields, in view of Lemma 7.7,

$$\begin{aligned}
\sup_{(G^\vartheta)^{-1}(w)\in S} \big|\hat{F}_{t_0,T}((G^\vartheta)^{-1}(w)) - G^\vartheta((G^\vartheta)^{-1}(w))\big| &= \sup_{(G^\vartheta)^{-1}(w)\in S} \big|\hat{F}_{t_0,T}((G^\vartheta)^{-1}(w)) - w\big| \\
&= O_{\mathbb{P}}(n^{-1/2}\sqrt{\log(n)}).
\end{aligned}$$

The first assertion of the lemma then follows from choosing

$$S = [(G^\vartheta)^{-1}(a) - Cn^{-1/2}\sqrt{\log(n)}, (G^\vartheta)^{-1}(b) + Cn^{-1/2}\sqrt{\log(n)}]$$

with an appropriate constant C. Turning to (ii), part (ii) of Lemma 7.7 entails, for any bounded set S,

$$\sup_{x\in S}\sup_{|y|\leq b_n} |\hat{F}_{t_0,T}(x+y) - \hat{F}_{t_0,T}(x) - G^\vartheta(x+y) + G^\vartheta(x)| = O_{\mathbb{P}}(\rho_n(b_n,\boldsymbol{\delta})).$$

Since G^ϑ is differentiable, with strictly positive density, $\inf_{[a-\gamma,b+\gamma]} g^\vartheta(x) > 0$. Hence, for any subset A of $[a-\gamma, b+\gamma]$ and appropriate constant C_A,

$$\sup_{u,v\in A} |(G^\vartheta)^{-1}(u) - (G^\vartheta)^{-1}(v)| \leq C_A|u-v|,$$

and therefore, with $y = (G^\vartheta)^{-1}(u) - (G^\vartheta)^{-1}(v)$ and $x = (G^\vartheta)^{-1}(v)$,

$$\sup_{\substack{u,v\in[a-\gamma,b+\gamma]\\|u-v|\leq b_n}} |\hat{F}_{t_0,T}((G^\vartheta)^{-1}(u)) - \hat{F}_{t_0,T}((G^\vartheta)^{-1}(v)) - u + v| = O_{\mathbb{P}}(\rho_n(b_n, \boldsymbol{\delta})) \tag{7.13}$$

We now apply Lemma 3.5 from Wendler (2011) with $F(w) = \hat{F}_{t_0,T}((G^\vartheta)^{-1}(w))$. Using the fact that, for any strictly increasing function G $(F\circ G^{-1})^{-1} = G\circ F^{-1}$ (see Exercise 3 in Chapter 1 of Shorak and Wellner (1986)), we get that the condition

$$\sup_{\substack{u,v\in[\hat{F}_{t_0,T}((G^\vartheta)^{-1}(C_1+2c+l)),\hat{F}_{t_0,T}((G^\vartheta)^{-1}(C_2-2c+l))]\\|u-v|\leq l}} \left|G^\vartheta(\hat{F}^{-1}_{t_0,T}(u)) - G^\vartheta(\hat{F}^{-1}_{t_0,T}(v)) - (u-v)\right| > c \tag{7.14}$$

implies

$$\sup_{\substack{u,v\in[a-\gamma,b+\gamma]\\|u-v|\leq l+2c}} |\hat{F}_{t_0,T}((G^\vartheta)^{-1}(u)) - \hat{F}_{t_0,T}((G^\vartheta)^{-1}(v)) - (u-v)| > c, \tag{7.15}$$

where γ is chosen such that $[a-\gamma, b+\gamma] \supset [C_1, C_2]$. Now, setting

$$C_1 = G^\vartheta(\hat{F}^{-1}_{t_0,T}(a)) - 2c - l, \quad C_2 = G^\vartheta(\hat{F}^{-1}_{t_0,T}(b)) + 2c + l, \quad l = b_n \quad \text{and} \quad c = D\rho_n(b_n, \delta),$$

inequality (7.15) for D large enough is in contradiction with inequality (7.13). Therefore, inequality (7.14) cannot be correct, which proves the claim. □

7.3 Details for the proof of Theorem 5.1 and Proposition 5.1

7.3.1 Proof of (6.2)

Observe that, by Lemma 7.8 in Section 7.2, we have, uniformly in τ and ω,

$$\boldsymbol{\delta}^\vartheta_{t_0,T}(\omega,\tau) = \text{diag}(1,2,2)\boldsymbol{\zeta}^\vartheta_{t_0,T}(\omega,\tau) + O_P(\sqrt{\log n}).$$

In order to establish (6.2), it is therefore sufficient to find a constant $A = A^\vartheta(\tau)$ such that

$$\mathbb{P}\Big(\sup_{\omega\in\mathcal{F}_n} \|\boldsymbol{\zeta}^\vartheta_{t_0,T}(\omega,\tau)\|_\infty > A^\vartheta(\tau)\sqrt{\log n}\Big) = \mathbb{P}\Big(\sup_{\omega\in\mathcal{F}_n} \big\|\sum_{u\in\mathcal{N}_{t_0,T}} H^\vartheta_{u,T}(\omega,\tau)\big\|_\infty \geq A^\vartheta(\tau)\sqrt{n\log n}\Big) = o(1) \tag{7.16}$$

where $H^\vartheta_{u,T}(\omega,\tau) := \mathbf{c}_u(\omega)(\tau - \mathbb{I}_{\{X_{u,T}\leq q_{u,T}(\tau)\}})$. To bound the probability on the right-hand side of (7.16), we apply the independent blocking technique from Lemma 7.4. Let us show that each component $H^\vartheta_{u,T,j}(\omega,\tau)$, $j=1,2,3$, of $H^\vartheta_{u,T}(\omega,\tau)$ satisfies the assumptions of Lemma 7.4.. Indeed, the $H^\vartheta_{u,T,j}(\omega,\tau)$'s form a $\beta-$ mixing triangular array of centered variables, and it follows from (7.21) and (A2) that $\sup_{\omega\in\mathcal{F}_n} |H^\vartheta_{u,T,j}(\omega,\tau)| \leq 1$. Therefore, conditions $(i)-(iii)$ are satisfied. To apply the blocking technique, set

$$a_n = \lceil n^{\frac{1}{\delta+1}}\rceil, \quad \mu_n = \lfloor\frac{n}{2a_n}\rfloor, \quad \text{and} \quad r_n = A^\vartheta(\tau)\sqrt{n\log n},$$

so that (a) $\mu_n \beta_{a_n} \xrightarrow{n\to\infty} 0$ and (b) $\frac{n}{2\mu_n} M_n \le C a_n \ll r_n$ hold. To bound the variance of each block, observe that, with $|t_1 - t_2| < a_n$,

$$\operatorname{Var}\left(\sum_{u=t_1}^{t_2} H^{\vartheta}_{u,T}(\omega,\tau)\right) = \sum_{u=t_1}^{t_2}\sum_{v=t_1}^{t_2} \mathbb{E}\big(H^{\vartheta}_{u,T}(\omega,\tau)(H^{\vartheta}_{v,T}(\omega,\tau))'\big)$$
$$\le \sum_{|h|<a_n} \big|\operatorname{Cov}(\mathbb{I}_{\{X_{u,T}\le q_{u,T}(\tau)\}}, \mathbb{I}_{\{X_{u+h,T}\le q_{u+h,T}(\tau)\}})\big| \sum_{u=t_1+(0\wedge h)}^{t_2-(0\vee h)} |\mathbf{c}_u(\omega)\mathbf{c}'_{u+h}(\omega)| = O(a_n)$$

since $\|\mathbf{c}_u(\omega)\|_\infty \le 1$ and (A1) implies that $\Big|\operatorname{Cov}\big((X_{u,T} \le q_{u,T}(\tau), \mathbb{I}_{\{X_{u+h,T}|\le q_{u+h,T}(\tau)\}}\big)\Big| \le |h|^{-\delta}$. Lemma 7.4 then yields

$$\mathbb{P}\Big(\sup_{\omega\in\mathcal{F}_n}\max_{j=1,2,3}\Big|\sum_{u\in\mathcal{N}_{t_0,T}} H^{\vartheta}_{u,T,j}(\omega,\tau)\Big| > A\sqrt{n\log n}\Big) \le 6n\exp\Big(-\frac{\log 2}{2}\Big(\frac{A^2 n\log n}{C a_n \mu_n} \wedge \frac{A\sqrt{n\log n}}{a_n}\Big)\Big),$$

which tends to zero for A large enough. □

7.3.2 Proof of (6.3)

First, note that, due to local strict stationarity, Lemma 7.7 and 7.8 still hold for $t \in \mathcal{N}_{t_0,T}$ if we exchange G^{ϑ} and $F_{t;T,\cdot}$. We have to show that

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}\|_\infty < A\log n}\big|\hat{Z}^R_{t_0,T}(\boldsymbol{\delta}) - \hat{Z}^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau)\big| = O_{\mathbb{P}}(n^{-\frac{1}{4}\frac{\delta-1}{\delta+1}}(\log n)^3).$$

Knight's identity (see p. 121 of Koenker (2005)) yields $\hat{Z}^R_{t_0,T}(\boldsymbol{\delta}) = \hat{Z}^R_{t_0,T,1}(\boldsymbol{\delta}) + \hat{Z}^R_{t_0,T,2}(\boldsymbol{\delta})$, where

$$\hat{Z}^R_{t_0,T,1}(\boldsymbol{\delta}) = -\boldsymbol{\delta}' n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}} \mathbf{c}_t(\omega)\big(\tau - \mathbb{I}_{\{U_{t,T}\le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau))\}}\big) \quad \text{and}$$
$$\hat{Z}^R_{t_0,T,2}(\boldsymbol{\delta}) = \sum_{t\in\mathcal{N}_{t_0,T}}\int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} \big(\mathbb{I}_{\{U_{t,T}\le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau+s))\}} - \mathbb{I}_{\{U_{t,T}\le F_{t,T}(\hat{F}^{-1}_{t_0,T}(\tau))\}}\big)ds.$$

A similar representation, namely

$$\hat{Z}^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau) = \hat{Z}^U_{t_0,T,1}(\boldsymbol{\delta},\omega,\tau) + \hat{Z}^U_{t_0,T,2}(\boldsymbol{\delta},\omega,\tau),$$

holds for $\hat{Z}^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau)$, where

$$\hat{Z}^U_{t_0,T,1}(\boldsymbol{\delta},\omega,\tau) = -\boldsymbol{\delta}' n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}} \mathbf{c}_t(\omega)\big(\tau - \mathbb{I}_{\{U_{t,T}\le\tau\}}\big) - \sqrt{n}\boldsymbol{e}'_1\big(F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau\big) \quad \text{and}$$
$$\hat{Z}^U_{t_0,T,2}(\boldsymbol{\delta},\omega,\tau) = \sum_{t\in\mathcal{N}_{t_0,T}}\int_0^{n^{-1/2}\mathbf{c}'_t(\omega)\boldsymbol{\delta}} \big(\mathbb{I}_{\{U_{t,T}\le\tau+s\}} - \mathbb{I}_{\{U_{t,T}\le\tau\}}\big)ds.$$

First consider $\big|\hat{Z}^R_{t_0,T,1}(\boldsymbol{\delta}) - \hat{Z}^U_{t_0,T,1}(\boldsymbol{\delta},\omega,\tau)\big|$. It is sufficient to show that

$$B_1 := \max_{k=2,3}\sup_{\omega\in\mathcal{F}_n} n^{-1/2}\Big|\sum_{t\in\mathcal{N}_{t_0,T}} \mathbf{c}_{t,k}(\omega)\big(\mathbb{I}_{\{U_{t,T}\le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau))\}} - \mathbb{I}_{\{U_{t,T}\le\tau\}}\big)\Big| = O_{\mathbb{P}}(n^{-\frac{1}{2}\frac{\delta}{1+\delta}}\log n) \tag{7.17}$$

and

$$
\begin{aligned}
B_2 &:= n^{-1/2} \sup_{\omega \in \mathcal{F}_n} \Big| \sum_{t \in \mathcal{N}_{t_0,T}} \big(\mathbb{I}_{\{U_{t,T} \le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau))\}} - \mathbb{I}_{\{U_{t,T} \le \tau\}} - (F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau) \big) \Big| \\
&= O_{\mathbb{P}}(n^{-\frac{1}{2}\frac{\delta}{1+\delta}} \log n). \qquad (7.18)
\end{aligned}
$$

It follows from Lemma 7.8(i) that $F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau = O_{\mathbb{P}}(n^{-1/2}\sqrt{\log n})$; hence,

$$
\begin{aligned}
B_1 \le n^{-1/2} \sup_{\omega \in \mathcal{F}_n} \sup_{|x-\tau| \le n^{-1/2}\sqrt{\log n}} \Big| \sum_{t \in \mathcal{N}_{t_0,T}} \mathbf{c}_{t,k}(\omega)(\mathbb{I}_{\{U_{t,T} \le x\}} - \mathbb{I}_{\{U_{t,T} \le \tau\}} - (x - \tau)) \Big| \\
+ n^{-1} \sup_{\omega \in \mathcal{F}_n} \sqrt{\log n} \Big| \sum_{t=1}^{n} \mathbf{c}_{t,k}(\omega) \Big|
\end{aligned}
$$

which coincides with equation (6.19) in Dette et al. (2014), so that (7.17) can be proven along the same lines by an application of the independent blocking technique from Lemma 7.4. To bound (7.18) we again apply Lemma 7.8(i)

$$
B_2 \le n^{-1/2} \sup_{|x-\tau| \le n^{-1/2}\sqrt{\log n}} \Big| \sum_{t \in \mathcal{N}_{t_0,T}} (\mathbb{I}_{\{U_{t,T} \le x\}} - \mathbb{I}_{\{U_{t,T} \le \tau\}} - (x - \tau)) \Big|
$$

so that the bound holds by an application of Lemma 7.7(ii).

The treatment of $|\hat{Z}^R_{t_0,T,2}(\boldsymbol{\delta}) - \hat{Z}^U_{t_0,T,2}(\boldsymbol{\delta}, \omega, \tau)|$, is more technical. Setting $b = n^{-1/2}\mathbf{c}_t'(\omega)\boldsymbol{\delta}$, observe that

$$
\begin{aligned}
&\hat{Z}^R_{t_0,T,2}(\boldsymbol{\delta}) - \hat{Z}^U_{t_0,T,2}(\boldsymbol{\delta}, \omega, \tau) \\
&= \sum_{t \in \mathcal{N}_{t_0,T}} \int_0^b \Big(\mathbb{I}_{\{U_{t,T} \le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau+s))\}} - \mathbb{I}_{\{U_{t,T} \le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau))\}} - \mathbb{I}_{\{U_{t,T} \le \tau+s\}} + \mathbb{I}_{\{U_{t,T} \le \tau\}} \Big) ds \\
&= n^{-1/2} \sum_{t \in \mathcal{N}_{t_0,T}} \int_0^{n^{1/2}b} \Big(\mathbb{I}_{\{U_{t,T} \le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau+n^{-1/2}s))\}} - \mathbb{I}_{\{U_{t,T} \le F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau))\}} - \mathbb{I}_{\{U_{t,T} \le \tau+n^{-1/2}s\}} + \mathbb{I}_{\{U_{t,T} \le \tau\}} \Big) ds \\
&=: \int_0^{n^{1/2}b} A(s) ds = \int_{\mathbb{R}} A(s)[\mathbb{I}_{\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}} - \mathbb{I}_{\{0 \ge s \ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}}] ds.
\end{aligned}
$$

Letting

$$
\begin{aligned}
S^+_{\boldsymbol{\delta}}(u, v; s) &:= n^{-1/2} \sum_{t \in \mathcal{N}_{t_0,T}} [\mathbb{I}_{\{U_{t,T} \le u\}} - \mathbb{I}_{\{U_{t,T} \le v\}} - (u - v)][\mathbb{I}_{\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}}] \qquad \text{and} \\
S^-_{\boldsymbol{\delta}}(u, v; s) &:= n^{-1/2} \sum_{t \in \mathcal{N}_{t_0,T}} [\mathbb{I}_{\{U_{t,T} \le u\}} - \mathbb{I}_{\{U_{t,T} \le v\}} - (u - v)][\mathbb{I}_{\{0 \ge s \ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}}],
\end{aligned}
$$

we obtain the decomposition

$$
\int_{\mathbb{R}} A(s)[\mathbb{I}_{\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}} - \mathbb{I}_{\{0 \ge s \ge \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}}] ds =: A^{1+} + A^{1-} + A^{2+} + A^{2-}, \quad \text{say,}
$$

where

$$
\begin{aligned}
A^{1+} &:= \int_{\mathbb{R}} \big(S^+_{\boldsymbol{\delta}}(F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau + n^{-1/2}s)), n^{-1/2}s + \tau; s) - S^+_{\boldsymbol{\delta}}(F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau)), \tau, s) \big) ds, \\
A^{2+} &:= n^{-1/2} \int_{\mathbb{R}} \sum_{t \in \mathcal{N}_{t_0,T}} \Big(F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau + n^{-1/2}s)) - (n^{-1/2}s + \tau) - (F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau)) - \tau \Big) \mathbb{I}_{\{0 \le s \le \mathbf{c}_t'(\omega)\boldsymbol{\delta}\}} ds
\end{aligned}
$$

and A^{1-}, A^{2-} are defined by replacing $S^+_{\boldsymbol{\delta}}$ by $S^-_{\boldsymbol{\delta}}$ and $\mathbb{I}_{\{0\le s\le \mathbf{c}'_t(\omega)\boldsymbol{\delta}\}}$ by $\mathbb{I}_{\{0\ge s\ge \mathbf{c}'_t(\omega)\boldsymbol{\delta}\}}$. In what follows, we derive upper bounds for A^{1+} and A^{2+} only, as A^{1-} and A^{2-} can be treated similarly. In view of Lemma 7.8(ii) and local stationarity,

$$\begin{aligned}|A^{2+}| &\le 4\|\boldsymbol{\delta}\|\sqrt{n}\max_{t\in\mathcal{N}_{t_0,T}}\sup_{|u-\tau|\le 2\|\boldsymbol{\delta}\|n^{-1/2}}|F_{t;T}(\hat F^{-1}_{t_0,T}(u)) - u - F_{t;T}(\hat F^{-1}_{t_0,T}(\tau)) + \tau|\\ &= O_{\mathbb{P}}(\rho_n(A(\log n)n^{-1/2},\delta)\sqrt{n}\log n) = O_{\mathbb{P}}((n^{-1/4}(\log n)^{3/2})\vee(n^{(1-\delta)/(2+2\delta)}(\log n)^2))\\ &= O_{\mathbb{P}}(n^{-\frac{1}{4}\frac{\delta-1}{\delta+1}}(\log n)^2),\end{aligned}$$

where δ is the exponent from the $\beta-$mixing rate. As for A^{1+}, still in view of Lemma 7.8,

$$\begin{aligned}&\Big|\int S^+_{\boldsymbol{\delta}}(F_{t;T}(\hat F^{-1}_{t_0,T}(\tau+n^{-1/2}s)), n^{-1/2}s+\tau; s)ds\Big|\\ &\le 2\int \sup_{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|n^{-1/2}}|S^+_{\boldsymbol{\delta}}(F_{t;T}(\hat F^{-1}_{t_0,T}(v)), v; s)|ds\\ &\le 2\int_{-2\|\boldsymbol{\delta}\|}^{2\|\boldsymbol{\delta}\|}\sup_{\{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|n^{-1/2}\}}\sup_{\{u:|u-v|\le n^{-1/2}\log n\}}|S^+_{\boldsymbol{\delta}}(u,v;s)|ds\\ &\le 8\|\boldsymbol{\delta}\|\sup_{\{s:|s|\le 2\|\boldsymbol{\delta}\|\}}\sup_{\{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|n^{-1/2}\}}\sup_{\{u:|u-v|\le n^{-1/2}\log n\}}|S^+_{\boldsymbol{\delta}}(u,v;s)|.\end{aligned}$$

An analogue inequality holds for $\int\big|S^+_{\boldsymbol{\delta}}(F_{t;T}(\hat F^{-1}_{t_0,T}(\tau)),\tau;s)\big|ds$.

We now can proceed with (6.3). Note that the dependence of $\big|\hat Z^R_{t_0,T}(\boldsymbol{\delta}) - \hat Z^U_{t_0,T}(\boldsymbol{\delta},\omega,\tau)\big|$ on $s,\boldsymbol{\delta}$ and ω only has an impact on which part of the sum in $S^+_{\boldsymbol{\delta}}$ is taken into account. For any $C>0$, we have $\mathbb{I}_{\{0\le s\le \mathbf{c}'_t(\omega)\boldsymbol{\delta}\}} = \mathbb{I}_{\{0\le Cs\le C\mathbf{c}'_t(\omega)\boldsymbol{\delta}\}}$, which means that we can restrict ourselves to $\|\boldsymbol{\delta}\|_2 = 1$ and $s\in[0,\sqrt{2}]$, as $\|\mathbf{c}_t(\omega)\|_2 = \sqrt{2}$. Furthermore, if $\mathbb{I}_{\{0\le s\le \mathbf{c}'_t(\omega)\boldsymbol{\delta}_1\}} = \mathbb{I}_{\{0\le s_2\le \mathbf{c}'_t(\omega)\boldsymbol{\delta}_2\}}$ for all $t = 1,\dots,n$, then also $S^+_{\boldsymbol{\delta}_1}(u,v;s_1) = S^+_{\boldsymbol{\delta}_2}(u,v;s_2)$. Thus, we need to prove that

$$\Delta_n := \sup_{S\in\mathscr{M}_n}\sup_{\substack{\{v:|v-\tau|\le 2\|\boldsymbol{\delta}\|n^{-1/2}\}\\ \{u:|u-v|\le n^{-1/2}\log n\}}}|\bar S^+_{\boldsymbol{\delta}}| = O_{\mathbb{P}}(n^{-\frac{1}{4}\frac{\delta-1}{\delta+1}}(\log n)^2), \tag{7.19}$$

where

$$\mathscr{M}_n = \{S = \{s\in\mathcal{N}_{t_0,T} : 0\le s\le \mathbf{c}'_t(\omega)\boldsymbol{\delta}\}|\omega\in\mathcal{F}_n, s\in(0,\sqrt{2}], \|\boldsymbol{\delta}\|_2 = 1\},$$

and

$$\bar S^+_{\boldsymbol{\delta}}(u,v;S) := n^{-1/2}\sum_{t\in S}[\mathbb{I}_{\{U_{t,T}\le u\}} - u - (\mathbb{I}_{\{U_{t,T}\le v\}} - v)] =: n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}} V_{t,S}(u,v).$$

Now (7.19) coincides with equation (6.22) in Dette et al. (2014) and follows along the same lines by an application of the independent blocking technique from Lemma 7.4. □

7.3.3 Proof of (6.4)

In order to establish (6.4), we use Knight's identity again, which yields

$$\begin{aligned}&\hat Z^U_{t_0,T}(\boldsymbol{\delta},\omega_{j,n},\tau) - Z^{U,\vartheta}_{t_0,T}(\boldsymbol{\delta},\omega_{j,n},\tau)\\ &\quad= \sum_{u\in\mathcal{N}_{t_0,T}} W_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau) + \delta_1\sqrt{n}\Big(n^{-1}\sum_{t\in\mathcal{N}_{t_0,T}} F_{t;T}(\hat F^{-1}_{t_0,T}(\tau)) - G^{\vartheta}(\hat F^{-1}_{t_0,T}(\tau))\Big)\end{aligned}$$

where

$$W_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau) := H_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau) - (\mathbf{c}'_u(\omega_{j,n})\boldsymbol{\delta})^2/2n$$

with

$$H_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau) := \int_0^{n^{-1/2}\mathbf{c}'_u(\omega_{j,n})\boldsymbol{\delta}} (\mathbb{I}_{\{U_{u,T}\le s+\tau\}} - \mathbb{I}_{\{U_{u,T}\le\tau\}})ds.$$

Observe that

$$n^{-1/2}\sum_{t\in\mathcal{N}_{t_0,T}}\big(F_{t;T}(\hat{F}^{-1}_{t_0,T}(\tau)) - G^{\vartheta}(\hat{F}^{-1}_{t_0,T}(\tau))\big) = O_{\mathbb{P}}(n^{-1/2}).$$

Thus, it is sufficient to bound the sum $\sum_{u\in\mathcal{N}_{t_0,T}} W_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau)$. To this end, we apply the blocking technique from Lemma 7.4 again, to the probability

$$\mathbb{P}\Big(\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}\|_\infty\le A\log(n)}\sum_{u\in\mathcal{N}_{t_0,T}}(W_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau)) > r_n\Big)$$

with a suitable r_n (to be chosen below).

First let us show that the supremum in (6.4) can be taken over a finite number of cases. As $\#\mathcal{F}_n < n$, we only have to deal with the supremum over $\{\boldsymbol{\delta} : \|\boldsymbol{\delta}\|_\infty \le A\log(n)\}$. One can construct $n^* = o(n^5)$ points $d_1,\dots,d_{n^*}$ such that, for every $\boldsymbol{\delta}$, there exists $j(\boldsymbol{\delta})$ with the property that $\|\boldsymbol{\delta} - d_{j(\boldsymbol{\delta})}\|_\infty \le n^{-3/2}$. For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$, consider

$$\begin{aligned}\big|W_{u,T}(\boldsymbol{a},\omega_{j,n},\tau) - W_{u,T}(\boldsymbol{b},\omega_{j,n},\tau)\big| \le& 2\Big|\int_{n^{-1/2}\mathbf{c}'_u(\omega_{j,n})\boldsymbol{a}}^{n^{-1/2}\mathbf{c}'_u(\omega_{j,n})\boldsymbol{b}} ds\Big| + |2(n)^{-1}(\mathbf{c}'_u(\omega)(a-b))^2| \\ \le& Cn^{-1}\big(\|\boldsymbol{a}-\boldsymbol{b}\|n^{1/2} + \|\boldsymbol{a}-\boldsymbol{b}\|^2\big)\end{aligned}$$

and

$$\Big|\sum_{u\in\mathcal{N}_{t_0,T}} W_{u,T}(\boldsymbol{\delta},\omega_{j,n},\tau) - \sum_{u\in\mathcal{N}_{t_0,T}} W_{u,T}(d_{j(\boldsymbol{\delta})},\omega_{j,n},\tau)\Big| = O_{\mathbb{P}}(n^{-1}). \tag{7.20}$$

In order to apply Lemma 7.4, define the centered random variables

$$\bar{W}_{u,T} := W_{u,T}(d_{j(\boldsymbol{\delta})},\omega_{j,n},\tau) - \mathbb{E}(W_{u,T}(d_{j(\boldsymbol{\delta})},\omega_{j,n},\tau))$$

and obtain, from Lemma 7.6,

$$\begin{aligned}M_n = \max_{\omega\in\mathcal{F}_n}\max_{1\le j\le n^*}|\bar{W}_{u,T}| &\le \max_{\omega\in\mathcal{F}_n}\max_{1\le j\le n^*}\big(|W_{u,T}(d_{j(\boldsymbol{\delta})},\omega,\tau)| + |\mathbb{E}(W_{u,T}(d_{j(\boldsymbol{\delta})},\omega,\tau))|\big) \\ &\le C\Big(\frac{\log(n)^2}{\sqrt{n}} + \frac{\log(n)^3}{n^{3/2}}\Big) = O\Big(\frac{\log(n)^2}{\sqrt{n}}\Big) \quad \text{a.s..}\end{aligned}$$

Set

$$a_n = \lceil n^{\frac{1}{\delta+1}}\rceil, \quad \mu_n = \lfloor\frac{n}{2a_n}\rfloor \quad \text{and} \quad r_n = D\log(n)^3(n^{-\frac{1}{4}} \vee n^{-\frac{1}{2}(\frac{\delta-1}{\delta+1})})$$

so that conditions (a) $\mu_n\beta_{a_n} \xrightarrow{n\to\infty} 0$ and (b) $\frac{n}{2\mu_n}M_n \le Ca_n\frac{\log(n)^2}{\sqrt{n}} \ll r_n$ from Lemma 7.4 are satisfied. As for (c), in order to bound the variance of each block, we again refer to Lemma 7.6 and obtain, for $|t_1 - t_2| < a_n$,

$$\begin{aligned}V_n = \mathrm{Var}(\sum_{k=t_1}^{t_2}\bar{W}_{u,T}) &\le \sum_{k=t_1}^{t_2}\sum_{l=t_1}^{t_2}\mathbb{E}(\bar{W}_{k,T}\bar{W}_{l,T}) \\ &\le C\log(n)^4(a_nn^{-3/2} + (a_n^2 - a_n)n^{-2}) = O(a_nn^{-3/2}\log(n)^4).\end{aligned}$$

Since $(i)-(iii)$ and $(a)-(c)$ in Lemma 7.4 hold, we conclude that

$$\mathbb{P}\Big(\max_{\omega\in\mathcal{F}_n}\max_{1\le j\le n^*}\Big|\sum_{u\in\mathcal{N}_{t_0,T}}\bar{W}_{u,T}\Big| > r_n\Big) \le nn^*\exp\Big(-\frac{\log 2}{2}\Big(\frac{r_n^2}{4\mu_nV_n}\wedge\frac{r_n}{2a_nM_n}\Big)\Big) + o(1).$$

Because $\frac{r_n^2}{4\mu_n V_n} \wedge \frac{r_n}{2a_n M_n} > C_D \log(n)$ for n sufficiently large, we obtain from (7.20) that

$$\mathbb{P}\Big(\sup_{\omega\in\mathcal{F}_n} \sup_{\|\boldsymbol{\delta}\|_\infty \le A^\vartheta(\tau)\sqrt{\log(n)}} \sum_{u\in\mathcal{N}_{t_0,T}} W_{u,T}(\boldsymbol{\delta},\omega,\tau) > r_n\Big)$$
$$\le \mathbb{P}\Big(\max_{\omega\in\mathcal{F}_n} \max_{1\le j\le n^*} \sum_{u\in\mathcal{N}_{t_0,T}} \bar{W}_{u,T} > r_n - n\mathbb{E}(W_{u,T}(d_j(\boldsymbol{\delta},\omega,\tau))\Big) + o(1) \le n^{-D+6} + o(1),$$

which converges to zero for sufficiently large D. □

7.3.4 Proof of (6.10)

Setting

$$\frac{4}{n} r_n := (\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau_1))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau_2) - (\boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_1))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_2)$$
$$= (\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau_1) - \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_1))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_2)$$
$$+ (\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau_1))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} (\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau_2) - \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_2))$$
$$+ (\hat{\boldsymbol{b}}_{t_0,T}(\varphi_n(\omega_{j,n}),\tau_1) - \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_1))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} (\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau_2) - \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau_2)),$$

we obtain from the definition of the local Laplace periodogram that

$$\hat{L}_{t_0,T}\big(\varphi_n(\omega_{j,n}),\tau_1,\tau_2\big) = \frac{n}{4}(\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau)$$
$$= \frac{n}{4}(\boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau))' \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau) + r_n$$
$$= \frac{1}{n} d^\vartheta_{t_0,T}(-\omega_{j,n},\tau_1) d^\vartheta_{t_0,T}(-\omega_{j,n},\tau_2) + r_n.$$

To complete the proof note, that by (6.1) and (6.2), we have

$$\sqrt{n} \sup_{\omega\in\mathcal{F}_n} \|\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau) - \boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau)\| = O_{\mathbb{P}}(n^{-\frac{1}{8}\frac{\delta-1}{\delta+1}}(\log n)^{3/2})$$

and

$$\sqrt{n} \sup_{\omega\in\mathcal{F}_n} \|\boldsymbol{b}^\vartheta_{t_0,T}(\omega_{j,n},\tau)\| = O_{\mathbb{P}}(\log n),$$

which yields $\|r_n\|_\infty = O_{\mathbb{P}}(n^{-\frac{1}{8}\frac{\delta-1}{\delta+1}}(\log n)^{5/2})$. □

7.3.5 Proof of (6.12)

In order to establish (6.12), we show that, uniformly in $j,k \in \{1,...,\lfloor\frac{n-1}{2}\rfloor\}$,

(i) $\mathbb{E}\big(\mathcal{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2)\big) = 2\pi\mathfrak{f}^\vartheta(\omega_{j,n},\tau_1,\tau_2) + o(1)$

(ii) $\mathrm{Cov}\big(\mathcal{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2), \mathcal{L}_{t_0,T}(\omega_{k,n},\tau_1,\tau_2)\big) = \begin{cases} \mathfrak{f}^\vartheta(\omega_{j,n},\tau_1,\tau_1)\mathfrak{f}^\vartheta(\omega_{j,n},\tau_2,\tau_2) + o(1) & j=k \\ o(1) & j\ne k. \end{cases}$

Defining $\Delta_n(\omega_{j,n},\tau_1,\tau_2) := \sum_{|k|\le K_n} W_{t_0,T}(k)\Big[\mathcal{L}_{t_0,T}(\omega_{j+k,n},\tau_1,\tau_2) - 2\pi\mathfrak{f}^{\vartheta}(\omega_{j+k,n},\tau_1,\tau_2)\Big]$: (i) and (ii) indeed imply that

$$\mathrm{Var}(\Delta_n(\omega_{j_n,n},\tau_1,\tau_2)) = \sum_{|k|\le K_n} (W_{t_0,T})^2(k)\mathrm{Var}(\mathcal{L}_{t_0,T}(\omega_{j_n+k,n},\tau_1,\tau_2))$$
$$+ \sum_{\substack{|k_1|\le K_n, |k_2|\le K_n\\ k_1\neq k_2}} W_{t_0,T}(k_1)W_{t_0,T}(k_2)\mathrm{Cov}(\mathcal{L}_{t_0,T}(\omega_{j_n+k_1,n}\tau_1,\tau_2),\mathcal{L}_{t_0,T}(\omega_{j_n+k_2,n},\tau_1,\tau_2)) = o(1)$$

and $\mathbb{E}(\Delta_n(\omega_{j_n,n},\tau_1,\tau_2)) = o(1)$; (6.12) follows.

We start with (i). Recalling that $t_{\min} := \min\{t\in\mathcal{N}_{t_0,T}\}$, consider the representation

$$d^{\vartheta}_{t_0,T}(\omega,\tau) = \sum_{t\in\mathcal{N}_{t_0,T}} e^{-i\omega\bar{t}}(\tau - \mathbb{I}_{\{X_{t,T}\le q_{t,T}(\tau)\}}),$$

where $\bar{t} = t - t_{\min} + 1$, and its stationary approximation

$$\mathfrak{d}^{\vartheta}_{t_0,T}(\omega,\tau) := \sum_{t\in\mathcal{N}_{t_0,T}} e^{-i\omega\bar{t}}(\tau - \mathbb{I}_{\{X^{\vartheta}_t\le q^{\vartheta}(\tau)\}}).$$

From equation (7.4) we obtain

$$\begin{aligned}\mathbb{E}(\mathcal{L}_{t_0,T}(\omega_{j_n,n},\tau_1,\tau_2)) &= \mathbb{E}\big(n^{-1}d^{\vartheta}_{t_0,T}(\omega_{j_n,n},\tau_1)d^{\vartheta}_{t_0,T}(-\omega_{j_n,n},\tau_2)\big)\\ &= n^{-1}\sum_{s\in\mathcal{N}_{t_0,T}}\sum_{t\in\mathcal{N}_{t_0,T}} F_{s,t;T}(q_{s,T}(\tau_1),q_{t,T}(\tau_2))e^{i(t-s)\omega_{j_n,n}}\\ &= n^{-1}\sum_{s\in\mathcal{N}_{t_0,T}}\sum_{t\in\mathcal{N}_{t_0,T}} G^{\vartheta}_{s-t}(q^{\vartheta}(\tau_1),q^{\vartheta}(\tau_2);\vartheta)e^{i(t-s)\omega_{j_n,n}} + o(1)\\ &= \mathbb{E}\big(n^{-1}\mathfrak{d}^{\vartheta}_{t_0,T}(-\omega_{j,n},\tau_1)\mathfrak{d}^{\vartheta}_{t_0,T}(-\omega_{j_n,n},\tau_2\big) + o(1),\end{aligned}$$

and Theorem 4.3.2 from Brillinger (1975) yields

$$\mathbb{E}\big(n^{-1}\mathfrak{d}^{\vartheta}_{t_0,T}(-\omega_{j_n,n},\tau_1)\mathfrak{d}^{\vartheta}_{t_0,T}(-\omega_{j_n,n},\tau_2)\big) = 2\pi\mathfrak{f}^{\vartheta}(\omega_{j_n,n},\tau_1,\tau_2) + O(n^{-1}),$$

which establishes (i).

Turning to (ii), set $Y_{t,T}(\tau) = \tau - \mathbb{I}_{\{X_{t,T}\le q_{t,T}(\tau)\}}$; we have

$$\begin{aligned}&\mathrm{Cov}\big(\mathcal{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2),\mathcal{L}_{t_0,T}(\omega_{k,n},\tau_1,\tau_2)\big)\\ &= n^{-2}\mathbb{E}\Big[d^{\vartheta}_{t_0,T}(-\omega_{j,n},\tau_1)d^{\vartheta}_{t_0,T}(\omega_{j,n},\tau_2)\overline{d^{\vartheta}_{t_0,T}(-\omega_{k,n},\tau_1)d^{\vartheta}_{t_0,T}(\omega_{k,n},\tau_2)}\Big]\\ &\qquad - \mathbb{E}\Big[d^{\vartheta}_{t_0,T}(-\omega_{j,n},\tau_1)d^{\vartheta}_{t_0,T}(\omega_{j,n},\tau_2)]\mathbb{E}[\overline{d^{\vartheta}_{t_0,T}(-\omega_{k,n},\tau_1)d^{\vartheta}_{t_0,T}(\omega_{k,n},\tau_2)}\Big]\\ &= n^{-2}\sum_{t_1,t_2,t_3,t_4\in\mathcal{N}_{t_0,T}}\big(\mathbb{E}[Y_{t_1,T}(\tau_1)Y_{t_2,T}(\tau_2)Y_{t_3,T}(\tau_1)Y_{t_4,T}(\tau_2)] - \mathbb{E}[Y_{t_1,T}(\tau_1)Y_{t_2,T}(\tau_2))\mathbb{E}(Y_{t_3,T}(\tau_1)Y_{t_4,T}(\tau_2)]\big)\\ &\qquad\qquad \times\exp(i\omega_{j,n}(t_2-t_1) + i\omega_{k,n}(t_3-t_4))\\ &= n^{-2}\sum_{t_1,t_2,t_3,t_4\in\mathcal{N}_{t_0,T}}\mathrm{cum}(Y_{t_1,T}(\tau_1)Y_{t_2,T}(\tau_2)Y_{t_3,T}(\tau_1)Y_{t_4,T}(\tau_2))e^{i\omega_{j,n}(t_2-t_1)+i\omega_{k,n}(t_3-t_4)}\\ &\quad + n^{-2}\sum_{t_1,t_2,t_3,t_4\in\mathcal{N}_{t_0,T}}\mathbb{E}(Y_{t_2,T}(\tau_2)Y_{t_3,T}(\tau_1))\mathbb{E}(Y_{t_1,T}(\tau_1)Y_{t_4,T}(\tau_2))e^{i\omega_{j,n}(t_2-t_1)+i\omega_{k,n}(t_3-t_4)}\\ &\quad + n^{-2}\sum_{t_1,t_2,t_3,t_4\in\mathcal{N}_{t_0,T}}\mathbb{E}(Y_{t_2,T}(\tau_2)Y_{t_4,T}(\tau_2))\mathbb{E}(Y_{t_1,T}(\tau_1)Y_{t_3,T}(\tau_1))e^{i\omega_{j,n}(t_2-t_1)+i\omega_{k,n}(t_3-t_4)}\\ &=: C_1 + C_2 + C_3, \quad \text{say.}\end{aligned}$$

An application of Lemma 7.1 (note that, by Assumption ($A6$), the triangular array $(Y_{t,T}(\tau))$ is β-mixing and satisfies $\alpha(T) \le \beta(T) = o(T^{-\delta})$) yields

$$n^{-2} \sum_{t_1,t_2,t_3,t_4 \in \mathcal{N}_{t_0,T}} |\mathrm{cum}(Y_{t_1,T}(\tau_1)Y_{t_2,T}(\tau_2)Y_{t_3,T}(\tau_1)Y_{t_4,T}(\tau_2))|$$
$$\le n^{-2} \sum_{t_1,t_2,t_3,t_4 \in \mathcal{N}_{t_0,T}} C\alpha(\lfloor \max(t_i - t_j)/4 \rfloor) = n^{-2} \sum_{m=0}^{n-1} \sum_{\max(t_i - t_j) = m} C\alpha(m/4)$$

where he right-hand side converges to zero because, in view of ($A6$), $\delta \ge 2$ and

$$\#\{t \in \mathcal{N}_{t_0,T}^4 \,|\, \max(t_i - t_j) = m\} \le 3nm^2.$$

Therefore, C_1 is $o(1)$, uniformly in j. For C_2 and C_3, consider their stationary approximations. Straightforward calculations and (7.4) yield

$$\begin{aligned} n^{-1} \sum_{s \in \mathcal{N}_{t_0,T}} \sum_{t \in \mathcal{N}_{t_0,T}} \mathbb{E}(Y_{s,T}(\tau_j)Y_{t,T}(\tau_k))e^{i(\omega\bar{t}+\mu\bar{s})} &= n^{-1} \sum_{s \in \mathcal{N}_{t_0,T}} \sum_{t \in \mathcal{N}_{t_0,T}} [F_{s,t;T}(q_{s,T}(\tau_j), q_{t,T}(\tau_k)) - \tau_j\tau_k]e^{i(\omega\bar{t}+\mu\bar{s})} \\ &= n^{-1} \sum_{s \in \mathcal{N}_{t_0,T}} \sum_{t \in \mathcal{N}_{t_0,T}} [G^{\vartheta}_{s,t}(q^{\vartheta}(\tau_j), q^{\vartheta}(\tau_k)) - \tau_j\tau_k]e^{i(\omega\bar{t}+\mu\bar{s})} + o(1) \\ &= n^{-1}\mathrm{cum}(\mathfrak{d}^{\vartheta}_{t_0,T}(\omega,\tau_j)), \mathfrak{d}^{\vartheta}_{t_0,T}(\mu,\tau_k)) + o(1), \end{aligned}$$

where the second equality follows from equation (7.4). Applying Theorem 4.3.2 from Brillinger (1975) again, we obtain, uniformly in $\omega, \mu \in \mathcal{F}_n$,

$$n^{-1}\mathrm{cum}(\mathfrak{d}^{\vartheta}_{t_0,T}(\omega,\tau_j)), \mathfrak{d}^{\vartheta}_{t_0,T}(\mu,\tau_k)) = \begin{cases} O(1/n) & \omega \ne -\mu \\ 2\pi\mathfrak{f}^{\vartheta}(\omega,\tau_j,\tau_k) + O(n^{-1}) & \omega = -\mu \end{cases}$$

so that (ii) is established. □

7.4 Details for Remark 5.1

Denote by

$$\mathring{\mathfrak{f}}^{\vartheta}(\omega,\tau_1,\tau_2) := \frac{\mathfrak{f}^{\vartheta}(\omega,\tau_1,\tau_2)}{g^{\vartheta}(q^{\vartheta}(\tau_1))g^{\vartheta}(q^{\vartheta}(\tau_2))}$$

the *rescaled time-varying spectral density.* The following two results give the asymptotic distribution of $\mathring{L}$ and show the consistency of a corresponding smoothed version.

Theorem 7.1. *Let $\Omega := \{\omega_1, \dots, \omega_\nu\} \subset (0,\pi)$ and $\mathcal{T} := \{\tau_1, \dots, \tau_p\} \subset (0,1)$ denote a ν-tuple of distinct frequencies and a p-tuple of distinct quantile orders, respectively. Let Assumptions ($A1$)-($A4$) be satisfied with ($A2$) and ($A3$) holding for every $\tau \in \mathcal{T}$. If, for T tending to infinity, $n \to \infty$, $nT^{-1/2} \to 0$ and $|t_0/T - \vartheta| = o(T^{-1/2})$, then $(\mathring{L}_{t_0,T}(\omega_1,\tau_1,\tau_2), \dots, \mathring{L}_{t_0,T}(\omega_\nu,\tau_1,\tau_2))$ converges in distribution, as $T \to \infty$, to $(\mathring{L}^{\vartheta}(\omega_1,\tau_1,\tau_2), \dots, \mathring{L}^{\vartheta}(\omega_\nu,\tau_1,\tau_2))$, where the random variables $\mathring{L}^{\vartheta}(\omega,\tau_1,\tau_2)$ associated with distinct frequencies are mutually independent and*

$$\mathring{L}^{\vartheta}(\omega,\tau_1,\tau_2) \stackrel{D}{=} \begin{cases} \pi\mathring{\mathfrak{f}}^{\vartheta}(\omega,\tau_1,\tau_2)\chi_2^2 & \text{if} \quad \tau_1 = \tau_2 \\ \frac{1}{4}(Z_{11}, Z_{12}) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} Z_{21} \\ Z_{22} \end{pmatrix} & \text{if} \quad \tau_1 \ne \tau_2, \end{cases}$$

where $(Z_{11}, Z_{12}, Z_{21}, Z_{22})' \sim \mathcal{N}(0, \mathbf{\Sigma}^{\vartheta}(\omega))$ *with covariance matrix*

$$\mathbf{\Sigma}^{\vartheta}(\omega, \tau_1, \tau_2) := 4\pi \begin{pmatrix} \mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_1) & 0 & \Re(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) & \Im(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) \\ 0 & \mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_1) & -\Im(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) & \Re(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) \\ \Re(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) & -\Im(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) & \mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_2, \tau_2) & 0 \\ \Im(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) & \Re(\mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2)) & 0 & \mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_2, \tau_2). \end{pmatrix}.$$

Proposition 7.1. *Under the assumptions of Theorem* 7.1 *and if* (*A4*)−(*A6*) *hold, then the smoothed periodogram*

$$\hat{f}^{S}_{t_0,T}(\omega, \tau_1, \tau_2) := \sum_{|k| \le K_n} W_{t_0,T}(k) \mathring{L}_{t_0,T}\Big(\phi_n(\omega) + \frac{2\pi k}{n}, \tau_1, \tau_2\Big)$$

is consistent. More precisely, for any fixed $\tau_1, \tau_2 \in (0,1), \omega \in (0, \pi)$

$$\hat{f}^{S}_{t_0,T}(\omega, \tau_1, \tau_2) = 2\pi \mathring{\mathfrak{f}}^{\vartheta}(\omega, \tau_1, \tau_2) + o_{\mathbb{P}}(1) \quad \textit{as } T \to \infty.$$

Proof of Theorem 7.1 and Proposition 7.1

The proofs are similar to those of Theorem 5.1 and Proposition 5.1, but somewhat simpler. For this reason we only provide an outline of the main arguments. The key idea is a linearization of $\mathring{\boldsymbol{b}}_{t_0,T}(\omega_{j,n}, \tau)$. For any $\tau \in (0,1), \omega \in (0, \pi), \boldsymbol{\delta} \in \mathbb{R}^3$ and $\vartheta \in (0,1)$, define the functions

$$\hat{Z}_{t_0,T}(\boldsymbol{\delta}, \omega, \tau) := \sum_{u \in \mathcal{N}_{t_0,T}} \rho_\tau \Big[X_{u,T} - q_{u,T}(\tau) - n^{-1/2}\mathbf{c}_u'(\omega)\boldsymbol{\delta}\Big] - \rho_\tau \left[X_{u,T} - q_{u,T}(\tau)\right].$$

$$Z^{\vartheta}_{t_0,T}(\boldsymbol{\delta}, \omega, \tau) := -\boldsymbol{\delta}'\boldsymbol{\zeta}_{t_0,T}(\omega, \tau) + \frac{1}{2}\boldsymbol{\delta}'\boldsymbol{Q}^{\vartheta}_{t_0,T}(\omega, \tau)\boldsymbol{\delta},$$

where

$$\boldsymbol{\zeta}_{t_0,T}(\omega, \tau) := \frac{1}{\sqrt{n}} \sum_{u \in \mathcal{N}_{t_0,T}} \mathbf{c}_u(\omega)(\tau - \mathbb{I}_{\{X_{u,T} \le q_{u,T}(\tau)\}})$$

$$\boldsymbol{Q}^{\vartheta}_{t_0,T}(\omega, \tau) := n^{-1} g^{\vartheta}(q^{\vartheta}(\tau)) \sum_{u \in \mathcal{N}_{t_0,T}} \mathbf{c}_u(\omega)\mathbf{c}_u'(\omega),$$

and $g^{\vartheta}(x)$ is the density of the strictly stationary approximating process $\{X^{\vartheta}_k\}_{k \in \mathbb{Z}}$ from Definition 2.1. For $\omega \in \mathcal{F}_n$ and T large enough, the matrix $\boldsymbol{Q}^{\vartheta}_{t_0,T}(\omega, \tau)$ equals

$$\boldsymbol{Q}^{\vartheta}_{t_0,T}(\omega, \tau) = g^{\vartheta}(q^{\vartheta}(\tau)) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{7.21}$$

It follows from Definition (5.4) that the components of $n^{1/2}\mathring{\boldsymbol{b}}_{t_0,T}(\omega_{j,n}, \tau)$ coincide with the last two components of $\mathring{\boldsymbol{\delta}}_{t_0,T}(\omega, \tau) := \operatorname{argmin}_{\boldsymbol{\delta} \in \mathbb{R}^3} \hat{Z}_{t_0,T}(\boldsymbol{\delta}, \omega, \tau)$. Next, show that $\mathring{\boldsymbol{\delta}}_{t_0,T}(\omega, \tau)$ is in probability close to

$$\mathring{\boldsymbol{\delta}}^{\vartheta}_{t_0,T}(\omega, \tau) := \operatorname*{argmin}_{\boldsymbol{\delta} \in \mathbb{R}^3} Z^{\vartheta}_{t_0,T}(\boldsymbol{\delta}, \omega, \tau) = (\boldsymbol{Q}^{\vartheta}_{t_0,T}(\omega, \tau))^{-1}\boldsymbol{\zeta}_{t_0,T}(\omega, \tau);$$

more precisely,

$$\sup_{\omega \in \mathcal{F}_n} \|\mathring{\boldsymbol{\delta}}_{t_0,T}(\omega, \tau) - \mathring{\boldsymbol{\delta}}^{\vartheta}_{t_0,T}(\omega, \tau)\| = O_{\mathbb{P}}\left(\log(n)(n^{-\frac{1}{8}} \vee n^{-\frac{1}{4}(\frac{\delta-1}{\delta+1})})\right). \tag{7.22}$$

To see this, note that, for fixed ϑ, Lemma 6.1 from Dette et al. (2014) applies, and therefore (7.22) holds if

$$\sup_{\omega\in\mathcal{F}_n}\sup_{\|\boldsymbol{\delta}-\mathring{\boldsymbol{\delta}}^{\vartheta}_{t_0,T}(\omega,\tau)\|\leq\epsilon}|\hat{Z}_{t_0,T}(\boldsymbol{\delta},\omega,\tau)-Z^{\vartheta}_{t_0,T}(\boldsymbol{\delta},\omega,\tau)|=O_{\mathbb{P}}\left(\log(n)^2(n^{-\frac{1}{4}}\vee n^{-\frac{1}{2}(\frac{\delta-1}{\delta+1})})\right). \tag{7.23}$$

Now, (7.23) follows by the same arguments as in the proof of (6.4), and thus (7.22) is established. Theorem 7.1 results from the linearization (7.22) by arguments similar to those considered in the proof of Theorem 5.1.

For Proposition 7.1, we proceed as in the proof of Proposition 5.1. Define $\mathcal{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2)$ as in (6.9) and, instead of equation (6.10), we show that

$$\mathring{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2)=\frac{\mathcal{L}_{t_0,T}(\omega_{j,n},\tau_1,\tau_2)}{g^{\vartheta}(q^{\vartheta}(\tau_1))g^{\vartheta}(q^{\vartheta}(\tau_2))}+O_{\mathbb{P}}(n^{-\frac{1}{8}\frac{\delta-1}{\delta+1}}(\log n)^{3/2}),$$

which follows along the same lines as in Section 7.3.4, by substituting $\mathring{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau)$ for $\hat{\boldsymbol{b}}_{t_0,T}(\omega_{j,n},\tau)$ and using equation (7.22) instead of (6.1). The rest of the proof goes as in Proposition 5.1. □

7.5 Assumptions and proof of Lemma 5.1

Assume that there exist functions $a(\cdot,j):(0,1)\to\mathbb{R}$ with

$$\sup_{t,T}|a_{t,T}(j)-a(\frac{t}{T},j)|\leq\frac{K}{Tl(j)},\quad \sup_{\vartheta\in(0,1)}\left|\frac{\partial a(\vartheta,j)}{\partial\vartheta}\right|\leq\frac{K}{l(j)},\quad\text{and}\quad\sup_{\vartheta\in(0,1)}\left|\frac{\partial\mu(\vartheta)}{\partial\vartheta}\right|\leq K \tag{7.24}$$

where K is a finite constant not depending on j and

$$l(j)=\begin{cases}1 & \text{if } |j|\leq 1\\ |j|\log^{1+\kappa}|j| & \text{if } |j|>1\end{cases} \tag{7.25}$$

for some $\kappa>0$. Then we can construct approximating processes by

$$X_t^{\vartheta}=\mu(\vartheta)+\sum_{j=0}^{\infty}a(\vartheta,j)\xi_{t-j}.$$

Let the random variables ξ_t have bounded density function f_ξ and finite expectation: $\mathbb{E}(|\xi_t|)<\infty$. Additionally we need the following technical assumptions: for some $y_0<\infty$ there exists $K<\infty$ such that

$$f_\xi(y)\leq K|y|^{-1}\quad\forall|y|\geq y_0, \tag{7.26}$$

$$\sup_{\vartheta\in(0,1)}\sum_{j=0}^{\infty}|a(\vartheta,j)|<\infty\quad\text{and}\quad\inf_{\vartheta\in(0,1)}|a(\vartheta,0)|>\delta>0. \tag{7.27}$$

Without loss of generality, we can assume that $\mu(\vartheta)=0$. Writing the distribution functions in terms of expectations, we obtain

$$\begin{aligned}&F_{s,t;T}(x,y)-G^{\vartheta}_{s-t}(x,y)\\&=\mathbb{E}\left[\mathbb{I}_{\{X_{s,T}\leq x\}}\mathbb{I}_{\{X_{t,T}\leq y\}}\right]-\mathbb{E}\left[\mathbb{I}_{\{X_s^{\vartheta}\leq x\}}\mathbb{I}_{\{X_t^{\vartheta}\leq y\}}\right]\\&=\mathbb{E}\left[\mathbb{I}_{\{X_{s,T}\leq x\}}(\mathbb{I}_{\{X_{t,T}\leq y\}}-\mathbb{I}_{\{X_t^{\vartheta}\leq y\}})\right]+\mathbb{E}\left[\mathbb{I}_{\{X_t^{\vartheta}\leq y\}}(\mathbb{I}_{\{X_{s,T}\leq x\}}-\mathbb{I}_{\{X_s^{\vartheta}\leq x\}})\right].\end{aligned}$$

To bound the first part of the sum, denote by σ_t the $\sigma-$field generated by the random variables $\{\xi_i | i \leq t\}$. Additionally, denote by F_ξ and f_ξ the distribution function of ξ and its density, respectively. We have

$$\begin{aligned}
&\Big|\mathbb{E}\big[\mathbb{I}_{\{X_{s,T}\leq x\}}(\mathbb{I}_{\{X_{t,T}\leq y\}} - \mathbb{I}_{\{X_t^\vartheta \leq y\}})\big]\Big| \\
&\quad \leq \mathbb{E}\big[\mathbb{E}(|\mathbb{I}_{\{X_{t,T}\leq y\}} - \mathbb{I}_{\{X_t^\vartheta \leq y\}}| \big| \sigma_{t-1})\big] \\
&\quad \leq \mathbb{E}\big[\mathbb{E}(|\mathbb{I}_{\{\xi_t \leq \frac{1}{a_{t,T}(0)}\{y - \sum_{j=1}^\infty a_{t,T}(j)\xi_{t-j}\}\}} - \mathbb{I}_{\{\xi_t \leq \frac{1}{a(\vartheta,0)}\{y - \sum_{j=1}^\infty a(\vartheta,j)\xi_{t-j}\}\}}| \big| \sigma_{t-1})\big] \\
&\quad = \mathbb{E}[|F_\xi(\frac{1}{a_{v,T}(0)}\{y - \sum_{j=1}^\infty a_{t,T}(j)\xi_{t-j}\}) - F_\xi(\frac{1}{a(\vartheta,0)}\{y - \sum_{j=1}^\infty a(\vartheta,j)\xi_{t-j}\})|] \\
&\quad \leq \mathbb{E}[f_\xi(\eta)|y|]\Big|\frac{1}{a_{t,T}(0)} - \frac{1}{a(\vartheta,0)}\Big| + C_f \mathbb{E}[|S_{t,T} - S_t^\vartheta|],
\end{aligned}$$

where C_f is an upper bound for the density f_ξ,

$$S_{t,T} := \frac{1}{a_{t,T}(0)}\sum_{j=1}^\infty a_{t,T}(j)\xi_{t-j}, \quad S_t^\vartheta := \frac{1}{a(\vartheta,0)}\sum_{j=1}^\infty a(\vartheta,j)\xi_{t-j},$$

and η_y denotes some intermediate point between $y/a_{t,T}(0)+S_{t,T}$ and $y/a(\vartheta,j)+S_t^\vartheta$. Straightforward calculations, under the assumptions made, lead to

$$\mathbb{E}[|S_{t,T} - S_t^\vartheta|] = \mathrm{O}(|t - \vartheta T^{-1}| + T^{-1})$$

and

$$\Big|\frac{1}{a_{t,T}(0)} - \frac{1}{a(\vartheta,0)}\Big| = \mathrm{O}(T^{-1}).$$

It thus remains to establish that

$$\sup_{y\in\mathbb{R}} \mathbb{E}[f_\xi(\eta_y)]|y| < \infty. \tag{7.28}$$

For this purpose, define $W := \max(|S_{v,T}|, |S_v^\vartheta|)$ and note that the inequality

$$(y/a_{v,T}(0) + S_{t,T})(y/a(\vartheta,0) + S_v^\vartheta) < 0$$

implies $W > |y/\max(a_{t,T}(0), a(\vartheta,j)|$. As the density f_ξ is bounded by a constant C_f, (7.28) follows via an application of the Markov inequality. On the other hand, assuming that

$$(y/a_{t,T}(0) + S_{t,T})(y/a(\vartheta,0) + S_t^\vartheta) > 0$$

and choosing T sufficiently large that $\frac{1}{2}|a_{t,T}(0)| \leq |a(\vartheta,0)| \leq 2|a_{t,T}(0)|$, we can bound $|\eta_y|$ through

$$|y/2a_{t,T}(0)| - W \leq |\eta_y| \leq |2y/a_{t,T}(0)| + W.$$

In this case we write $\mathbb{E}[f_\xi(\eta)]|y| = \mathbb{E}[|y|f_\xi(\eta)\mathbb{I}_{\{|\eta|\geq y_0\}}] + \mathbb{E}[|y|f_\xi(\eta)\mathbb{I}_{\{|\eta|<y_0\}}] =: E_1 + E_2$, say. For E_2, since $|y| \leq 2|a_{v,T}(0)|(y_0 + W)$ whenever $|\eta| \leq y_0$, we obtain

$$E_2 \leq C_f \mathbb{E}[2|a_{v,T}(0)|(y_0 + W)] < \infty.$$

As for E_1, let us split it further into

$$E_1 \leq \mathbb{E}[|y|f_\xi(\eta)\mathbb{I}_{\{|\eta|\geq y_0\}}\mathbb{I}_{\{|y|\leq 4|a_{t,T}(0)|W\}}] + \mathbb{E}[|y|f_\xi(\eta)\mathbb{I}_{\{|\eta|\geq y_0\}}\mathbb{I}_{\{|y|>4|a_{t,T}(0)|W\}}] =: E_{11} + E_{12}.$$

The first term E_{11} is easily bounded by $C_f \mathbb{E}[4|a_{t,T}(0)|W] < \infty$. We now apply assumption (7.26) and get

$$E_{12} \leq |y| \mathbb{E}\Big[\frac{K}{|\eta_y|} \mathbb{I}_{\{|y|>4|a_{t,T}(0)|W\}} \Big] \leq \frac{2|a_{t,T}|(0)K|y|}{|y|} \mathbb{E}\Big[\frac{1}{1 - \frac{2|a_{t,T}(0)|W}{|y|}} \mathbb{I}_{\{|y|>4|a_{t,T}(0)|W\}} \Big]$$
$$\leq 4|a_{t,T}(0)|K|y|/|y| < \infty.$$

Therefore, $\sup_{y\in\mathbb{R}} \mathbb{E}[f_\xi(\eta_y)]|y| < \infty$, which leads to

$$\left| \mathbb{E}\left[\mathbb{I}_{\{X_{s,T}\leq x\}} (\mathbb{I}_{\{X_{t,T}\leq y\}} - \mathbb{I}_{\{X_t^\vartheta \leq y\}}) \right] \right| \leq \mathrm{O}(|t/T - \vartheta| + T^{-1}).$$

With the same arguments, we obtain

$$\left| \mathbb{E}\left[\mathbb{I}_{\{X_t^\vartheta \leq y\}} (\mathbb{I}_{\{X_{s,T}\leq x\}} - \mathbb{I}_{\{X_s^\vartheta \leq x\}}) \right] \right| = \mathrm{O}(|s/T - \vartheta| + T^{-1}).$$

Combining these two inequalities yields

$$\|F_{s,t;T} - G_{s-t}^\vartheta\|_\infty = O\left(|\max(s/T - \vartheta|, |t/T - \vartheta|) + \frac{1}{T} \right),$$

which completes the proof. □