Range-Based Estimation of Quadratic Variation*

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Abstract

This paper proposes using realized range-based estimators to draw inference about the quadratic variation of jump-diffusion processes. We also construct a range-based test of the hypothesis that an asset price has a continuous sample path. Simulated data shows that our approach is efficient, the test is well-sized and more powerful than a return-based t-statistic for sampling frequencies normally used in empirical work. Applied to equity data, we show that the intensity of the jump process is not as high as previously reported.

JEL Classification: C10; C22; C80.

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1. Introduction

Modern financial econometrics has largely been developed from the presumption that return-generating processes have continuous sample paths. The workhorse of both applied and theoretical papers is the continuous time stochastic volatility model. These models, however, are contrasted by the many abrupt changes found in empirical data, and a series of recent papers has therefore estimated jump-diffusion processes and/or proposed jump detection tests using i) low-frequency data (e.g., Aït-Sahalia (2002), Andersen, Benzoni & Lund (2002), Pan (2002), Chernov, Gallant, Ghysels & Tauchen (2003), Eraker, Johannes & Polson (2003), Johannes (2004)), or ii) high-frequency data (e.g., Barndorff-Nielsen & Shephard (2004, 2006), henceforth BN-S, Huang & Tauchen (2005), Jiang & Oomen (2005), Andersen, Bollerslev & Diebold (2006)).

Information in high-frequency data, in particular, has provided strong support for jumps in asset prices. The jump component appears to account for a significant proportion of quadratic variation. An asymptotic distribution theory for the preferred test was derived in BN-S (2006), which is based on the ratio of realized variance and bipower variation, suitably normalized. In the presence of microstructure noise, these return-based statistics are often sampled at a moderate frequency to reduce the impact of the noise (e.g., sampling 5-minute returns). This principle, of course, entails a loss of information and much research has been devoted to develop estimators that are more robust to microstructure noise (e.g., Zhang, Mykland & Aït-Sahalia (2004) or Barndorff-Nielsen, Hansen, Lunde & Shephard (2006), among others).

In this paper, we propose a framework using realized range-based estimators to draw inference about the quadratic variation, and we construct a new non-parametric test for jump detection. The motivation for using the range is that intraday range-based estimation of integrated variance is very efficient (see, e.g., Parkinson (1980), Christensen & Podolskij (2006) or Dijk & Martens (2006)). By replacing returns with ranges, we can extract most of the information contained in the data not used by a sparsely sampled realized variance and bipower variation, but without inducing more microstructure noise or relying on complicated corrections to reduce its impact. Hence, we would expect that range-based inference about the jump component is powerful. The properties of the high-low has, however, been neglected in the context of jump-diffusion processes.

Our paper makes several contributions. First, we extend the asymptotic results on the realized range-based variance in Christensen & Podolskij (2006) to cover the jump-diffusion setting. It turns out that this estimator is inconsistent for the quadratic variation of these processes. Second, we introduce range-based bipower variation, derive its probability limit, and asymptotic distribution under the null of a continuous sample path. Third, we use range-based bipower variation to modify the realized range-based variance and restore consistency for the quadratic variation. Fourth, we develop a range-based test of the hypothesis of no jump component.

The paper proceeds as follows. In section 2, we set notation and invoke a standard arbitrage-free continuous time semimartingale framework. We review the theory of realized variance within jump-diffusion models and then switch to realized range-based variance. In section 3, we conduct a Monte Carlo study to inspect the finite sample properties of range-based bipower variation and the new jump detection test. In section 4, we progress with some empirical results using high-frequency data from New York Stock Exchange (NYSE). In section 5, we conclude and offer directions for future research. An appendix contains the derivations of our results.

2. A Jump-Diffusion Semimartingale

In this section, we propose a non-parametric method based on the price range for consistently estimating the components of quadratic variation. Moreover, we introduce a new test for drawing inference about the jump part. The theory is developed for a univariate log-price, say $p = (p_t)_{t \geq 0}$, defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$. p evolves in continuous time and is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, i.e. a collection of σ-fields with $\mathcal{F}_u \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $u \leq t < \infty$.

Throughout the paper, we assume that p is a member of the class of jump-diffusion semimartingales that satisfy the generic representation:[1]

$$p_t = p_0 + \int_0^t \mu_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + \sum_{i=1}^{N_t} J_i, \qquad \textbf{(2.1)}$$

where $\mu = (\mu_t)_{t \geq 0}$ is locally bounded and predictable, $\sigma = (\sigma_t)_{t \geq 0}$ is càdlàg, $W = (W_t)_{t \geq 0}$ a standard Brownian motion, $N = (N_t)_{t \geq 0}$ a finite-activity simple counting process, and $J = \{J_i\}_{i=1,\ldots,N_t}$ is a sequence of non-zero random variables.[2] Equation **(2.1)** with $N = 0$ is called a Brownian semimartingale and we write $p \in BSM$ to reflect this in the following.

We assume that high-frequency data are available through $[0, t]$, which is the sampling period and is thought of as being a trading day. At sampling times t_{i-1} and t_i, such that $0 \leq t_{i-1} \leq t_i \leq t$, we define the intraday return of p over $[t_{i-1}, t_i]$ by:

$$r_{t_i, \Delta_i} = p_{t_i} - p_{t_{i-1}}, \qquad \textbf{(2.2)}$$

where $\Delta_i = t_i - t_{i-1}$.

[1] Asset prices must be semimartingales under rather weak conditions (e.g., Back (1991)).

[2] A simple counting process, N, is of finite-activity provided that $N_t < \infty$ for $t \geq 0$, almost surely. In this paper, we do not explore infinite-activity processes, although these models have been studied in the context of realized multipower variation (e.g., Barndorff-Nielsen, Shephard & Winkel (2006) or Woerner (2004*a*, 2004*b*)).

With this notation, we can introduce the object of interest; the quadratic variation. The theory of stochastic integration states that this process exists for all semimartingales. Its relevance to financial economics is stressed in several papers (e.g., Andersen, Bollerslev & Diebold (2002)). The definition of quadratic variation is given by:

$$\langle p \rangle_t = \operatorname*{p-lim}_{n\to\infty} \sum_{i=1}^{n} r_{t_i,\Delta_i}^2, \tag{2.3}$$

for any sequence of partitions $0 = t_0 < t_1 < \ldots < t_n = t$ such that $\max_{1\le i\le n}\{\Delta_i\} \to 0$ as $n \to \infty$ (e.g., Protter (2004, pp. 66-77)). In our setting, $\langle p \rangle_t$ reduces to:

$$\langle p \rangle_t = \int_0^t \sigma_u^2 \mathrm{d}u + \sum_{i=1}^{N_t} J_i^2, \tag{2.4}$$

the integrated variance and squared jumps.

The econometric problem is that $\langle p \rangle_t$ is latent. We will estimate $\langle p \rangle_t$, its two components and test $H_0 : p \in BSM$ against $H_a : p \notin BSM$ from discrete high-frequency data. The basis for our analysis is an equidistant grid $t_i = i/mn$, $i = 0, 1, \ldots, [mnt]$, where n is the sampling frequency and $[x]$ is the integer part of x.[3] We then construct intraday returns and ranges:

$$r_{i\Delta,\Delta} = p_{i/n} - p_{(i-1)/n}, \tag{2.5}$$

$$s_{p_{i\Delta,\Delta},m} = \max_{(i-1)/n\le s,t\le i/n} \{p_t - p_s\}, \tag{2.6}$$

for $i = 1, \ldots, [nt]$. Below, we also use the range of a standard Brownian motion, which is denoted by $s_{W_{i\Delta,\Delta},m}$, simply replacing p with W in Equation **(2.6)**.

Our assumptions on the data imply that each interval $[(i-1)/n, i/n]$ contains $m + 1$ ultra high-frequency recordings of p at time points $t_{(i-1)/n+j/mn}$, $j = 0, 1, \ldots, m$. Of course, the notation $s_{p_{i\Delta,\Delta},m}$ reflects that each range is based on the corresponding m returns. Note that $r_{i\Delta,\Delta}$ is not exhausting the data, which motivates our approach. This is related to market microstructure noise and will be discussed below. We do not explicitly model the noise in this paper. Instead, we assume that n is chosen such that potential biases from the noise can be ignored.[4]

[3] In practice, high-frequency data are irregularly spaced and equidistant prices are imputed from the observed ones. Two approaches are linear interpolation (e.g., Andersen & Bollerslev (1997)) or the previous-tick method suggested by Wasserfallen & Zimmermann (1985). The former has an unfortunate property in connection with quadratic variation, see Hansen & Lunde (2006, Lemma 1).

[4] A short remark about the setup is appropriate. In the appendix, we start by assuming that m is infinity. In that setting, we suppress dependence on m to write $s_{p_{i\Delta,\Delta}} = \sup_{(i-1)/n\le s,t\le i/n} \{p_t - p_s\}$, using the same convention for $s_{W_{i\Delta,\Delta}}$. We then relax this to finite m. Moreover, we only assume that $[0, t]$ is divided into $[nt]$ equidistant subintervals $[(i-1)/n, i/n]$, $i = 1, \ldots, [nt]$, for simplicity. The asymptotic results for irregular $[t_{i-1}, t_i]$ can be derived as in Christensen & Podolskij (2006). Under suitable conditions, we can also allow for varying number of points and positions in the subintervals. Finally, m can be a function of n, but this dependence is also dropped for notational ease.

2.1. Realized Variance and Bipower Variation

The availability of high-frequency data in financial economics has inspired the development of a powerful toolkit for measuring the variation of asset price processes. Under the heading realized multipower variation, this framework builds on powers of absolute returns over non-overlapping intervals (e.g., BN-S (2007)).

More formally, we define realized multipower variation by setting:

$$MPV^n_{(r_1,\ldots,r_k),t} = n^{r_+/2-1} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} \frac{1}{\mu_{r_j}} |r_{(i+j-1)\Delta,\Delta}|^{r_j}, \qquad \textbf{(2.7)}$$

with $k \in \mathbb{N}$, $r_j \geq 0$ for all j, $r_+ = \sum_{j=1}^{k} r_j$, $\mu_{r_j} = \mathbb{E}(|\,\phi\,|^{r_j})$, and $\phi \sim N(0,1)$.[5]

Equation (**2.7**) boils down to many econometric estimators for suitable choices of k and the r_k's. The most popular is *realized variance* ($k = 1$ and $r_1 = 2$):

$$RV^n_t = \sum_{i=1}^{[nt]} r^2_{i\Delta,\Delta}. \qquad \textbf{(2.8)}$$

RV^n_t is the sum of squared returns and by definition consistent for $\langle\, p\,\rangle_t$ of all semimartingales as $n \to \infty$. It follows from Equation (**2.4**) that:

$$RV^n_t \xrightarrow{p} \int_0^t \sigma^2_u \mathrm{d}u + \sum_{i=1}^{N_t} J^2_i. \qquad \textbf{(2.9)}$$

RV^n_t measures the total variation induced by the diffusive and jump component. BN-S (2004) introduced (realized) *bipower variation* that can be used to separate these parts. The estimator was extended in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) to weaker conditions. The (first-order) bipower variation is defined as ($k = 2$, $r_k = 1$):

$$BV^n_t = \frac{1}{\mu_1^2} \sum_{i=1}^{[nt]-1} |r_{i\Delta,\Delta}||r_{(i+1)\Delta,\Delta}|. \qquad \textbf{(2.10)}$$

Then it holds that:

$$BV^n_t \xrightarrow{p} \int_0^t \sigma^2_u \mathrm{d}u. \qquad \textbf{(2.11)}$$

Intuitively, the boundedness of N ensures that the probability of jumps in consecutive returns goes to zero as $n \to \infty$. Thus, for n sufficiently large, all returns with a jump are paired with continuous returns. The latter converges in probability to zero, so the limit is unaffected by the product.

[5] In the simulation study and empirical application, a small sample correction $n/(n - k + 1)$ is applied to the realized range- and return-based estimators. We omit it in this section to ease notation.

2.1.1. A Return-Based Theory for Jump Detection

BN-S (2004) coupled the stochastic convergence in (**2.11**) with a central limit theorem (CLT) for (RV_t^n, BV_t^n), computed under the null of a continuous sample path:

$$\sqrt{n}\begin{pmatrix} RV_t^n - \int_0^t \sigma_u^2 \mathrm{d}u \\ BV_t^n - \int_0^t \sigma_u^2 \mathrm{d}u \end{pmatrix} \xrightarrow{d} MN\left(\mathbf{0}, \int_0^t \sigma_u^4 \mathrm{d}u \begin{bmatrix} 2 & 2 \\ 2 & 2+\nu_1 \end{bmatrix}\right), \tag{2.12}$$

where $\nu_1 = \left(\pi^2/4\right) + \pi - 5 \simeq 0.6091$ and $\int_0^t \sigma_u^4 \mathrm{d}u$ is the *integrated quarticity*. Note that RV_t^n is more efficient than BV_t^n. Applying the delta-method to the joint asymptotic distribution of (RV_t^n, BV_t^n), we can construct a non-parametric test of H_0 as:

$$\frac{\sqrt{n}\left(RV_t^n - BV_t^n\right)}{\sqrt{\nu_1 \int_0^t \sigma_u^4 \mathrm{d}u}} \xrightarrow{d} N(0,1). \tag{2.13}$$

The CLT in (**2.13**) is infeasible, however, because it depends on $\int_0^t \sigma_u^4 \mathrm{d}u$. To implement a feasible test, we replace $\int_0^t \sigma_u^4 \mathrm{d}u$ with a consistent estimator computed directly from the data. In order not to erode the power of the test, it is important to use an estimator that is robust to the jump component under H_a. One such statistic is *quad-power quarticity* ($k = 4$; $r_k = 1$):

$$QQ_t^n = \frac{1}{\mu_1^4} \sum_{i=1}^{[nt]-3} |r_{i\Delta,\Delta}||r_{(i+1)\Delta,\Delta}||r_{(i+2)\Delta,\Delta}||r_{(i+3)\Delta,\Delta}|. \tag{2.14}$$

Now, it holds both under H_0 and H_a that $QQ_t^n \xrightarrow{p} \int_0^t \sigma_u^4 \mathrm{d}u$ as $n \to \infty$. Hence, this allows us to construct a feasible test:

$$z_{RV_t^n, BV_t^n, QQ_t^n} = \frac{\sqrt{n}\left(RV_t^n - BV_t^n\right)}{\sqrt{\nu_1 QQ_t^n}} \xrightarrow{d} N(0,1). \tag{2.15}$$

The linear t-statistic in Equation (**2.15**) can be interpreted as a Hausman (1978) test. Under H_a, $RV_t^n - BV_t^n \xrightarrow{p} \sum_{i=1,\ldots,N_t} J_i^2 \geq 0$, so the test is one-sided and positive outcomes go against H_0.[6] Thus, we reject H_0 if $z_{RV_t^n, BV_t^n, QQ_t^n}$ exceeds some critical value, $z_{1-\alpha}$, in the right-hand tail of the $N(0,1)$ (α is the significance level). Simulation studies in Huang & Tauchen (2005) and BN-S (2006), however, show that (**2.15**) is a poor description for sampling frequencies used in practice. BN-S (2006) suggested a modified ratio-statistic to improve the asymptotic approximation:

$$z_{RV_t^n, BV_t^n, QQ_t^n}^{ar} = \frac{\sqrt{n}\left(1 - BV_t^n / RV_t^n\right)}{\sqrt{\nu_1 \max\left\{QQ_t^n / \left(BV_t^n\right)^2, 1/t\right\}}} \xrightarrow{d} N(0,1), \tag{2.16}$$

[6] Recently, Jiang & Oomen (2005) proposed a two-sided swap-variance test that exploits information in the higher-order moments of asset returns.

where the maximum correction is based on the inequality:

$$\frac{QQ_t^n}{\left(BV_t^n\right)^2} \xrightarrow{p} \frac{\int_0^t \sigma_u^4 \mathrm{d}u}{\left(\int_0^t \sigma_u^2 \mathrm{d}u\right)^2} \geq 1/t, \tag{2.17}$$

2.2. Realized Range-Based Variance and Bipower Variation

In theory, the efficient return-based estimators exhaust the data, so we should compute RV_t^{nm}, BV_t^{nm} and QQ_t^{nm}. It is well-known that RV_t^{nm} is the maximum likelihood estimator of the quadratic variation in the parametric version of this problem. Notwithstanding this result, RV_t^{nm} is probably the worst estimator in practice, as microstructure noise corrupts high-frequency data, which leads to bad inference about $\langle p \rangle_t$ if n is too high.

There are some non-parametric estimators that are consistent under various assumptions on the noise process (including the case of endogenous noise), e.g., the subsampler of Zhang et al. (2004) or multiscale estimator of Zhang (2004). This is related to the kernel-based framework studied in Barndorff-Nielsen, Hansen, Lunde & Shephard (2006). The sparsely sampled RV_t^n is, however, still the most widely used volatility statistic in empirical work.

In an earlier paper, we proposed an estimator of the integrated variance, which is based on high-frequency price ranges, see Christensen & Podolskij (2006) and also Dijk & Martens (2006) for related work. This estimator is more efficient than RV_t^n, when microstructure noise is not too severe. Intuitively, a range extracts some of the information about volatility in data interior to RV_t^n. Range-based volatility, of course, has deep roots in finance and traces back to Parkinson (1980), who studied the scaled Brownian motion, $p_t = \sigma W_t$.[7] *Realized range-based variance* is the high-frequency version of his estimator, though we were able to handle, essentially, all Brownian semimartingales and discretely sampled high-frequency data. A drawback of the analysis was that we excluded the jump component of Equation (**2.1**). We close that gap here, among other things.

Realized range-based variance is defined as:

$$RRV_{b,t}^{n,m} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^{[nt]} s_{p_{i\Delta,\Delta},m}^2, \tag{2.18}$$

where $\lambda_{r,m} = \mathbb{E}\left(s_{W,m}^r\right)$ and $s_{W,m} = \max_{0 \leq s,t \leq m} \left\{W_{t/m} - W_{s/m}\right\}$ is the range of a Brownian motion based on m equidistant increments over $[0,1]$. $\lambda_{r,m}$ removes the bias from discrete data and are related to the work of Garman & Klass (1980) and Rogers & Satchell (1991). Note that $\lambda_{r,m}$ is not necessarily finite for all $r \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{\infty\}$. Lemma 1 presents a sufficient condition to ensure this property.

[7]Further readings on range-based volatility estimation can be found in, e.g., Feller (1951), Garman & Klass (1980), Rogers & Satchell (1991) or Alizadeh, Brandt & Diebold (2002).

Lemma 1 *With $r > -m$, it holds that*

$$\lambda_{r,m} < \infty. \tag{2.19}$$

It is worth pointing out that $\lambda_r \equiv \lambda_{r,\infty}$ is finite for all $r \in \mathbb{R}$ and $\lambda_{r,1} = \mu_r$ is not. This result follows, because for $m = 1$ the range equals the absolute return.

Now, we review the asymptotic results developed for $RRV_{b,t}^{n,m}$ and extend these in a number of ways. To prove a CLT, we impose some regularity conditions on the σ process:

(V) σ is everywhere invertible (V_1) and satisfies:

$$\sigma_t = \sigma_0 + \int_0^t \mu'_u \mathrm{d}u + \int_0^t \sigma'_u \mathrm{d}W_u + \int_0^t v'_u \mathrm{d}B'_u, \tag{$\mathbf{V_2}$}$$

where $\mu' = (\mu'_t)_{t \geq 0}$, $\sigma' = (\sigma'_t)_{t \geq 0}$, $v' = (v'_t)_{t \geq 0}$ are adapted càdlàg processes with μ' also predictable and locally bounded, and $B' = (B'_t)_{t \geq 0}$ is a Brownian motion independent of W.

Assumption V_1 is a rather technical condition required in the proofs, but it is satisfied for almost all Brownian semimartingales. V_2 is sufficient, but not necessary, and could be weakened to include a jump process in σ.

The next proposition is adapted from Christensen & Podolskij (2006).

Proposition 1 *Assume that $p \in BSM$. Then, as $n \to \infty$*

$$RRV_{b,t}^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u, \tag{2.20}$$

*where the convergence holds locally uniform in t and uniformly in m. Moreover, if condition **(V)** holds and $m \to c \in \mathbb{N} \cup \{\infty\}$:*

$$\sqrt{n} \left(RRV_{b,t}^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u \right) \xrightarrow{d} MN\left(0, \Lambda_c^R \int_0^t \sigma_u^4 \mathrm{d}u \right), \tag{2.21}$$

where $MN(\cdot, \cdot)$ denotes a mixed Gaussian distribution and $\Lambda_c^R = \left(\lambda_{4,c} - \lambda_{2,c}^2\right) / \lambda_{2,c}^2$.

Note that c affects the asymptotic variance of $RRV_{b,t}^{n,m}$, and so its efficiency relative to RV_t^n. If $m \to 1$ as $n \to \infty$, $\Lambda_m^R \to 2$. If $m \to \infty$ as $n \to \infty$, $\Lambda_m^R \to 0.4073$ (roughly), so $RRV_{b,t}^{n,m}$ is up to five times more accurate than RV_t^n, which is an extension of Parkinson (1980).

Maintaining the assumption that $p \in BSM$, a consistent estimator of $\int_0^t \sigma_u^4 \mathrm{d}u$ is given by the *realized range-based quarticity*:

$$RRQ_t^{n,m} = \frac{n}{\lambda_{4,m}} \sum_{i=1}^{[nt]} s_{p_{i\Delta},\Delta,m}^4 \xrightarrow{p} \int_0^t \sigma_u^4 \mathrm{d}u, \tag{2.22}$$

so

$$\frac{\sqrt{n} \left(RRV_{b,t}^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u \right)}{\sqrt{\Lambda_m^R RRQ_t^{n,m}}} \xrightarrow{d} N(0,1). \tag{2.23}$$

With this result, we can construct confidence intervals for $\int_0^t \sigma_u^2 \mathrm{d}u$. It will be clear, however, that neither $RRV_{t,b}^{n,m}$ nor $RRQ_t^{n,m}$ are appropriate choices, if p exhibits discontinuities.

2.2.1. Extension to Jump-Diffusion Processes

To the best of our knowledge, there is no theory for estimating quadratic variation of jump-diffusion processes with the price range. This raises the question of whether the convergence in probability extends to that situation. The answer, unfortunately, is negative. In fact, $RRV_{b,t}^{n,m}$ is downward biased if $N \neq 0$ (and $m \neq 1$), as the subscript b indicates.

Theorem 1 *If p satisfies* (**2.1**)*, then as $n \to \infty$:*

$$RRV_{b,t}^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u + \frac{1}{\lambda_{2,m}} \sum_{i=1}^{N_t} J_i^2, \tag{2.24}$$

where the convergence holds locally uniform in t and uniformly in m.

Theorem 1 shows that $RRV_{b,t}^{n,m}$ is inconsistent, except for Brownian semimartingales or $m = 1$. Nonetheless, the structure of the problem opens the route for a modified intraday high-low statistic that is also consistent for the quadratic variation of the jump component.

Inspired by bipower variation, we might exploit the corollary:

$$BV_t^n + \lambda_{2,m} \left(RRV_{b,t}^{n,m} - BV_t^n \right) \xrightarrow{p} \langle p \rangle_t \,. \tag{2.25}$$

This defies the nature of our approach, however, so we opt for other ways of correcting $RRV_{b,t}^{n,m}$. In particular, we introduce the idea of (realized) *range-based bipower variation.*

Definition 1 *Range-based bipower variation with parameter $(r, s) \in \mathbb{R}_+^2$ is defined as:*

$$RBV_{(r,s),t}^{n,m} = n^{(r+s)/2-1} \frac{1}{\lambda_{r,m}} \frac{1}{\lambda_{s,m}} \sum_{i=1}^{[nt]-1} s_{p_{i\Delta,\Delta},m}^r s_{p_{(i+1)\Delta,\Delta},m}^s, \tag{2.26}$$

Remark 1 *In the definition, $(i+1)$ may be replaced with $(i+q)$, for any finite positive integer q. Such "staggering" has been suggested for BV_t^n in Andersen et al. (2006) and BN-S (2006). Moreover, Huang & Tauchen (2005) show that extra lagging can alleviate the impact of microstructure noise by breaking the serial correlation in returns.*

$RBV_{(r,s),t}^{n,m}$ is composed of range-based cross-terms raised to the powers (r, s) and constitutes a direct analogue to the general definition of bipower variation from BN-S (2004). The parameter sets $n^{(r+s)/2-1}$, which is required to balance the order of the estimator and produce non-trivial limits.

Theorem 2 *If $p \in BSM$, then as $n \to \infty$*

$$RBV^{n,m}_{(r,s),t} \xrightarrow{p} \int_0^t |\sigma_u|^{r+s} \mathrm{d}u, \tag{2.27}$$

where the convergence holds locally uniform in t and uniformly in m.

Corollary 1 *Set $r = 0$:*

$$RPV^{n,m}_{(s),t} \xrightarrow{p} \int_0^t |\sigma_u|^{s} \mathrm{d}u, \tag{2.28}$$

with the convention $RPV^{n,m}_{(s),t} \equiv RBV^{n,m}_{(0,s),t}$. This estimator is called realized range-based power variation with parameter $s \in \mathbb{R}_+$.

Theorem 2 implies that for $r \in (0, 2)$

$$RBV^{n,m}_{(r,\,2-r),t} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u, \tag{2.29}$$

so $RBV^{n,m}_{(r,\,2-r),t}$ provides an alternative way of drawing inference about $\int_0^t \sigma_u^2 \mathrm{d}u$. Moreover, it will be shown below that $RBV^{n,m}_{(r,\,2-r),t}$ continues to estimate $\int_0^t \sigma_u^2 \mathrm{d}u$ under H_a.

In this paper, we mainly focus on the first-order range-based bipower variation, defined as $RBV^{n,m}_{(1,1),t} \equiv RBV^{n,m}_{t}$. Obviously:

$$RBV^{n,m}_{t} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u. \tag{2.30}$$

This subsection is closed by introducing a new range-based estimator that is consistent for $\langle p \rangle_t$ of the jump-diffusion semimartingale in (**2.1**):

$$RRV^{n,m}_{t} \equiv \lambda_{2,m} RRV^{n,m}_{b,t} + (1 - \lambda_{2,m})\, RBV^{n,m}_{t} \xrightarrow{p} \langle p \rangle_t\,, \tag{2.31}$$

i.e. we combine $RRV^{n,m}_{b,t}$ and $RBV^{n,m}_{t}$ using the weights $\lambda_{2,m}$ and $1 - \lambda_{2,m}$. This estimator is used below to compare with RV^{n}_{t}.

2.2.2. Asymptotic Distribution Theory

The consistency of $RBV^{n,m}_{(r,s),t}$ does not offer any information about the rate of convergence. Moreover, in practice market microstructure noise effectively puts a bound on n (e.g., at the 5-minute frequency) and it is therefore of interest to know more about the sampling errors. Theorem 3 extends the convergence in probability of $RBV^{n,m}_{(r,s),t}$ to a CLT.[8]

[8] To prove the CLT, we use the concept of stable convergence. A sequence of random variables, $(X_n)_{n \in \mathbb{N}}$, *converges stably in law with limit* X, defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, if and only if for every $\mathcal{F}$-measurable, bounded random variable Y and any bounded, continuous function g, the convergence $\lim_{n \to \infty} \mathbb{E}[Y g(X_n)] = \mathbb{E}[Y g(X)]$ holds. Throughout the paper, $X_n \xrightarrow{d_s} X$ is used to denote stable convergence. Note that it implies weak convergence in distribution by setting $Y = 1$ (see, e.g., Rényi (1963) or Aldous & Eagleson (1978) for more details).

Theorem 3 *Given $p \in BSM$ and **(V)** are satisfied, then as $n \to \infty$ and $m \to c \in \mathbb{N} \cup \{\infty\}$*

$$\sqrt{n}\left(RBV_{(r,s),t}^{n,m} - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u\right) \xrightarrow{d_s} \sqrt{\Lambda_c^{B_{r,s}}}\int_0^t |\sigma_u|^{r+s}\mathrm{d}B_u, \tag{2.32}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, defined on an extension of $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, that is independent from the σ-field $\mathcal{F}$, and

$$\Lambda_c^{B_{r,s}} = \frac{\lambda_{2r,c}\lambda_{2s,c} + 2\lambda_{r,c}\lambda_{s,c}\lambda_{r+s,c} - 3\lambda_{r,c}^2\lambda_{s,c}^2}{\lambda_{r,c}^2\lambda_{s,c}^2}. \tag{2.33}$$

Remark 2 *Note that the rate of convergence is not influenced by m and no assumptions on the ratio n/m are required.*

Remark 3 *Suppose that $p_t = \int_0^t \sigma_u \mathrm{d}W_u$, where σ is independent of W and bounded away from zero. If we have slightly more data than the moment condition in Lemma 1 requires (e.g., $r, s > -m+1$), then Theorem 2 and 3 allows for negative values of (r,s). In principle, this means $RBV_{(r,s),t}^{n,m}$ can estimate integrals with negative powers of σ, e.g., $\int_0^t \sigma_u^{-2}\mathrm{d}u$. Unfortunately, it does not seem possible for general processes without further assumptions. Nevertheless, it is an intriguing feature of $RBV_{(r,s),t}^{n,m}$, as bipower variation cannot estimate such quantities.*

The critical feature of Theorem 3 is that B is independent of σ. This implies that the limit process in Equation (**2.32**) has a mixed normal distribution:

$$\sqrt{n}\left(RBV_{(r,s),t}^{n,m} - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u\right) \xrightarrow{d} MN\left(0, \Lambda_c^{B_{r,s}}\int_0^t |\sigma_u|^{2(r+s)}\mathrm{d}u\right). \tag{2.34}$$

[INSERT FIGURE 1 ABOUT HERE]

$\Lambda_m^B \equiv \Lambda_m^{B_{1,1}}$ is plotted in Figure 1 for all values of m that integer divide 23,400. As m increases, there is less sampling variation, as $s_{p_{i\Delta,\Delta},m}$, $i = 1, \ldots, [nt]$, is based on more increments. A striking result is that $\Lambda_m^B \to \left(\lambda_2^2 + 2\lambda_1^2\lambda_2 - 3\lambda_1^4\right)/\lambda_1^4 \simeq 0.3631$ as $m \to \infty$, which is lower than the asymptote of Λ_m^R of about 0.4073. The break-even point, $\Lambda_m^R \simeq \Lambda_m^B$, is a stunning low $m = 3$. This means that $RBV_t^{n,m}$ is more efficient than $RRV_{b,t}^{n,m}$ for almost every m under H_0, which contradicts both the comparison of (RV_t^n, BV_t^n) and our intuition. Note that $\Lambda_1^B = 2.6091$ is the constant appearing in the CLT of BV_t^n. Hence, $RBV_t^{n,m}$ is up to 7.2 times more efficient than BV_t^n (as $m \to \infty$).

2.2.3. A Range-Based Theory for Jump Detection

Now, the univariate convergence in distribution from Proposition 1 and Theorem 3 is extended to the bivariate distribution of $\left(RRV_{b,t}^{n,m}, RBV_t^{n,m}\right)$. This result is used to propose a new test of H_0.

Theorem 4 *If $p \in BSM$ and **(V)** holds, then as $n \to \infty$ and $m \to c \in \mathbb{N} \cup \{\infty\}$*

$$\sqrt{n}\begin{pmatrix} RRV_{b,t}^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u \\ RBV_t^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u \end{pmatrix} \xrightarrow{d} MN\left(\mathbf{0}, \int_0^t \sigma_u^4 \mathrm{d}u \begin{bmatrix} \Lambda_c^R & \Lambda_c^{RB} \\ \Lambda_c^{RB} & \Lambda_c^B \end{bmatrix}\right), \tag{2.35}$$

with

$$\Lambda_c^{RB} = \frac{2\lambda_{3,c}\lambda_{1,c} - 2\lambda_{2,c}\lambda_{1,c}^2}{\lambda_{2,c}\lambda_{1,c}^2}. \tag{2.36}$$

The proof of Theorem 4 is a simple extension of Equation (**2.21**) and (**2.34**), so we omit it. By the delta-method, it follows that under H_0 (note the subscripting):

$$\frac{\sqrt{n}\left(RRV_t^{n,m} - RBV_t^{n,m}\right)}{\sqrt{\nu_m \int_0^t \sigma_u^4 \mathrm{d}u}} \xrightarrow{d} N(0,1), \tag{2.37}$$

where $\nu_m = \lambda_{2,m}^2\left(\Lambda_m^R + \Lambda_m^B - 2\Lambda_m^{RB}\right)$.

We noticed in Equation (**2.31**) that $RRV_t^{n,m} \xrightarrow{p} \langle p \rangle_t$. The next theorem shows that, under H_a, $RBV_t^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}s$. Thus, to implement a feasible range-based test, we need only to substitute $\int_0^t \sigma_u^4 \mathrm{d}s$ with a consistent estimator that is robust to jumps. As noted $RRQ_t^{n,m}$ is not a suitable choice, because it explodes under H_a.

Theorem 5 *If p satisfies* (**2.1**)*, then:*

$$RBV_{(r,s),t}^{n,m} \xrightarrow{p} \begin{cases} \int_0^t |\sigma_u|^{r+s} \mathrm{d}u, & \max(r,s) < 2, \\ X_t^*, & \max(r,s) = 2, \\ \infty, & \max(r,s) > 2, \end{cases} \tag{2.38}$$

where X_t^* is some stochastic process.

The proof follows the logic of Theorem 5 in BN-S (2004) and is omitted. Note that $RRQ_t^{n,m} \xrightarrow{p} \infty$ as $n \to \infty$ under H_a and $RBV_{(r,s),t}^{n,m}$ fails to estimate $\int_0^t \sigma_u^4 \mathrm{d}u$, as $\max(r,s) < 2$ restricts that $r + s < 4$. It is straightforward, however, to define range-based multipower variation analogous to Equation (**2.7**). Provided that $\max(r_1, \ldots, r_k) < 2$, such estimators are robust to the jump component and can estimate higher-order integrated power variation. We postpone an in-depth treatment of these concepts to later work. In this paper, we only introduce *range-based quad-power quarticity*:

$$RQQ_t^{n,m} = \frac{n}{\lambda_{1,m}^4} \sum_{i=1}^{[nt]-3} s_{p_{i\Delta},\Delta,m} s_{p_{(i+1)\Delta},\Delta,m} s_{p_{(i+2)\Delta},\Delta,m} s_{p_{(i+3)\Delta},\Delta,m}. \tag{2.39}$$

Now, both under H_0 and H_a: $RQQ_t^{n,m} \xrightarrow{p} \int_0^t \sigma_u^4 \mathrm{d}u$, so

$$z_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}} = \frac{\sqrt{n}\left(RRV_t^{n,m} - RBV_t^{n,m}\right)}{\sqrt{\nu_m RQQ_t^{n,m}}} \xrightarrow{d} N(0,1). \qquad \textbf{(2.40)}$$

This constitutes our new jump detection test. The intuition is exactly as for realized multipower variation. Based on the above, we would expect a transformation of the t-statistic to improve the size properties in finite samples.[9] Here we adopt the modified ratio-statistic:

$$z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}} = \frac{\sqrt{n}\left(1 - RBV_t^{n,m}/RRV_t^{n,m}\right)}{\sqrt{\nu_m \max\left\{RQQ_t^{n,m}/\left(RBV_t^{n,m}\right)^2, 1/t\right\}}} \xrightarrow{d} N(0,1). \qquad \textbf{(2.41)}$$

3. Monte Carlo Simulation

In this section, a Monte Carlo simulation is used to inspect the small sample properties of the asymptotic results. We untangle the two parts of $\langle p \rangle_t$ with $RBV_t^{n,m}$ and evaluate the new t-statistic for jump detection, $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$. We also compare our test with the return-based t-statistic $z^{ar}_{RV_t^n,BV_t^n,QQ_t^n}$. The simulated Brownian semimartingale is given by:

$$\begin{aligned} \mathrm{d}p_u &= \sigma_u \mathrm{d}W_u \\ \mathrm{d}\ln\sigma_u^2 &= \theta(\omega - \ln\sigma_u^2)\mathrm{d}u + \eta \mathrm{d}B_u, \end{aligned} \qquad \textbf{(3.1)}$$

where W and B are independent Brownian motions. In this model, the log-variance evolves as a mean reverting Orstein-Uhlenbeck process with parameters (θ, ω, η). We use estimates from Andersen, Benzoni & Lund (2002), setting $(\theta, \omega, \eta) = (0.032, -0.631, 0.374)$.

To produce a discontinuous sample path for p, we follow BN-S (2006) and allocate j jumps uniformly in each unit of time, $j = 1, 2$. Hence, the reported power is the conditional probability of rejecting the null, given j jumps. We generate jump sizes by drawing independent $N(0, \sigma_J^2)$ variables and set $\sigma_J^2 = 0.05, 0.10, \ldots, 0.25$ to uncover the impact on power of varying this parameter.

The remaining settings are: $T = 100,000$ replications of **(3.1)** are made for all σ_J^2. The proportion of trading each day amounts to 6.5 hours, or 23,400 seconds. This choice reflects the length of regular trading at NYSE, from which our empirical data are collected. We set $p_0 = 0$, $\ln\sigma_0^2 = \omega$ and generate a realization of **(3.1)** such that a new observation of p is recorded every 20th second ($mn = 1170$). Again, this is calibrated to match our real data. $RRV_t^{n,m}$, $RBV_t^{n,m}$ and $RQQ_t^{n,m}$ are then computed for $n = 39, 78, 390$ ($m = 30, 15, 3$), corresponding to 10-, 5-, and 1-minute sampling.

[9] In unreported simulations, we found that the t-statistic is Equation **(2.40)** is highly oversized for sampling frequencies that are common in applied work.

3.1. Simulation Results

In the first row of Figure 2, we plot $RBV_t^{n,m}$ and the integrated variance for 200 iterations of the model with $j = 1$ and $\sigma_J^2 = 0.10$. The second row shows $(RRV_t^{n,m} - RBV_t^{n,m})^+$ against the squared jump, where $(x)^+ = \max(0, x)$. The maximum correction applied to $RRV_t^{n,m} - RBV_t^{n,m}$ was suggested by BN-S (2004) in the context of realized variance, and as

$$RRV_t^{n,m} - RBV_t^{n,m} \xrightarrow{p} \sum_{i=1}^{N_t} J_i^2 \geq 0, \qquad \textbf{(3.2)}$$

we also expect a better finite sample behavior here, although the modified estimator has the disadvantage of being biased.

[INSERT FIGURE 2 ABOUT HERE]

As n increases, both statistics converge to their population counterparts. At $n = 78$, they are usually quite accurate, although $RBV_t^{n,m}$ has a larger RMSE relative to $(RRV_t^{n,m} - RBV_t^{n,m})^+$. According to the CLT, the conditional variance of $RBV_t^{n,m}$ is $\Lambda_m^B \int_0^t \sigma_u^4 \mathrm{d}u$ for Brownian semimartingales. There is some indication that the errors bounds of $RBV_t^{n,m}$ increase with σ - most pronounced at $n = 390$ - but, of course, in our setting jumps are interacting.

[INSERT FIGURE 3 ABOUT HERE]

The distribution of $z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ is plotted in Figure 3 with $\sigma_J^2 = 0.00$. From Equation **(2.41)**, the t-statistic then converges to the $N(0, 1)$ as $n \to \infty$, which the kernel-based densities confirm. The approximation is not impressive for moderate n, but the focal point is the right-hand tail, where the rejection region is located. Testing at a nominal level of $\alpha = 0.01$ with critical value $z_{1-\alpha} = 2.326$, for example, yields actual sizes of 2.236, 1.856 and 1.292 percent, respectively. At $\alpha = 0.05$ - or $z_{1-\alpha} = 1.645$ - the type I errors are 6.497, 6.071 and 5.504 percent, in both situations leading to a modest over-rejection. This finding is consistent with the Monte Carlo studies on $z^{ar}_{RV_t^n, BV_t^n, QQ_t^n}$ in Huang & Tauchen (2005) and BN-S (2006).

[INSERT TABLE 1 ABOUT HERE]

The bottom part of Table 1 shows the power of $z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ across the jump variances $\sigma_J^2 = 0.00, 0.05, \ldots, 0.25$ with $j = 1$ and $j = 2$. The numbers reflect the proportion of t-statistics that exceeded $z_{1-\alpha} = 2.326$ (i.e. no size-correction).

There is a substantial type II error for $j = 1$ and small σ_J^2, but it diminishes as we depart from the null. At $\sigma_J^2 = 0.10$, the rejection rates are 0.234, 0.321 and 0.490 for $n = 39, 78, 390$. The power improves more quickly for $j = 2$, reflecting the increase in the jump variation. Consistent with BN-S (2006), we find that power is roughly equal for $\sigma_J^2 = x$ and $j = 1$ compared to $\sigma_J^2 = x/2$ and $j = 2$, showing that the main constituent affecting the properties of $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$ under the alternative is the variance of the jump process: $j\sigma_J^2$. As an example, consider $j = 2$ and $\sigma_J^2 = 0.05$; here the fraction of t-statistics above $z_{1-\alpha} = 2.326$ is 0.218, 0.340 and 0.585.

Note that the relationship is much weaker at $n = 390$. Across simulations, there is a pronounced pattern that - keeping $j\sigma_J^2$ fixed - the t-statistic tends to prefer a higher value of σ_J^2 at the expense of j for low n, while the opposite holds for large n. Intuitively, at higher sampling frequencies two small breaks in p appear more abrupt, while they are drowned by the variation of the continuous part for infrequent sampling.

As the simulation is designed, m is greater than 1. Hence, the range-based t-statistic ought to be more powerful than the return-based version. We construct RV_t^n, BV_t^n, QQ_t^n and report $z^{ar}_{RV_t^n,BV_t^n,QQ_t^n}$ in the right-hand side of Table 1. In general, the range-based t-statistic is more powerful, in particular, has a much higher probability of detecting small jumps at lower sampling frequencies. Interestingly, though, the size of $z^{ar}_{RV_t^n,BV_t^n,QQ_t^n}$ is slightly better than the size of $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$.

4. Empirical Application

We illustrate some features of the theory for a component of the Dow Jones Industrial Average as of April 8, 2004. Our exposition is based on Merck (MRK).

High-frequency data for Merck was extracted from the Trade and Quote (TAQ) database for the sample period January 3, 2000 to December 31, 2004. A total of 1,253 trading days. We restrict attention to midquote data from NYSE.[10] The raw data were filtered for outliers and we discarded updates outside the regular trading session from 9:30AM to 4:00PM EST.

[INSERT TABLE 2 ABOUT HERE]

Table 2 reports the amount of tick data. We exclude zero returns - $r_{\tau_i} = 0$ - and non-zero returns that are reversals - $r_{\tau_i} \neq 0$ but $\Delta r_{\tau_i} = 0$ - when computing mn. Here $r_{\tau_i} = p_{\tau_i} - p_{\tau_{i-1}}$ and τ_i is the arrival time of the ith tick. There is a lot of empirical support for adopting this convention, because counting such returns induce an upward bias in mn - due to transaction price/midquote repetitions and bounces - thus a downward bias in the range-based estimates. Hence, for price changes to affect mn, we require that both $r_{\tau_i} \neq 0$ and $\Delta r_{\tau_i} \neq 0$. On average, this reduces the mn numbers by one-third (one-half) for the quote (trade) data relative to using $r_{\tau_i} \neq 0$.

[10] The analysis based on transaction data is available at request.

To account for the irregular spacing of high-frequency data, we use *tick-time sampling* (e.g., Hansen & Lunde (2006)). We set the sampling times t_i at every 15th new quotation ($m = 15$), which corresponds to 5-minute sampling on average for our sample and stock. This procedure has the advantage, apart from end effects, of fixing the number of returns in each interval $[t_{i-1}, t_i]$. Tick-time sampling is irregular in calendar-time, but this is not a problem provided that we also use a tick-time estimator of the conditional variance. Finally, following the recommendation of Huang & Tauchen (2005), we calculate the jump robust estimators by staggering ranges and returns using a "skip-one" approach.

[INSERT TABLE 3 ABOUT HERE]

In Table 3, we report some sample statistics of the time series used here. The variance of the realized range-based estimators are smaller compared to the return-based statistics. The reduction is most pronounced for the robust $RBV_t^{n,m}$ and $RQQ_t^{n,m}$. There is a high positive correlation between $(RRV_t^{n,m}, RV_t^n)$, $(RBV_t^{n,m}, BV_t^n)$ and $(RQQ_t^{n,m}, QQ_t^n)$, which reflects that they are estimating the same part of p. Note the large differences in the mean and variance of $RQQ_t^{n,m}$ and QQ_t^n. The maximum QQ_t^n is twice that of $RQQ_t^{n,m}$. We return to this below.

[INSERT FIGURE 4 ABOUT HERE]

Figure 4 plots $RRV_t^{n,m}$ ($RBV_t^{n,m}$) against the left (right) y-axis. Both series are reported as annualized standard deviations. The correlation coefficient of the two series is a high 0.901. Moreover, they exhibit a strong own serial dependence, reflecting the volatility clustering in the data. The first five autocorrelations of $RRV_t^{n,m}$ are 0.523, 0.461, 0.383, 0.361 and 0.383, compared to 0.722, 0.644, 0.564, 0.539 and 0.563 for $RBV_t^{n,m}$. Intuitively, $RBV_t^{n,m}$ is the most persistent process, because it is robust against the (less persistent) jump component. The most important feature of this graph is that some of the spikes appearing in $RRV_t^{n,m}$ are not matched by $RBV_t^{n,m}$. Here the estimators associate a large proportion of $\langle p \rangle_t$ to the jump process, which we now review in more detail.

[INSERT FIGURE 5 ABOUT HERE]

In Figure 5, we plot $z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ and $z^{ar}_{RV_t^n, BV_t^n, QQ_t^n}$. The y-axis is truncated at 0, as negative outcomes never go against H_0. The horizontal line represents a critical value of $z_{1-\alpha} = 2.326$, which is the 0.99 quantile of the $N(0,1)$ density. Figure 5 shows that there is a significant difference between $z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ and $z^{ar}_{RV_t^n, BV_t^n, QQ_t^n}$.

[INSERT TABLE 4 ABOUT HERE]

This is underscored in Table 4, where the number of rejections at the 5- and 1% level is shown. We also compute the fraction of $\langle p \rangle_t$ explained by the jump process. At the 5% level, $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ rejects H_0 141 times, while $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ does so 289 times (out of 1,253). Without any jumps at all in the data, we would expect the t-statistics to reject 63 times. At the 1% level, the numbers are down to 48 and 151 rejections, as opposed to the 13 expected under H_0.

Nonetheless, $RRV^{n,m}_t - RBV^{n,m}_t$ induces a higher proportion of $\langle p \rangle_t$ - 16.9% - when all positive, also insignificant, jump terms are counted. This is because the means of $RRV^{n,m}_t$ and $RBV^{n,m}_t$ differ more than those of RV^n_t and BV^n_t. $RV^n_t - BV^n_t$ explains 10.9% of $\langle p \rangle_t$. Taking sampling variation into account the numbers are aligned, as the range-based t-statistic regards many more of the small jump contributions to be insignificant. At the 1% level, $RRV^{n,m}_t - RBV^{n,m}_t$ ($RV^n_t - BV^n_t$) accounts for 5.6% (6.1%) of $\langle p \rangle_t$.

4.1. Estimation of Integrated Quarticity

How much the jump process induces of $\langle p \rangle_t$ empirically is an open question. It is unlikely, though, that $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ picks out so many small jumps $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ fails to detect. Our simulations reveal that the latter has higher power to unearth these. To conclude our paper, we therefore inspect this finding a little further. In Figure 6, we draw the data for Thursday, August 24, 2000.

[INSERT FIGURE 6 ABOUT HERE]

There are several downticks at the beginning of trading, after which the price slopes upward until trading stops. This day has some rapid changes in p, but there is no jump. Still $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t} = 3.688$, which is a huge rejection. By contrast, $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t} = 0.560$. Thus, the t-statistics lead to opposite conclusions about the sample path. We studied Merck's price at the days in our sample, where $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ rejected at the 1% level. A majority has slides as in Figure 6 and almost none were falsely rejected by $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$, pointing to more robustness here. This feature, of course, is relevant to empirical work, because specialists at NYSE are charged with maintaining a smooth price sequence and to avoid large changes between transactions.

We believe that estimation of integrated quarticity is key here. Because of sampling variation, QQ^n_t is going to deviate somewhat from $\int_0^t \sigma^4_u \mathrm{d}u$ relative to $RQQ^{n,m}_t$. Notice from Table 3 that, compared to $RQQ^{n,m}_t$, the variance of QQ^n_t is three times larger. Indeed, the asymptotic variance of QQ^n_t is more than 9.7 times bigger than that of $RQQ^{n,m}_t$ (as $m \to \infty$). On days where $1 - BV^n_t / RV^n_t$ is small, a too low QQ^n_t can still move the t-statistic into the rejection region, although the lower bound $1/t$ does provide some protection here. We tried to replace QQ^n_t with $RQQ^{n,m}_t$ in the return-based t-statistic, and, in fact, found that the number of jumps were almost equal.

Recently, Barndorff-Nielsen, Hansen, Lunde & Shephard (2006) proposed an estimator of integrated quarticity that is more robust to microstructure noise, which may enable researchers to further reduce the sampling errors of such return-based estimators by helping to increase the sampling frequency. But realized range-based estimation offers a simple, efficient framework for conducting such inference.

5. Conclusions and Directions for Future Research

This paper proposes using realized range-based estimators to conduct inference about the quadratic variation of asset prices and derives a new test for jump detection. The Monte Carlo study indicates that these estimators are quite efficient at sampling frequencies normally used in applied work, and our empirical results confirm this.

The theory developed here casts new light on the properties of the price range, but there are still several problems left hanging for ongoing and future research. First, the realized range-based estimators were somewhat informally motivated by appealing to the sparse sampling of realized variance caused by microstructure noise. It is not clear how severely microstructure noise affects the range, and we are currently pursuing a paper on this topic. Second, Garman & Klass (1980), among others, construct estimators of a constant diffusion coefficient by combining the daily range and return. Their procedure extends to general semimartingales and intraday data, which suggests that further efficiency gains are waiting. Indeed, other (non-standard) functionals of the sample path could be more informative about the quadratic variation. Third, it will be interesting to connect non-parametric historical volatility measurements using the intraday high-low statistics with model-based forecasting. Fourth, it is also worth considering bootstrap methods to refine the asymptotic normality approximation, as recently suggested by Gonçalves & Meddahi (2005) in the context of realized variance.

A. **Appendix**

Note that as $t \mapsto \sigma_t$ is càdlàg, all powers of σ are locally integrable with respect to the Lebesgue measure, so that for any t and $s > 0$, $\int_0^t \sigma_u^s \mathrm{d}u < \infty$. Moreover, we can restrict the functions μ, σ, μ', σ', v' and σ^{-1} to be bounded, without loss of generality (e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)).

A.1. **Proof of Lemma 1**

The assertion is trivial for $r \geq 0$ and is a consequence of Burkholder's inequality (e.g., Revuz & Yor (1998, pp. 160)). Now, assume $-m < r < 0$ and note that:

$$s_{W,m} = \max_{0 \leq s,t \leq m} \left\{ W_{t/m} - W_{s/m} \right\} \geq \max_{1 \leq i \leq m} \left\{ |W_{i/m} - W_{(i-1)/m}| \right\}$$
$$\stackrel{d}{=} \frac{1}{\sqrt{m}} \max_{1 \leq i \leq m} \left\{ |\phi_i| \right\} \equiv M_\phi,$$

where ϕ_i, $i = 1, \ldots, m$, are IID standard normal random variables. Then, we have the inequality $\lambda_{r,m} \leq \mathbb{E}\big(M_\phi^r\big) < \infty$ (for $-m < r < 0$). ■

A.2. **Proof of Theorem 1**

This theorem is proved by decomposing $RRV_{b,t}^{n,m}$ into a continuous, jump and mixed part. We adopt the additional notation:

$$p_t^b = p_0 + \int_0^t \mu_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u, \qquad p_t^j = p_0 + \sum_{i=1}^{N_t} J_i^2.$$

Then, using the finite-activity property of N_t, it follows that:

$$\sum_{i=1}^{[nt]} s^2_{p^j_{i\Delta,\Delta},m} \xrightarrow{p} \sum_{i=1}^{N_t} J_i^2, \qquad \sum_{i=1}^{[nt]} s^2_{p^b_{i\Delta,\Delta},m} \xrightarrow{p} \lambda_{2,m} \int_0^t \sigma_u^2 \mathrm{d}u, \qquad 2\sum_{i=1}^{[nt]} s_{p^b_{i\Delta,\Delta},m} s_{p^j_{i\Delta,\Delta},m} \xrightarrow{p} 0,$$

uniformly in m, where the second convergence is from Christensen & Podolskij (2006). ■

In the upcoming theorems, we first prove the result with $m = \infty$ and then extend this to $m < \infty$. To simplify notation, we make the replacements:

$$g(x) = \frac{1}{\lambda_{r,m}} x^r, \qquad h(x) = \frac{1}{\lambda_{s,m}} x^s,$$

for $x \in \mathbb{R}_+$. We also fix some notation before proceeding. For the processes X^n and X, we denote by $X^n \xrightarrow{p} X$, the uniform convergence:

$$\sup_{s \leq t} |X_s^n - X_s| \xrightarrow{p} 0,$$

for all $t > 0$. When X^n has the form:

$$X_t^n = \sum_{i=1}^{[nt]} \zeta_i^n,$$

for an array (ζ_i^n) and $X^n \xrightarrow{p} 0$, we say that (ζ_i^n) is asymptotically negligible (AN). The constants appearing below are denoted by C, or C_p if they depend on an external parameter p. Finally, to prove our asymptotic results some technical preliminaries are required.

A.3. **Preliminaries I**

First, we define:

$$\beta_i^n = \sqrt{n}|\sigma_{\frac{i-1}{n}}|s_{W_{i\Delta,\Delta}}, \qquad \beta_i'^n = \sqrt{n}|\sigma_{\frac{i-1}{n}}|s_{W_{(i+1)\Delta,\Delta}}, \tag{A.1}$$

and

$$\rho_x(f) = \mathbb{E}\left[f\left(|x|s_W\right)\right],$$

where $s_W = \sup_{0 \leq s,t \leq 1} \{W_t - W_s\}$ and f is a real-valued function. Note that

$$\rho_x(g) = |x|^r.$$

We consider an adapted càdlàg and bounded process ν, and the function $f(x) = x^p$, for $p > 0$. Then, we prove a central limit theorem for the quantities

$$U_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \nu_{\frac{i-1}{n}} \left(f\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(f) \right), \tag{A.2}$$

$$U_t'^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left(g\left(\beta_i^n\right) h\left(\beta_i'^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\, \rho_{\sigma_{\frac{i-1}{n}}}(h) \right). \tag{A.3}$$

Lemma 2 *If $p \in BSM$:*

$$U_t^n \xrightarrow{d_s} U_t = \sqrt{\lambda_{2p} - \lambda_p^2} \int_0^t \nu_u \sigma_u^p \mathrm{d}B_u, \tag{A.4}$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, defined on an extension of the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ and independent from the σ-field $\mathcal{F}$.

Lemma 3 *If $p \in BSM$:*

$$U_t'^n \xrightarrow{d_s} \sqrt{\frac{\lambda_{2r}\lambda_{2s} + 2\lambda_r\lambda_s\lambda_{r+s} - 3\lambda_r^2\lambda_s^2}{\lambda_r^2\lambda_s^2}} \int_0^t |\sigma_u|^{r+s} \mathrm{d}B_u.$$

Here we prove Lemma 3, leaving Lemma 2 that can be shown with similar techniques. But before doing so, note that the following estimate holds under $p \in BSM$:

$$\mathbb{E}\left[|\beta_i^n|^q\right] + \mathbb{E}\left[|\beta_i'^n|^q\right] \leq C_q, \tag{A.5}$$

for all $q > 0$. Lemma 2 and 3 also imply that

$$\frac{1}{n}\sum_{i=1}^{[nt]} \nu_{\frac{i-1}{n}} f\left(\beta_i^n\right) \xrightarrow{p} \int_0^t \nu_u \rho_{\sigma_u}(f)\,\mathrm{d}u, \tag{A.6}$$

and

$$\frac{1}{n}\sum_{i=1}^{[nt]} g\left(\beta_i^n\right) h\left(\beta_i'^n\right) \xrightarrow{p} \int_0^t \rho_{\sigma_u}(g)\,\rho_{\sigma_u}(h)\,\mathrm{d}u. \tag{A.7}$$

Proof of Lemma 3

We decompose $U_t'^n$ into

$$U_t'^n = \sum_{i=2}^{[nt]+1} \zeta_i^n + \gamma_1^n - \gamma_{[nt]+1}^n,$$

with

$$\begin{aligned}
\zeta_i^n &= \frac{1}{\sqrt{n}}\left(g\left(\beta_{i-1}^n\right)\left(h\left(\beta_{i-1}'^n\right) - \rho_{\sigma_{\frac{i-2}{n}}}(h)\right) + \left(g\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\rho_{\sigma_{\frac{i-1}{n}}}(h)\right),\\
\gamma_i^n &= \frac{1}{\sqrt{n}}\left(g\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\rho_{\sigma_{\frac{i-1}{n}}}(h).
\end{aligned}$$

Now, we set

$$\rho_{i-2,i-1}^n(g,h) = \int g\left(\sigma_{\frac{i-1}{n}}x\right) h\left(\sigma_{\frac{i-2}{n}}x\right)\delta(\mathrm{d}x),$$

where

$$\delta(x) = 8\sum_{j=1}^{\infty}(-1)^{j+1} j^2 \phi(jx),$$

is the density function of s_W (e.g., Feller (1951)), and we note the identity

$$\begin{aligned}
\mathbb{E}\left[|\zeta_i^n|^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] = \frac{1}{n}\Bigg(& g\left(\beta_{i-1}^n\right)^2\left(\rho_{\sigma_{\frac{i-2}{n}}}\left(h^2\right) - \rho_{\sigma_{\frac{i-2}{n}}}^2(h)\right) + \rho_{\sigma_{\frac{i-1}{n}}}^2(h)\left(\rho_{\sigma_{\frac{i-1}{n}}}\left(g^2\right) - \rho_{\sigma_{\frac{i-1}{n}}}^2(g)\right)\\
& + 2g\left(\beta_{i-1}^n\right)\rho_{\sigma_{\frac{i-1}{n}}}(h)\left(\rho_{i-2,i-1}^n(g,h) - \rho_{\sigma_{\frac{i-2}{n}}}(h)\,\rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\Bigg).
\end{aligned}$$

Since

$$\sup_{i\le[nt]+1}\left|\rho_{i-2,i-1}^n(gh) - \rho_{\sigma_{\frac{i-2}{n}}}(gh)\right| \xrightarrow{p} 0,$$

it holds by (A.6) that

$$\sum_{i=2}^{[nt]+1}\mathbb{E}\left[|\zeta_i^n|^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] \xrightarrow{p} \frac{\lambda_{2r}\lambda_{2s} + 2\lambda_r\lambda_s\lambda_{r+s} - 3\lambda_r^2\lambda_s^2}{\lambda_r^2\lambda_s^2}\int_0^t |\sigma_u|^{2(r+s)}\mathrm{d}u,$$

and

$$\sup_{i\le[nt]}|\gamma_i^n| \xrightarrow{p} 0,$$

for any t. Moreover:

$$\mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0.$$

As $W \overset{d}{=} -W$, we also get

$$\mathbb{E}\left[\zeta_i^n \left(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0.$$

Next, assume that N is a bounded martingale on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, which is orthogonal to W (i.e. with quadratic covariation $\langle W, N\rangle_t = 0$, almost surely). As $g\left(\beta_i^n\right)$ is a functional of W times $|\sigma_{\frac{i-1}{n}}|^r$, it follows from Clark's representation theorem (e.g., Karatzas & Shreve (1998, Appendix E)):

$$g\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g) = \frac{1}{\lambda_r}|\sigma_{\frac{i-1}{n}}|^r \int_{\frac{i-1}{n}}^{\frac{i}{n}} H_u^n \mathrm{d}W_u,$$

for some predictable function H_u^n. This also holds for $h\left(\beta_{i-1}'^n\right) - \rho_{\sigma_{\frac{i-2}{n}}}(h)$. Hence,

$$\mathbb{E}\left[\left(g\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\left(N_{\frac{i}{n}} - N_{\frac{i-1}{n}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0,$$

$$\mathbb{E}\left[\left(h\left(\beta_{i-1}'^n\right) - \rho_{\sigma_{\frac{i-2}{n}}}(h)\right)\left(N_{\frac{i}{n}} - N_{\frac{i-1}{n}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0,$$

as N is orthogonal to W. Finally

$$\mathbb{E}\left[\zeta_i^n \left(N_{\frac{i}{n}} - N_{\frac{i-1}{n}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0. \tag{A.8}$$

Now, Lemma 3 follows from Theorem IX 7.28 in Jacod & Shiryaev (2002). ■

A.4. **Preliminaries II**

We define the process

$$\begin{aligned} U(g,h)_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \Big\{ & g\left(\sqrt{n} s_{p_{i\Delta,\Delta}}\right) h\left(\sqrt{n} s_{p_{(i+1)\Delta,\Delta}}\right) \\ & - \mathbb{E}\left[g\left(\sqrt{n} s_{p_{i\Delta,\Delta}}\right) h\left(\sqrt{n} s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]\Big\}, \end{aligned} \tag{A.9}$$

In this subsection, we show that

$$U(g,h)_t^n - U_t'^n \xrightarrow{p} 0, \tag{A.10}$$

and, therefore,

$$U(g,h)_t^n \xrightarrow{d_s} \sqrt{\frac{\lambda_{2r}\lambda_{2s} + 2\lambda_r\lambda_s\lambda_{r+s} - 3\lambda_r^2\lambda_s^2}{\lambda_r^2\lambda_s^2}} \int_0^t |\sigma_u|^{r+s} \mathrm{d}B_u.$$

We begin with:

$$\xi_i^n = \sqrt{n} s_{p_{i\Delta,\Delta}} - \beta_i^n, \qquad \xi_i'^n = \sqrt{n} s_{p_{(i+1)\Delta,\Delta}} - \beta_i'^n, \tag{A.11}$$

and note that

$$\xi_i^n \leq \sqrt{n}\left(\sup_{(i-1)/n \leq s,t \leq i/n} \left|\int_s^t \mu_u \mathrm{d}u\right| + \sup_{(i-1)/n \leq s,t \leq i/n} \left|\int_s^t \left(\sigma_{u-} - \sigma_{\frac{i-1}{n}}\right) \mathrm{d}W_u\right|\right).$$

A similar inequality holds for $\xi_i'^n$.

Lemma 4 *If $p \in BSM$, it holds that*

$$\frac{1}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\xi_i^n|^2 + |\beta_{i+1}^n - \beta_i'^n|^2\right] \to 0, \tag{A.12}$$

for all $t > 0$.

Proof of Lemma 4

The boundedness of μ and Burkholder's inequality yield

$$\mathbb{E}\left[|\xi_i^n|^2\right] \leq C\left(\frac{1}{n} + n\mathbb{E}\left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\sigma_{u-} - \sigma_{\frac{i-1}{n}}|^2 \mathrm{d}u\right]\right).$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[|\beta_{i+1}^n - \beta_i'^n|^2\right] &\leq C\mathbb{E}\left[|\sigma_{\frac{i}{n}} - \sigma_{\frac{i-1}{n}}|^2\right] \\ &\leq Cn\mathbb{E}\left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(|\sigma_{u-} - \sigma_{\frac{i-1}{n}}|^2 + |\sigma_{u-} - \sigma_{\frac{i}{n}}|^2\right) \mathrm{d}u\right].\end{aligned}$$

Hence, the left-hand side of (A.12) is smaller than

$$C\left(\frac{t}{n} + \int_0^t \mathbb{E}\left[|\sigma_{u-} - \sigma_{\frac{[nu]}{n}}|^2 + |\sigma_{u-} - \sigma_{\frac{[nu]+1}{n}}|^2\right] \mathrm{d}u\right).$$

As σ is càdlàg, the last expectation converges to 0 for almost all u and is bounded by a constant. Thus, the assertion follows from Lebesgue's theorem. ∎

To prove the convergence in Equation (A.10), we need the univariate version of Lemma 6.2 and 4.7 from Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006). These are reproduced here.

Lemma 5 *Let (ζ_i^n) be an array of random variables satisfying*

$$\sum_{i=1}^{[nt]}\mathbb{E}\left[|\zeta_i^n|^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] \xrightarrow{p} 0, \tag{A.13}$$

for all t. If further each ζ_i^n is $\mathcal{F}_{\frac{i+1}{n}}$-measurable:

$$\sum_{i=1}^{[nt]}\left(\zeta_i^n - \mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]\right) \xrightarrow{p} 0.$$

Lemma 6 *Assume that for all $q > 0$*

1. *f and k are functions on $\mathbb{R}$ of at most polynomial growth.*

2. *γ_i^n, $\gamma_i'^n$, $\gamma_i''^n$ are $\mathbb{R}$-valued random variables.*

3. *The process*

$$Z_i^n = 1 + \left|\gamma_i^n\right| + |\gamma_i'^n| + |\gamma_i''^n|,$$

satisfies

$$\mathbb{E}\left[(Z_i^n)^q\right] \leq C_q.$$

If k is continuous and

$$\frac{1}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\gamma_i'^n - \gamma_i''^n|^2\right] \to 0, \tag{A.14}$$

then for all $t > 0$

$$\frac{1}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[f^2\left(\gamma_i^n\right)\left(k\left(\gamma_i'^n\right) - k\left(\gamma_i''^n\right)\right)^2\right] \to 0. \tag{A.15}$$

Now, we prove (A.10). We define:

$$\zeta_i^n = \frac{1}{\sqrt{n}}\Big(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) - g\left(\beta_i^n\right)h\left(\beta_i'^n\right)\Big), \tag{A.16}$$

and note that ζ_i^n is $\mathcal{F}_{\frac{i+1}{n}}$-measurable and

$$U\left(g,h\right)_t^n - U_t'^n = \sum_{i=1}^{[nt]}\Big(\zeta_i^n - \mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]\Big).$$

Appealing to Lemma 5, it is enough to show that:

$$\sum_{i=1}^{[nt]}\mathbb{E}\left[|\zeta_i^n|^2\right] \to 0. \tag{A.17}$$

Recall the identity:

$$\sqrt{n}s_{p_{i\Delta,\Delta}} = \beta_i^n + \xi_i^n,$$

and, therefore,

$$|\zeta_i^n|^2 \leq \frac{C}{n}\Big(h^2\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right)\left(g\left(\beta_i^n + \xi_i^n\right) - g\left(\beta_i^n\right)\right)^2$$
$$+ g^2\left(\beta_i^n\right)\left(h\left(\beta_{i+1}^n + \xi_{i+1}^n\right) - h\left(\beta_{i+1}^n\right)\right)^2 + g^2\left(\beta_i^n\right)\left(h\left(\beta_{i+1}^n\right) - h\left(\beta_i'^n\right)\right)^2\Big).$$

(A.17) is now a consequence of (A.5), Lemma 4 and 6. ■

A.5. **Proof of Theorem 2**

$m = \infty$: We set

$$V_t^n = \frac{1}{n}\sum_{i=1}^{[nt]}\eta_i^n,$$

$$V_t'^n = \frac{1}{n}\sum_{i=1}^{[nt]}\eta_i'^n,$$

with

$$\eta_i^n = \mathbb{E}\left[g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

$$\eta_i'^n = \rho_{\sigma_{\frac{i-1}{n}}}(g)\,\rho_{\sigma_{\frac{i-1}{n}}}(h).$$

The convergence in (A.10) means that:

$$RBV_{(r,s),t}^n - V_t^n \xrightarrow{p} 0,$$

and by Riemann integrability:

$$V_t'^n \xrightarrow{p} \int_0^t |\sigma_u|^{r+s}\mathrm{d}u.$$

Thus, if we can show that:

$$V_t^n - V_t'^n \xrightarrow{p} 0.$$

the proof is complete. Further, as

$$\eta_i^n - \eta_i'^n = \sqrt{n}\mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

a sufficient condition is that:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\zeta_i^n|\right] \to 0. \tag{A.18}$$

Using the Cauchy-Schwarz inequality:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\zeta_i^n|\right] \leq \left(t\sum_{i=1}^{[nt]}\mathbb{E}\left[|\zeta_i^n|^2\right]\right)^{\frac{1}{2}},$$

and so (A.18) is implied by (A.17).

$m < \infty$: We define

$$\beta_i^{n,m} = \sqrt{n}|\sigma_{\frac{i-1}{n}}|s_{W_{i\Delta,\Delta},m}, \qquad \beta_i'^{n,m} = \sqrt{n}|\sigma_{\frac{i-1}{n}}|s_{W_{(i+1)\Delta,\Delta},m}, \tag{A.19}$$

which are discrete versions of β_i^n and $\beta_i'^n$ from (A.1). Also, we set

$$\rho_x^m(f) = \mathbb{E}\left[f\left(|x|s_{W,m}\right)\right],$$

where $s_{W,m}$ was defined in (**2.2**). Note that

$$\rho_x^m(g) = |x|^r,$$

We proceed with

$$RBV_{(r,s),t}^{n,m} - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u = U_t^{n,m}(1) + U_t^{n,m}(2) + U_t^{n,m}(3),$$

where $U_t^{n,m}(k)$ are given by:

$$U_t^{n,m}(1) = \frac{1}{n}\sum_{i=1}^{[nt]}\left(g\left(\sqrt{n}s_{p_{i\Delta},\Delta,m}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta},\Delta,m}\right) - g\left(\beta_i^{n,m}\right)h\left(\beta_i'^{n,m}\right)\right),$$

$$U_t^{n,m}(2) = \frac{1}{n}\sum_{i=1}^{[nt]}\left(g\left(\beta_i^{n,m}\right)h\left(\beta_i'^{n,m}\right) - \rho^m_{\sigma_{\frac{i-1}{n}}}(g)\,\rho^m_{\sigma_{\frac{i-1}{n}}}(h)\right),$$

$$U_t^{n,m}(3) = \frac{1}{n}\sum_{i=1}^{[nt]}\rho^m_{\sigma_{\frac{i-1}{n}}}(g)\,\rho^m_{\sigma_{\frac{i-1}{n}}}(h) - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u.$$

Then

$$U_t^{n,m}(3) = \lambda_{r,m}\lambda_{s,m}\left(\frac{1}{n}\sum_{i=1}^{[nt]}|\sigma_{\frac{i-1}{n}}|^{r+s} - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u\right).$$

The boundedness of $\lambda_{r,m}$ (for fixed r) yields the convergence:

$$U_t^{n,m}(3) \xrightarrow{p} 0 \quad \text{as } n \to \infty,$$

uniformly in m. From the calculation of the conditional variance in the proof of Lemma 3, we also get

$$U_t^{n,m}(2) \xrightarrow{p} 0 \quad \text{as } n \to \infty,$$

uniformly in m. We split $U_t^{n,m}(1)$ further into:

$$U_t^{n,m}(1) = U_t^{n,m}(1.1) + U_t^{n,m}(1.2),$$

where

$$U_t^{n,m}(1.1) = \frac{1}{n}\sum_{i=1}^{[nt]} h\Big(\beta_i'^{n,m}\Big)\Big(g\left(\sqrt{n}s_{p_{i\Delta},\Delta,m}\right) - g\left(\beta_i^{n,m}\right)\Big),$$

$$U_t^{n,m}(1.2) = \frac{1}{n}\sum_{i=1}^{[nt]} g\Big(\beta_i^{n,m}\Big)\Big(h\left(\sqrt{n}s_{p_{(i+1)\Delta},\Delta,m}\right) - h\left(\beta_i'^{n,m}\right)\Big).$$

Here we show that:

$$U_t^{n,m}(1.1) \xrightarrow{p} 0,$$

uniformly in m. The corresponding result for $U_t^{n,m}(1.2)$ can be proved with identical methods. First, we assume $r \geq 1$. Then it follows that

$$|h\left(\beta_i'^{n,m}\right)\left(g\left(\sqrt{n}s_{p_{i\Delta},\Delta,m}\right) - g\left(\beta_i^{n,m}\right)\right)| \leq \frac{r}{\lambda_{r,m}}h\left(\beta_i'^{n,m}\right)\left(\sqrt{n}s_{p_{i\Delta},\Delta,m} + \beta_i^{n,m}\right)^{r-1}|\sqrt{n}s_{p_{i\Delta},\Delta,m} - \beta_i^{n,m}|.$$

The estimate:

$$\mathbb{E}\left[|\sqrt{n}s_{p_{i\Delta},\Delta,m}|^q + |\beta_i^{n,m}|^q + |\beta_i'^{n,m}|^q\right] \leq C_q,$$

holds for all $q > 0$. Thus:

$$\mathbb{E}\Big[|U_t^{n,m}(1.1)|\Big] \leq \frac{C}{n}\sum_{i=1}^{[nt]}\Big(\mathbb{E}\Big[|\sqrt{n}s_{p_{i\Delta,\Delta,m}} - \beta_i^{n,m}|^2\Big]\Big)^{\frac{1}{2}}.$$

By Hölder's inequality:

$$\mathbb{E}\Big[|U_t^{n,m}(1.1)|\Big] \leq \left(\frac{Ct}{n}\sum_{i=1}^{[nt]}\mathbb{E}\Big[|\sqrt{n}s_{p_{i\Delta,\Delta,m}} - \beta_i^{n,m}|^2\Big]\right)^{\frac{1}{2}}. \tag{A.20}$$

Note

$$|\sqrt{n}s_{p_{i\Delta,\Delta,m}} - \beta_i^{n,m}| \leq \sqrt{n}\left(\sup_{(i-1)/n\leq s,t\leq i/n}|\int_s^t \mu_u \mathrm{d}u| + \sup_{(i-1)/n\leq s,t\leq i/n}|\int_s^t\Big(\sigma_{u-} - \sigma_{\frac{i-1}{n}}\Big)\mathrm{d}W_u|\right),$$

with the right-hand side independent of m. Now, from (A.20), (A.5) and Lemma 4:

$$U_t^{n,m}(1.1) \xrightarrow{p} 0,$$

uniformly in m. Second, assume $r < 1$. Then

$$|g\left(\sqrt{n}s_{p_{i\Delta,\Delta,m}}\right) - g\left(\beta_i^{n,m}\right)| \leq \frac{1}{\lambda_{r,m}}|\sqrt{n}s_{p_{i\Delta,\Delta,m}} - \beta_i^{n,m}|^r.$$

We get the inequality:

$$\mathbb{E}\left[|U_t^{n,m}(1.1)|\right] \leq \left(\frac{Ct^{\frac{2}{r}-1}}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\sqrt{n}s_{p_{i\Delta,\Delta,m}} - \beta_i^{n,m}|^2\right]\right)^{\frac{r}{2}}.$$

and therefore

$$U_t^{n,m}(1.1) \xrightarrow{p} 0,$$

uniformly in m, which completes the proof. ■

A.6. **Proof of Theorem 3**

In light of the previous results, Theorem 3 follows from the convergence

$$\sqrt{n}\left(RBV_{(r,s),t}^n - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u\right) - U(g,h)_t^n \xrightarrow{p} 0,$$

which is shown by proving that

$$\zeta_i^n = \frac{1}{\sqrt{n}}\mathbb{E}\left[g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] - \sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\rho_{\sigma_u}(g)\,\rho_{\sigma_u}(h)\,\mathrm{d}u,$$

is AN. To accomplish this, we split ζ_i^n into:

$$\zeta_i^n = \zeta_i'^n + \zeta_i''^n,$$

where

$$\zeta_i'^n = \frac{1}{\sqrt{n}}\left(\mathbb{E}\left[g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] - \mathbb{E}\left[g\left(\beta_i^n\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]\mathbb{E}\left[h\left(\beta_i'^n\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]\right), \tag{A.21}$$

$$\zeta_i''^n = \sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\rho_{\sigma_u}(g)\,\rho_{\sigma_u}(h) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\,\rho_{\sigma_{\frac{i-1}{n}}}(h)\right)\mathrm{d}u. \tag{A.22}$$

It follows from Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) that $\zeta_i''^n$ is AN. Next, we prove that the sequence $\zeta_i'^n$ is AN. Using V_2, we introduce the random variables:

$$\begin{aligned}\zeta(1)_i^n = {} & \sqrt{n}\sup_{(i-1)/n\le s,t\le i/n}\Bigg(\int_s^t \sigma_{\frac{i-1}{n}}\mathrm{d}W_u + \int_s^t \mu_{\frac{i-1}{n}}\mathrm{d}u \\ & + \int_s^t\left\{\sigma'_{\frac{i-1}{n}}\left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}}\left(B'_u - B'_{\frac{i-1}{n}}\right)\right\}\mathrm{d}W_u\Bigg) - \beta_i^n,\end{aligned} \tag{A.23}$$

$$\begin{aligned}\zeta(2)_i^n = {} & \sqrt{n}\Bigg\{\sup_{(i-1)/n\le s,t\le i/n}\Bigg(\int_s^t \mu_u\mathrm{d}u + \int_s^t\sigma_u\mathrm{d}W_u\Bigg) - \sup_{(i-1)/n\le s,t\le i/n}\Bigg(\int_s^t\sigma_{\frac{i-1}{n}}\,\mathrm{d}W_u + \int_s^t\mu_{\frac{i-1}{n}}\mathrm{d}u \\ & + \int_s^t\left\{\sigma'_{\frac{i-1}{n}}\left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}}\left(B'_u - B'_{\frac{i-1}{n}}\right)\right\}\mathrm{d}W_u\Bigg)\Bigg\}.\end{aligned} \tag{A.24}$$

We get that

$$\xi_i^n = \zeta(1)_i^n + \zeta(2)_i^n,$$

and a similar decomposition holds for $\xi_i'^n$. The next lemma is shown at the end of this subsection.

Lemma 7 *If $p \in BSM$ and assumption V_2 holds, then for any $q > 0$*

$$\mathbb{E}\left[|\xi_i^n|^q\right] \le Cn^{-\frac{q}{2}}, \tag{A.25}$$

uniformly in i.

We have

$$\zeta_i'^n = \mathbb{E}\left[\delta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

with δ_i^n defined by:

$$\delta_i^n = \frac{1}{\sqrt{n}}\left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) - g\left(\beta_i^n\right)h\left(\beta_i'^n\right)\right).$$

Observe that

$$\begin{aligned}\delta_i^n &= \frac{1}{\sqrt{n}}g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)\left(h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) - h\left(\beta_i'^n\right)\right) + \frac{1}{\sqrt{n}}\left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) - g\left(\beta_i^n\right)\right)h\left(\beta_i'^n\right) \\ &\equiv \delta_i'^n + \delta_i''^n.\end{aligned}$$

We show that

$$\mathbb{E}\left[\delta_i''^n \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

is AN, but the omit the proof for $\mathbb{E}\left[\delta_i'^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ to save space. We define

$$A_i^n = \left\{|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n| > \frac{\beta_i^n}{2}\right\}.$$

We find that

$$\begin{aligned} g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) - g\left(\beta_i^n\right) &= \left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) - g\left(\beta_i^n\right)\right) I_{A_i^n} - \nabla g\left(\beta_i^n\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right) I_{A_i^n} \\ &\quad + \left(\nabla g\left(\bar{\gamma}_i^n\right) - \nabla g\left(\beta_i^n\right)\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right) I_{\left(A_i^n\right)^c} + \nabla g\left(\beta_i^n\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right) \\ &\equiv \vartheta_i^n(1) + \vartheta_i^n(2) + \vartheta_i^n(3) + \vartheta_i^n(4), \end{aligned} \tag{A.26}$$

where $\bar{\gamma}_i^n$ is some random variable located between $\sqrt{n}s_{p_{i\Delta,\Delta}}$ and β_i^n. Then

$$\delta_i''^n = \delta_i''^n(1) + \delta_i''^n(2) + \delta_i''^n(3) + \delta_i''^n(4),$$

with

$$\delta_i''^n(k) = \frac{1}{\sqrt{n}} h\left(\beta_i^n\right)\vartheta_i^n(k).$$

To complete the proof, it therefore suffices that

$$\mathbb{E}\left[\delta_i''^n(k) \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

are AN for $k = 1$, 2, 3, and 4.

The term $\mathbb{E}\left[\delta_i''^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$:

With $r \geq 1$:

$$|\vartheta_i^n(1)| \leq C|\sqrt{n}s_{p_{i\Delta,\Delta}} + \beta_i^n|^{r-1}\frac{|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+s}}{\left(\beta_i^n\right)^s},$$

for some $s \in (0,1)$. As μ and σ are bounded:

$$\mathbb{E}\left[|\sqrt{n}s_{p_{i\Delta,\Delta}}|^p\right] \leq C_p, \tag{A.27}$$

for all $p > 0$. With $r < 1$:

$$|\vartheta_i^n(1)| \leq C\frac{|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+r/2}}{\left(\beta_i^n\right)^{1-r/2}}. \tag{A.28}$$

Now

$$\mathbb{E}\left[\delta_i''^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = \frac{1}{\sqrt{n}}\rho_{\sigma_{\frac{i-1}{n}}}(h)\,\mathbb{E}\left[\vartheta_i^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right].$$

For all $r > 0$, it follows by (A.5), (A.27), (**2.19**), Lemma 7 and Hölder's inequality that:

$$\mathbb{E}\left[\left|\vartheta_i^n(1)\right|\right] \leq C n^{-\frac{q}{2}},$$

for some $q > 1$, so $\mathbb{E}\left[\delta_i''^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is AN.

The term $\mathbb{E}\left[\delta_i''^n(2) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$**:**

As before, for some $s \in (0,1)$,

$$\begin{aligned} |\vartheta_i^n(2)| &\leq C\left(\beta_i^n\right)^{r-1-s} |\sqrt{n} s_{p_i\Delta,\Delta} - \beta_i^n|^{1+s}, \quad \text{for } r \geq 1, \\ |\vartheta_i^n(2)| &\leq C\left(\beta_i^n\right)^{r/2-1} |\sqrt{n} s_{p_i\Delta,\Delta} - \beta_i^n|^{1+r/2}, \quad \text{for } r < 1. \end{aligned} \tag{A.29}$$

The AN property of $\mathbb{E}\left[\delta_i''^n(2) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is now a consequence of Equation (A.5), (**2.19**), Lemma 7 and Hölder's inequality.

The term $\mathbb{E}\left[\delta_i''^n(3) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$**:**

For $r \geq 2$:

$$|\vartheta_i^n(3)| \leq C|\sqrt{n} s_{p_i\Delta,\Delta} + \beta_i^n|^{r-2} |\sqrt{n} s_{p_i\Delta,\Delta} - \beta_i^n|^2.$$

For $r < 2$:

$$|\vartheta_i^n(3)| \leq C\left(\beta_i^n\right)^{r-2} |\sqrt{n} s_{p_i\Delta,\Delta} - \beta_i^n|^2 I_{\left(A_i^n\right)^c}.$$

By the definition of A_i^n:

$$|\vartheta_i^n(3)| \leq C\left(\beta_i^n\right)^{r/2-1} |\sqrt{n} s_{p_i\Delta,\Delta} - \beta_i^n|^{1+r/2}, \tag{A.30}$$

for $r < 2$. That $\mathbb{E}\left[\delta_i''^n(3) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is AN follows from the above.

The term $\mathbb{E}\left[\delta_i''^n(4) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$**:**

First, we find a stochastic expansion for

$$\xi_i^n = \sqrt{n} s_{p_i\Delta,\Delta} - \beta_i^n,$$

defined in (A.11). Recall that

$$\xi_i^n = \zeta(1)_i^n + \zeta(2)_i^n,$$

with $\zeta(1)_i^n$ and $\zeta(2)_i^n$ defined by (A.23) and (A.24), respectively. Set

$$\begin{aligned} f_{in}(s,t) &= \sqrt{n}\sigma_{\frac{i-1}{n}}(W_t - W_s), \\ g_{in}(s,t) &= n\int_s^t \mu_{\frac{i-1}{n}}\mathrm{d}u + n\int_s^t \left\{\sigma'_{\frac{i-1}{n}}\left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}}\left(B'_u - B'_{\frac{i-1}{n}}\right)\right\}\mathrm{d}W_u \\ &= \mu_{\frac{i-1}{n}}g^1_{in}(s,t) + \sigma'_{\frac{i-1}{n}}g^2_{in}(s,t) + v'_{\frac{i-1}{n}}g^3_{in}(s,t), \end{aligned}$$

to achieve the identity:

$$\zeta(1)_i^n = \sup_{(i-1)/n\le s,t\le i/n}\left(f_{in}(t,s) + \frac{1}{\sqrt{n}}g_{in}(t,s)\right) - \sup_{(i-1)/n\le s,t\le i/n} f_{in}(t,s).$$

Imposing assumption V_1:

$$\begin{aligned} (t^*_{in}(W), s^*_{in}(W)) &= \arg\sup_{(i-1)/n\le s,t\le i/n} f_{in}(s,t) \\ &= \arg\sup_{(i-1)/n\le s,t\le i/n} \sqrt{n}(W_t - W_s) \\ &\overset{d}{=} \arg\sup_{0\le s,t\le 1}(W_t - W_s). \end{aligned}$$

A standard result states that the pair $(t^*_{in}(W), s^*_{in}(W))$ is unique, almost surely (e.g., Revuz & Yor (1998)). In Lemma 8, a stochastic expansion of $\zeta(1)_i^n$ is given.

Lemma 8 *Given assumption V_1*

$$\zeta(1)_i^n = \frac{1}{\sqrt{n}}\left\{g_{in}(t^*_{in}(W), s^*_{in}(W)) + \tilde{g}_{in}\right\},$$

where

$$\mathbb{E}\left[|\tilde{g}_{in}|^p\right] = \mathrm{o}(1), \tag{A.31}$$

for all $p > 0$ and uniformly in i.

Note also that

$$(t^*_{in}(W), s^*_{in}(W)) = (s^*_{in}(-W), t^*_{in}(-W)).$$

As $(W, B') \overset{d}{=} -(W, B')$ and $\nabla g(\beta_i^n)$ is an even functional of W:

$$\mathbb{E}\left[\nabla g(\beta_i^n)\, g^k_{in}(t^*_{in}(W), s^*_{in}(W)) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0,$$

for $k = 1, 2, 3$. Hence

$$\mathbb{E}\left[\nabla g(\beta_i^n)\, g_{in}(t^*_{in}(W), s^*_{in}(W)) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0. \tag{A.32}$$

For $\zeta(2)_i^n$, we get the estimate

$$\zeta(2)_i^n \leq \sqrt{n}\Bigg(\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\mu_u - \mu_{\frac{i-1}{n}}| \mathrm{d}u \tag{A.33}$$

$$+ \sup_{(i-1)/n \leq s,t \leq i/n} \int_s^t \left\{\int_{\frac{i-1}{n}}^u \mu_r' \mathrm{d}r + \int_{\frac{i-1}{n}}^u \left(\sigma_{r-}' - \sigma_{\frac{i-1}{n}}'\right) \mathrm{d}W_r + \int_{\frac{i-1}{n}}^u \left(v_{r-}' - v_{\frac{i-1}{n}}'\right) \mathrm{d}B_r'\right\} \mathrm{d}W_u\Bigg).$$

Lemma 9 *For $q \geq 2$, it then holds that*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left(\mathbb{E}\left[|\zeta(2)_i^n|^q\right]\right)^{\frac{1}{q}} \to 0,$$

for all $t > 0$.

Using Hölder's inequality, it follows that

$$|\,\mathbb{E}\left[\delta_i^{\prime\prime n}(4) \mid \mathcal{F}_{\frac{i-1}{n}}\right]| = \frac{1}{\sqrt{n}} \rho_{\sigma_{\frac{i-1}{n}}}(h)\, |\,\mathbb{E}\left[\nabla g(\beta_i^n)\left(\zeta(1)_i^n + \zeta(2)_i^n\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]|$$

$$\leq \frac{1}{\sqrt{n}} \rho_{\sigma_{\frac{i-1}{n}}}(h) \Bigg(|\,\mathbb{E}\left[\nabla g(\beta_i^n)\, \zeta(1)_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]|$$

$$+ \left(\mathbb{E}\left[(\nabla g(\beta_i^n))^p\right]\right)^{\frac{1}{p}} \left(\mathbb{E}\left[|\zeta(2)_i^n|^q\right]\right)^{\frac{1}{q}}\Bigg), \tag{A.34}$$

for some $p > 1, q \geq 2$ with $(r-1)p > -1$ and $1/p + 1/q = 1$. Finally, by combining (**2.19**), (A.31), (A.32) and Lemma 9, we get the AN property of the sequence $\mathbb{E}\left[\delta_i^{\prime\prime n}(4) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$. Hence, Theorem 3 with $m = \infty$ has been proven.

$m < \infty$: To show the theorem with $m < \infty$, the main structure of the previous proof can be adapted directly. The difference lies in the moment condition:

$$\lambda_{r,m} < \infty,$$

for $r > -m$. The estimates (A.28), (A.29), (A.30) and (A.34), however, were formulated such that this condition can be used without changing the proof (for all $m \in \mathbb{N}$). ■

Proof of Lemma 7

Note that:

$$\mathbb{E}\left[|\zeta(1)_i^n|^q\right] \leq C n^{\frac{q}{2}} \Bigg(\sup_{(i-1)/n \leq s,t \leq i/n} |\int_s^t \mu_{\frac{i-1}{n}} \mathrm{d}u|^q$$

$$+ \sup_{(i-1)/n \leq s,t \leq i/n} |\int_s^t \left\{\sigma_{\frac{i-1}{n}}' \left(W_u - W_{\frac{i-1}{n}}\right) + v_{\frac{i-1}{n}}' \left(B_u' - B_{\frac{i-1}{n}}'\right)\right\} \mathrm{d}W_u|^q\Bigg).$$

The boundedness of μ, σ', v' and Burkholder's inequality give

$$\mathbb{E}\left[\left|\zeta\left(1\right)_{i}^{n}\right|^{q}\right] \leq Cn^{-\frac{q}{2}}.$$

$\zeta\left(2\right)_{i}^{n}$ is handled equivalently. ■

Proof of Lemma 8

We need a deterministic version of Lemma 8:

Lemma 10 *Given two continuous functions $f, g : I \rightarrow \mathbb{R}$ on a compact set $I \subseteq \mathbb{R}^{n}$, assume t^{*} is the only point where the maximum of the function f on I is achieved. Then, it holds:*

$$M_{\epsilon}\left(g\right) \equiv \frac{1}{\epsilon}\left[\sup_{t \in I}\left\{f\left(t\right)+\epsilon g\left(t\right)\right\}-\sup_{t \in I}\left\{f\left(t\right)\right\}\right] \rightarrow g\left(t^{*}\right) \quad \textit{as} \quad \epsilon \downarrow 0.$$

Proof of Lemma 10

Construct the set

$$\bar{G}=\Big\{h \in C\left(I\right) \mid h \text{ is constant on } B_{\delta}\left(t^{*}\right) \cap I \text{ for some } \delta>0\Big\}.$$

As usual, $C\left(I\right)$ is the set of continuous functions on I and $B_{\delta}\left(t^{*}\right)$ is an open ball of radius δ centered at t^{*}. Take $\bar{g} \in \bar{G}$ and recall $\bar{g}$ is bounded on I. Thus, for ϵ small enough:

$$\begin{aligned}
\sup_{t \in I}\left\{f\left(t\right)+\epsilon \bar{g}\left(t\right)\right\} &= \max \left\{\sup_{t \in I \cap B_{\delta}\left(t^{*}\right)}\left\{f\left(t\right)+\epsilon \bar{g}\left(t\right)\right\}, \sup_{t \in I \cap B_{\delta}^{c}\left(t^{*}\right)}\left\{f\left(t\right)+\epsilon \bar{g}\left(t\right)\right\}\right\} \\
&= \sup_{t \in I \cap B_{\delta}\left(t^{*}\right)}\left\{f\left(t\right)+\epsilon \bar{g}\left(t\right)\right\} \\
&= f\left(t^{*}\right)+\epsilon \bar{g}\left(t^{*}\right).
\end{aligned}$$

So

$$M_{\epsilon}\left(\bar{g}\right) \rightarrow \bar{g}\left(t^{*}\right),$$

$\forall\, \bar{g} \in \bar{G}$. Now, let $g \in C\left(I\right)$. As $\bar{G}$ is dense in $C\left(I\right)$, $\exists\, \bar{g} \in \bar{G}: \bar{g}\left(t^{*}\right)=g\left(t^{*}\right)$ and $\left|\bar{g}-g\right|_{\infty}<\epsilon'$ ($\left|\cdot\right|_{\infty}$ is the sup-norm). We see that $\left|M_{\epsilon}\left(\bar{g}\right)-M_{\epsilon}\left(g\right)\right|<\epsilon'$, and

$$\left|M_{\epsilon}\left(g\right)-g\left(t^{*}\right)\right| \leq\left|M_{\epsilon}\left(\bar{g}\right)-\bar{g}\left(t^{*}\right)\right|+\left|M_{\epsilon}\left(g\right)-M_{\epsilon}\left(\bar{g}\right)\right| \rightarrow 0.$$

Thus, the assertion is established. ■

Now, $\left(t_{in}^{*}\left(W\right), s_{in}^{*}\left(W\right)\right)$ is unique and the functions g_{in}, f_{in} are continuous, both almost surely. Thus, Lemma 8 is shown by replicating the proof of Lemma 10 for g_{in} and f_{in}. More precisely, the random function $\bar{g}_{in} \in \bar{G}$ that is constant in a neighbourhood of $(t^*_{in}(W), s^*_{in}(W))$ must be constructed. The rest goes along the lines of Lemma 10. ■

Proof of Lemma 9

From (A.33) and repeated use of the Hölder and Burkholder inequalities plus the boundedness of μ', we get:

$$\mathbb{E}\left[|\zeta(2)^n_i|^q\right] \leq C_q n^{\frac{q}{2}} \left(n^{-q+1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |\mu_u - \mu_{\frac{i-1}{n}}|^q \mathrm{d}u + n^{-\frac{3q}{2}} + n^{-q+1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |\sigma'_{u-} - \sigma'_{\frac{i-1}{n}}|^q \mathrm{d}u \right.$$
$$\left. + n^{-q+1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |v'_{u-} - v'_{\frac{i-1}{n}}|^q \mathrm{d}u \right).$$

Thus,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left(\mathbb{E}\left[|\zeta(2)^n_i|^q\right]\right)^{\frac{1}{q}} \leq C_q t^{\frac{q-1}{q}} \left(\sum_{i=1}^{[nt]} \mathbb{E}\left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\mu_u - \mu_{\frac{i-1}{n}}|^q \mathrm{d}u + \int_{\frac{i-1}{n}}^{\frac{i}{n}} |\sigma'_{u-} - \sigma'_{\frac{i-1}{n}}|^q \mathrm{d}u \right.\right.$$
$$\left.\left. + \int_{\frac{i-1}{n}}^{\frac{i}{n}} |v'_{u-} - v'_{\frac{i-1}{n}}|^q \mathrm{d}u \right] \right)^{\frac{1}{q}} + \mathrm{o}(1)$$
$$= C_q t^{\frac{q-1}{q}} \left(\mathbb{E}\left[\int_0^t |\mu_u - \mu_{\frac{[nu]}{n}}|^q + |\sigma'_{u-} - \sigma'_{\frac{[nu]}{n}}|^q \right.\right.$$
$$\left.\left. + |v'_{u-} - v'_{\frac{[nu]}{n}}|^q \mathrm{d}u \right] \right)^{\frac{1}{q}} + \mathrm{o}(1).$$

As σ' and v' are càdlàg, the proof is complete. ■

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Table 1: Finite sample properties of t-statistics for jump detection.

	$z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$				$z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$		
Size:	$n = 39$	$n = 78$	$n = 390$		$n = 39$	$n = 78$	$n = 390$
$\alpha = 0.01$	2.236	1.856	1.292		1.417	1.291	1.164
0.05	6.497	6.071	5.504		5.607	5.447	5.114
0.10	10.769	10.673	10.282		10.379	10.186	10.008
Power:				$j = 1$			
$\sigma^2_J = 0.05$	11.982	18.385	33.970		3.805	6.777	23.314
0.10	23.440	32.096	48.995		8.215	15.317	38.592
0.15	31.882	41.576	57.492		12.895	22.694	47.714
0.20	37.890	47.217	62.343		17.144	28.172	53.138
0.25	42.582	51.886	65.832		20.873	32.916	57.572
				$j = 2$			
$\sigma^2_J = 0.05$	21.767	33.971	58.519		6.003	12.673	43.301
0.10	40.940	55.274	75.695		14.091	27.806	63.913
0.15	53.817	66.833	83.029		22.654	39.789	73.940
0.20	61.641	73.426	86.930		29.419	48.280	79.307
0.25	67.074	77.683	89.234		35.410	54.375	82.817

The table reports small sample properties of the jump detection t-statistics at the sampling frequencies $n = 39, 78, 390$ ($m = 30, 15, 3$). We show the actual size of the tests at an $\alpha = 0.01, 0.05, 0.10$ nominal level of significance. Power is computed at the $\alpha = 0.01$ nominal level with $j = 1$ or $j = 2$ IID $N\big(0, \sigma^2_J\big)$ jumps added to the continuous process, and we set $\sigma^2_J = 0.05, 0.10, \ldots, 0.25$.

Table 2: Number of tick data pr. trading day.

Ticker	Trades			Quotes		
	All	$\#r_{\tau_i} \neq 0$	$\#\Delta r_{\tau_i} \neq 0$	All	$\#r_{\tau_i} \neq 0$	$\#\Delta r_{\tau_i} \neq 0$
MRK	2891	1314	706	5537	1750	1246

The table contains information about the filtering of the Merck high-frequency data. All numbers are averages across the 1,253 trading days in our sample from January 3, 2000 through December 31, 2004. $\#r_{\tau_i} \neq 0$ is the amount of tick data left after skipping transaction price (midquote) repetitions in consecutive ticks. $\#\Delta r_{\tau_i} \neq 0$ also ignores reversals.

Table 3: Sample statistics for estimators of $\langle p \rangle_t$, $\int_0^t \sigma_u^2 \mathrm{d}u$ and $\int_0^t \sigma_u^4 \mathrm{d}u$.

	Mean	Var.	Skew.	Kurt.	Min.	Max.	Correlation					
$RRV_t^{n,m}$	7.266	64.143	5.554	52.411	0.597	117.679	1.000	*0.986*	0.901	0.888	0.798	0.727
RV_t^n	7.063	67.200	5.271	47.118	0.372	116.096		1.000	0.916	0.927	0.810	0.747
$RBV_t^{n,m}$	6.077	30.446	3.681	24.774	0.495	61.589			1.000	*0.976*	0.843	0.759
BV_t^n	6.459	48.860	4.353	31.893	0.284	82.273				1.000	0.855	0.796
$RQQ_t^{n,m}$	0.286	0.843	10.751	154.839	0.001	16.114					1.000	*0.972*
QQ_t^n	0.381	2.598	13.280	231.051	0.000	33.576						1.000

Sample statistics for the annualized percentage $RRV_t^{n,m}$, $RBV_t^{n,m}$, $RQQ_t^{n,m}$, RV_t^n, BV_t^n and QQ_t^n of Merck from January 3, 2000 through December 31, 2004. The table shows the mean, variance, skewness, kurtosis, minimum and maximum of the various time series, plus the correlation matrix. $RQQ_t^{n,m}$ and QQ_t^n are further multiplied by 100 to improve the scale.

Table 4: Proportion of $\langle p \rangle_t$ induced by the jump process.

		$\alpha = 0.05$		$\alpha = 0.01$	
	$\%JV$	#rej	$\%JV_s$	#rej	$\%JV_s$
$RRV_t^{n,m}$	16.966	141	7.984	48	5.559
RV_t^n	10.938	289	8.012	151	6.091

The proportion of $\langle p \rangle_t$ of Merck induced by the jump process is reported using three criteria. $\%JV = \sum_{t=1}^{T} (RRV_t^{n,m} - RBV_t^{n,m})^+ / \sum_{t=1}^{T} RRV_t^{n,m}$ is an aggregate measure for the $T = 1,253$ trading days in our sample period from January 3, 2000 to December 31, 2004 (the definition for RV_t^n is identical). $\%JV_s$ sums only significant terms - at the $\alpha = 0.05$ or $\alpha = 0.01$ nominal level - in the numerator of $\%JV$. #rej is the number of rejections made by $z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ and $z^{ar}_{RV_t^n, BV_t^n, QQ_t^n}$.

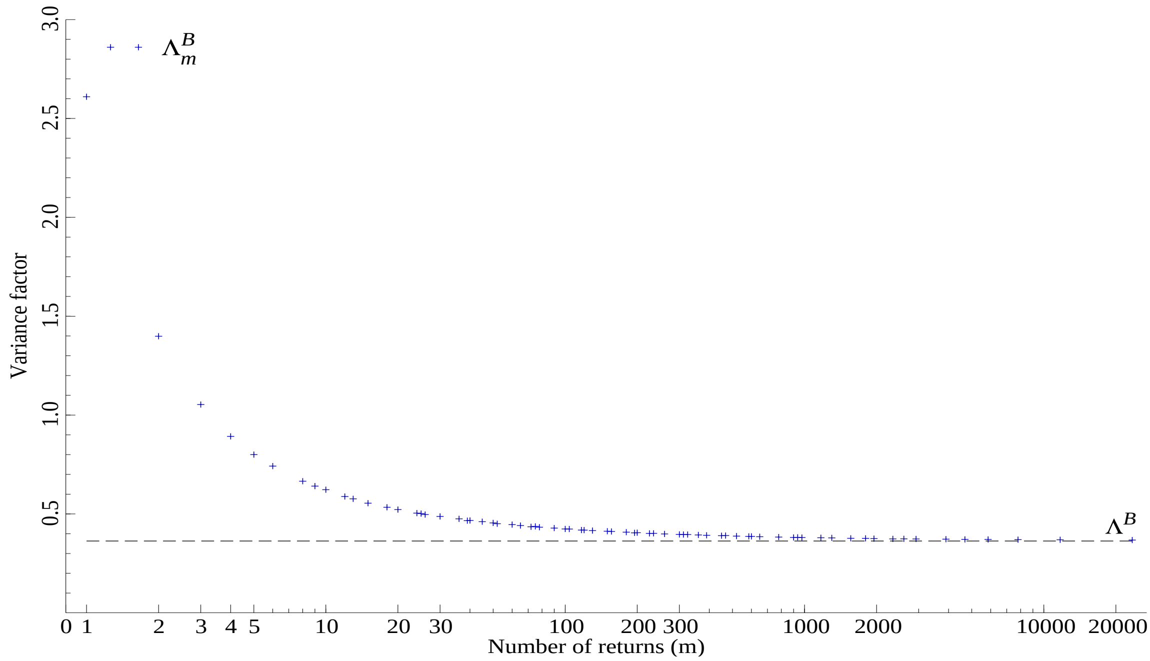


Figure 1: Λ_m^B against m on a log-scale. All estimates are based on a simulation with 1,000,000 repetitions, and the dashed line represents the asymptotic value.

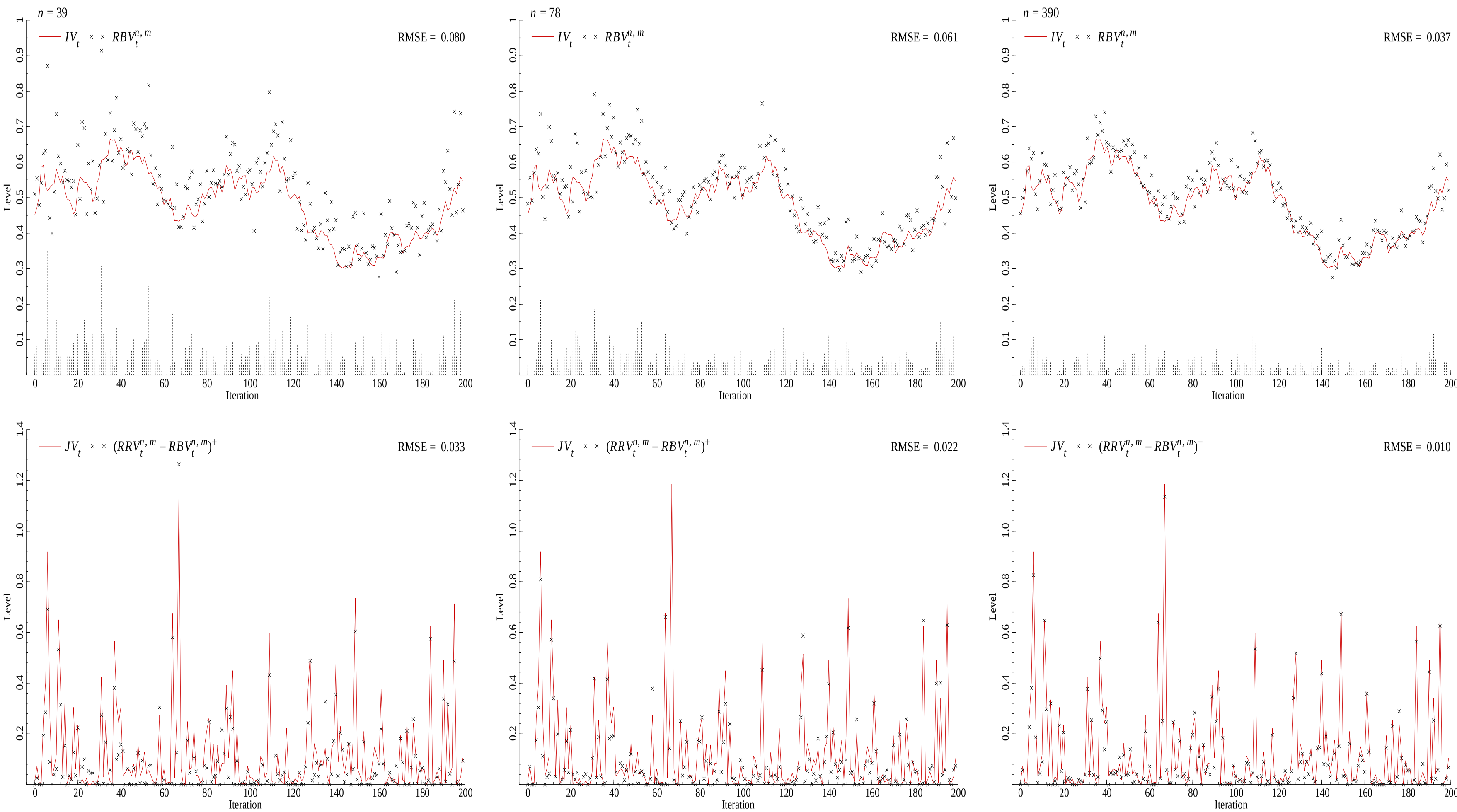


Figure 2: $RBV_t^{n,m}$ and $(RRV_t^{n,m} - RBV_t^{n,m})^+$ are plotted against the population value of the integrated variance (IV_t) and jump variation (JV_t) for 200 iterations with $j = 1$ and $\sigma_J^2 = 0.10$, using $n = 39, 78, 390$ (column-wise). The vertical dashed lines in the first row are the absolute sampling errors of $RBV_t^{n,m}$. We also report the root mean squared error (RMSE) across all 100,000 repetitions.

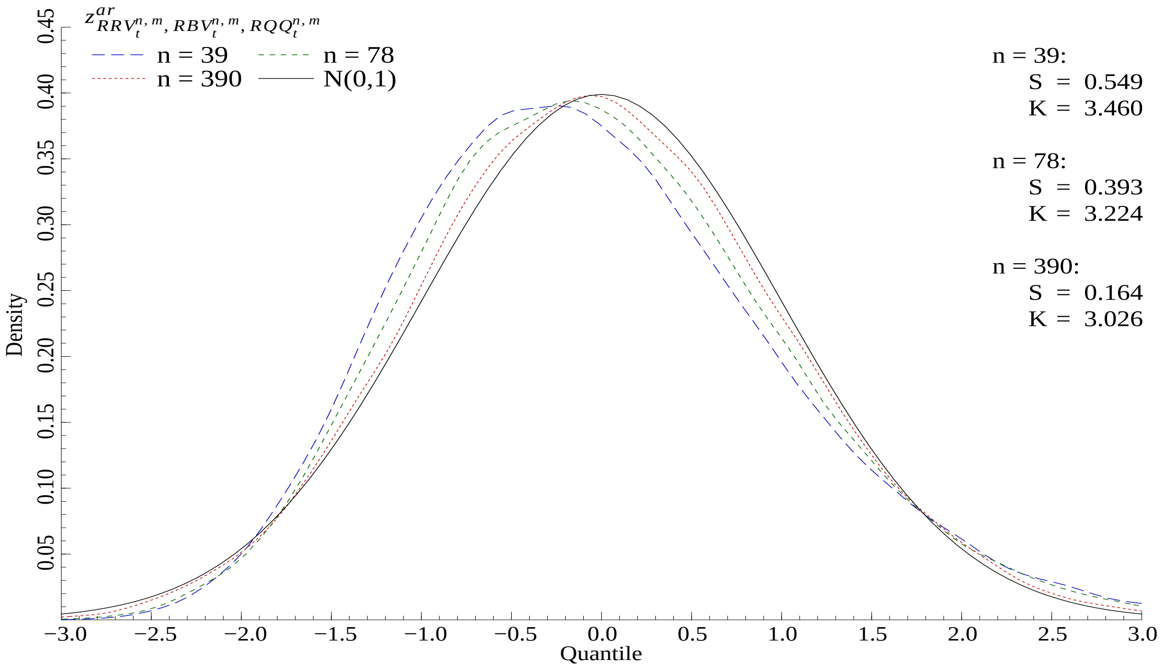


Figure 3: The distribution of $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$ under the null is drawn against the sampling frequencies $n = 39, 78, 390$ ($m = 30, 15, 3$). We compute the coefficient of skewness and kurtosis for each n and add a standard normal distribution for visual reference (solid line). The figure is based on a simulation with 100,000 repetitions, as detailed in the main text.

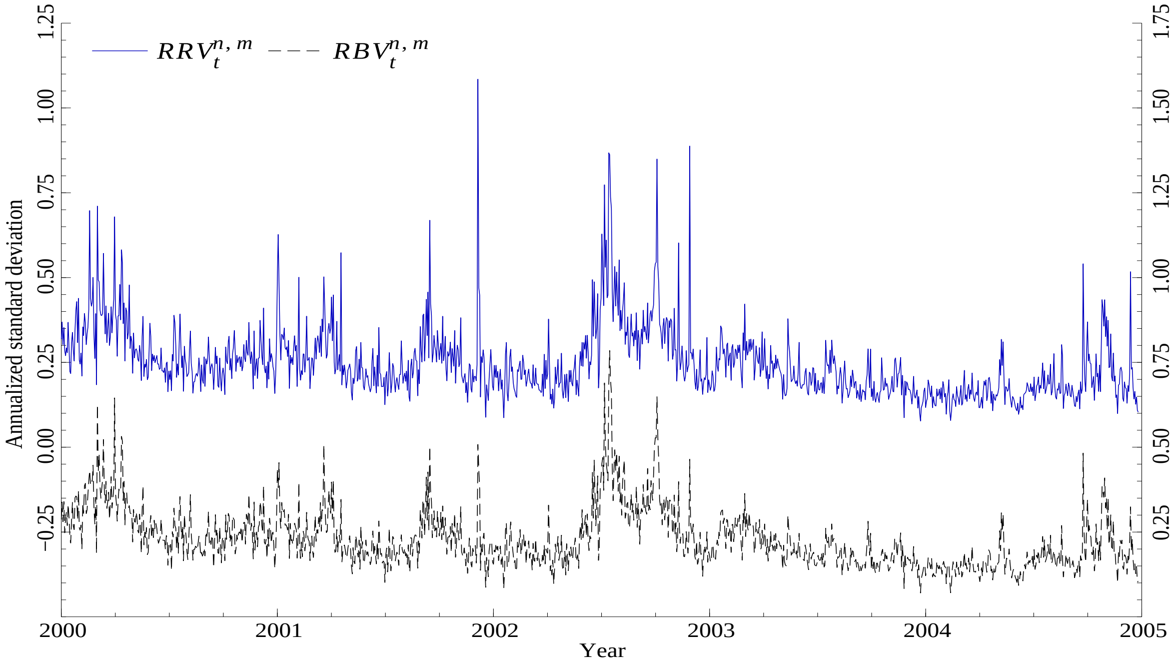
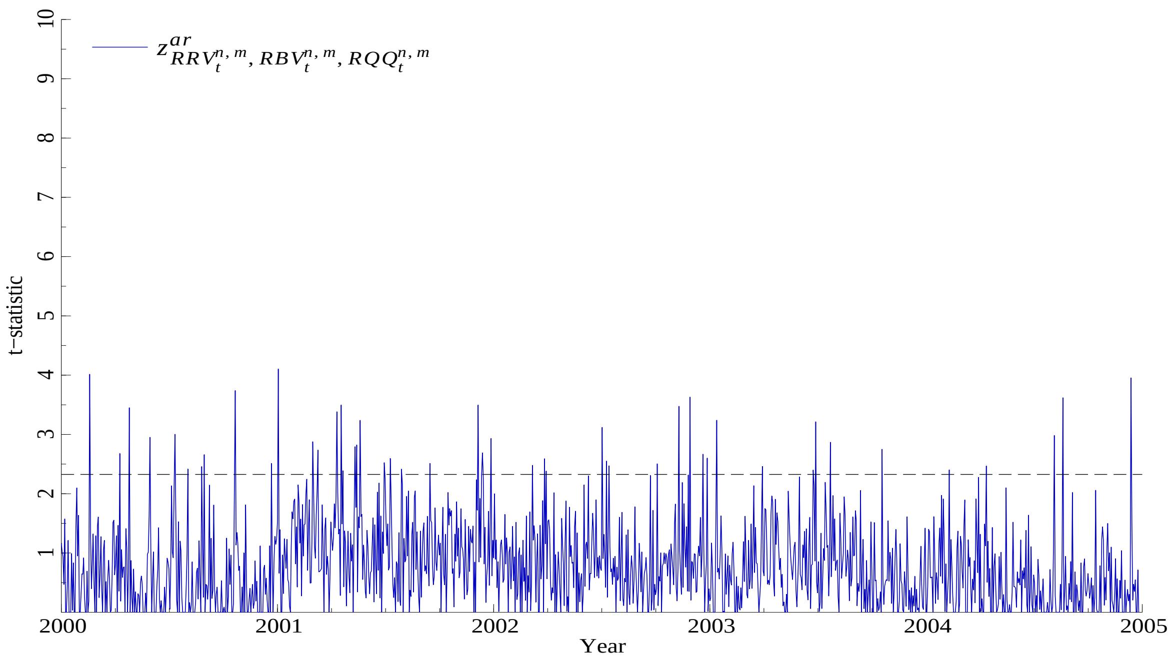


Figure 4: $RRV_t^{n,m}$ and $RBV_t^{n,m}$ are shown for the period January 3, 2000 through December 31, 2004. The series are based on tick-time sampled ranges of Merck, setting $m = 15$, and reported as annualized standard deviations. $RRV_t^{n,m}$ ($RBV_t^{n,m}$) is read off from the left (right) y-axis.



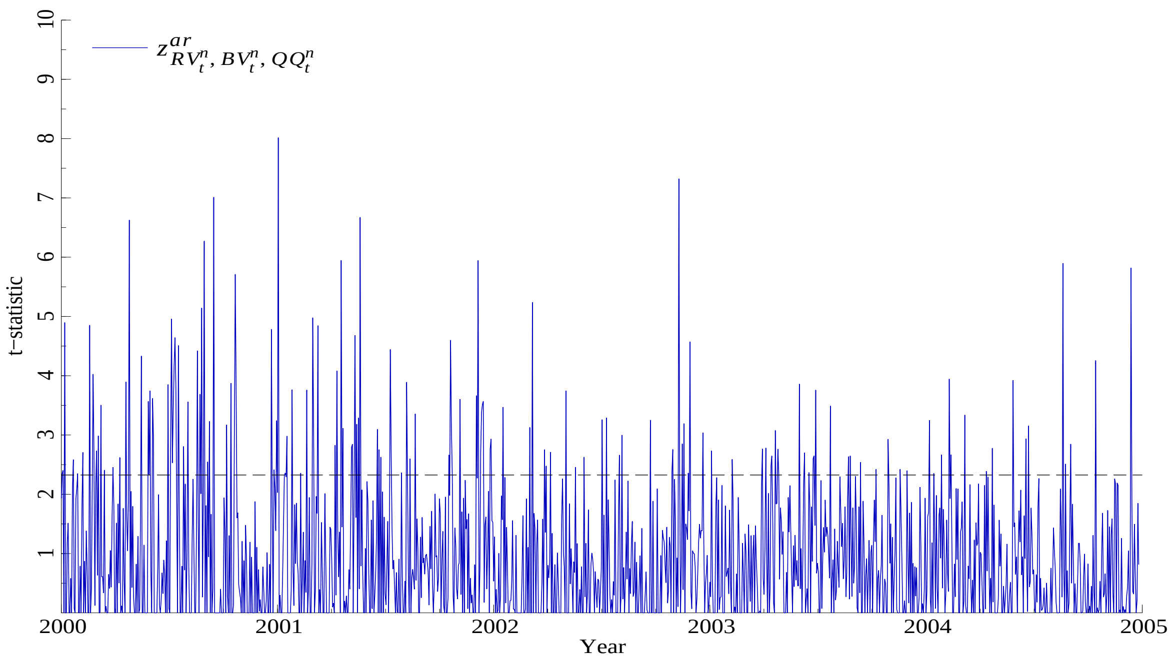


Figure 5: We plot the jump detection t-statistics $z^{ar}_{RRV^{n,m}_t,RBV^{n,m}_t,RQQ^{n,m}_t}$ and $z^{ar}_{RV^n_t,BV^n_t,QQ^n_t}$. The horizontal dashed line is the 0.99 quantile of the standard normal distribution.

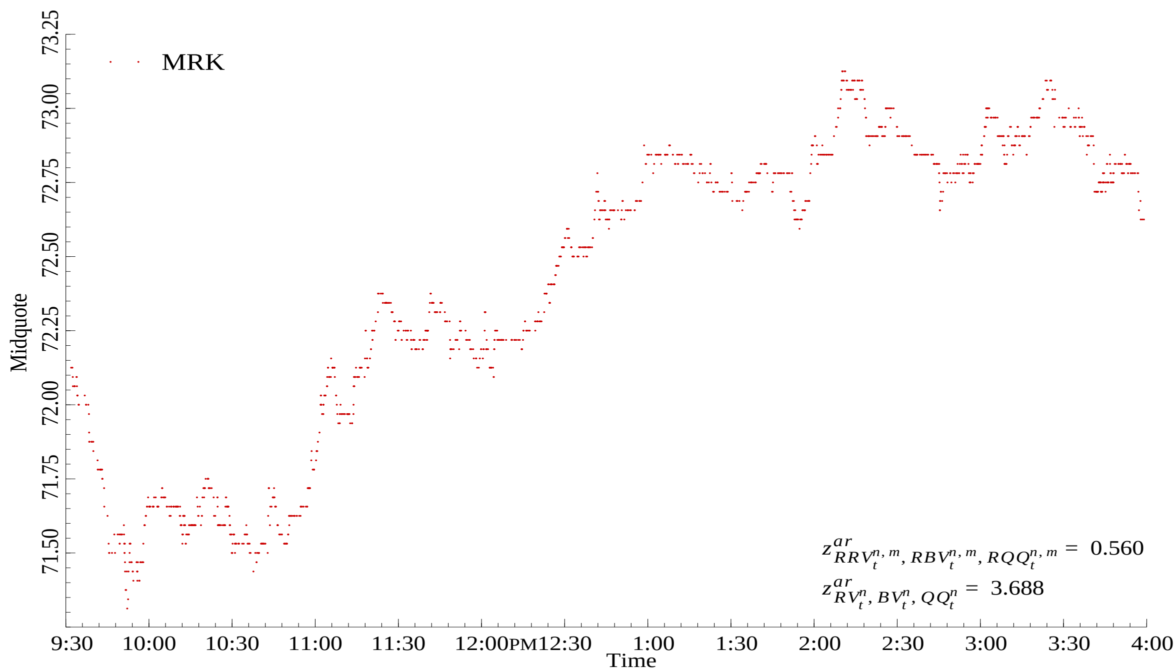


Figure 6: The midquote data of Merck on August 24, 2000 is shown. In the lower right-hand corner, we report the outcome of $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ and $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$.