Random block matrices and matrix orthogonal polynomials

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Abstract

In this paper we consider random block matrices, which generalize the general beta ensembles, which were recently investigated by Dumitriu and Edelmann (2002, 2005). We demonstrate that the eigenvalues of these random matrices can be uniformly approximated by roots of matrix orthogonal polynomials which were investigated independently from the random matrix literature. As a consequence we derive the asymptotic spectral distribution of these matrices. The limit distribution has a density, which can be represented as the trace of an integral of densities of matrix measures corresponding to the Chebyshev matrix polynomials of the first kind. Our results establish a new relation between the theory of random block matrices and the field of matrix orthogonal polynomials, which have not been explored so far in the literature.

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1 Introduction

The classical orthogonal, unitary and symplectic ensembles of random matrices have been studied extensively in the literature, and there are numerous directions to generalize these investigations to the case of random block matrices. Several authors have studied the limiting spectra of such matrices under different conditions on the blocks [see e.g. Girko (2000), Oraby (2006 a,b) among others].

In the present paper we study a class of random block matrices which generalize the tridiagonal matrices corresponding to the classical β-Hermite ensemble defined by its density

$$c_\beta \cdot \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^n \frac{\lambda_i^2}{2}} , \tag{1.1}$$

where $\beta > 0$ is a given parameter and c_β a normalizing constant. It is known for a long time that for $\beta = 1, 2, 4$ this density corresponds to the Gaussian ensembles [see Dyson (1962)], which have been studied extensively in the literature on random matrices [see Mehta (1967)]. It was recently shown by Dumitriu and Edelmann (2002) that for any $\beta > 0$ the joint density of the eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ of the $n \times n$ symmetric matrix

$$G_n^{(1)} = \begin{pmatrix} N_1 & \frac{1}{\sqrt{2}}X_{(n-1)\beta} & & & \\ \frac{1}{\sqrt{2}}X_{(n-1)\beta} & N_2 & \frac{1}{\sqrt{2}}X_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\sqrt{2}}X_{2\beta} & N_{n-1} & \frac{1}{\sqrt{2}}X_\beta \\ & & & \frac{1}{\sqrt{2}}X_\beta & N_n \end{pmatrix} \tag{1.2}$$

is given by (1.1) where $X_\beta, \dots, X_{(n-1)\beta}$, $N_1, \dots, N_n$ are independent random variables with $X_{j\beta}^2 \sim \mathcal{X}_{j\beta}^2$ and $N_j \sim \mathcal{N}(0,1)$. Here $\mathcal{X}_{j\beta}^2$ and $\mathcal{N}(0,1)$ denote a chi-square (with $j\beta$ degrees of freedom) and a standard normal distributed random variable, respectively. In the present paper we consider random block matrices of the form

$$G_n^{(p)} = \begin{pmatrix} B_0^{(p)} & A_1^{(p)} & & & \\ A_1^{(p)} & B_1^{(p)} & A_2^{(p)} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{\frac{n}{p}-2}^{(p)} & B_{\frac{n}{p}-2}^{(p)} & A_{\frac{n}{p}-1}^{(p)} \\ & & & A_{\frac{n}{p}-1}^{(p)} & B_{\frac{n}{p}-1}^{(p)} \end{pmatrix}, \tag{1.3}$$

where $n = mp \in \mathbb{N},\ m \in \mathbb{N},\ p \in \mathbb{N}$ and the $p \times p$ blocks $A_i^{(p)}$ and $B_i^{(p)}$ are defined by

$$B_i^{(p)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}N_{ip+1} & X_{\gamma_1(n-ip-1)} & X_{\gamma_2(n-ip-2)} & \cdots & X_{\gamma_{p-1}(n-(i+1)p+1)} \\ X_{\gamma_1(n-ip-1)} & \sqrt{2}N_{ip+2} & X_{\gamma_1(n-ip-2)} & \cdots & X_{\gamma_{p-2}(n-(i+1)p+1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X_{\gamma_{p-2}(n-(i+1)p+2)} & \cdots & X_{\gamma_1(n-(i+1)p+1)} & \sqrt{2}N_{(i+1)p-1} & X_{\gamma_1(n-(i+1)p+1)} \\ X_{\gamma_{p-1}(n-(i+1)p+1)} & \cdots & X_{\gamma_2(n-(i+1)p+1)} & X_{\gamma_1(n-(i+1)p+1)} & \sqrt{2}N_{(i+1)p} \end{pmatrix} \tag{1.4}$$

and

$$A_i^{(p)} = \frac{1}{\sqrt{2}} \begin{pmatrix} X_{\gamma_p(n-ip)} & X_{\gamma_{p-1}(n-ip)} & X_{\gamma_{p-2}(n-ip)} & \cdots & X_{\gamma_1(n-ip)} \\ X_{\gamma_{p-1}(n-ip)} & X_{\gamma_p(n-ip-1)} & X_{\gamma_{p-1}(n-ip-1)} & \cdots & X_{\gamma_2(n-ip-1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X_{\gamma_2(n-ip)} & \cdots & X_{\gamma_{p-1}(n-(i+1)p+3)} & X_{\gamma_p(n-(i+1)p+2)} & X_{\gamma_{p-1}(n-(i+1)p+2)} \\ X_{\gamma_1(n-ip)} & \cdots & X_{\gamma_{p-2}(n-(i+1)p+3)} & X_{\gamma_{p-1}(n-(i+1)p+2)} & X_{\gamma_p(n-(i+1)p+1)} \end{pmatrix}, \quad (1.5)$$

respectively, $\gamma_1, \ldots, \gamma_p > 0$ are given constants and all random variables are independent. Note that in the case $p = 1$ we have $G_n^{(p)} = G_n^{(1)}$ and that for $p > 1$ the elements $g_{ij}^{(p)}$ of the matrix $G_n^{(p)}$ are standard normally distributed if $i = j$ and $\mathcal{X}_k$-distributed else, where the degrees of freedom depend on the position of the element in the matrix $G_n^{(p)}$. In the present paper we investigate the limiting spectral behaviour of matrices of the form (1.3). In particular it is demonstrated that the eigenvalues of the matrix $G_n^{(p)}$ are closely related to roots of matrix orthogonal polynomials, which have been studied independently from the theory of random matrices [see e.g. Duran (1996, 1999), Duran and Daneri-Vias (2001), Duran and Lopez-Rodriguez (1996), Duran, Lopez-Rodriguez and Saff (1999), Sinap and van Assche (1996) or Zygmunt (2002) among others]. To our knowledge this is the first paper, which relates the field of matrix orthogonal polynomials to the theory of random matrices.

In Section 2 we review some basic facts on matrix orthogonal polynomials. We prove a new result on the asymptotic behaviour of the roots of such polynomials which is of own interest and the basis for the investigation of the limiting spectral properties of the matrix $G_n^{(p)}$. In Section 3 we provide a strong uniform approximation of the random eigenvalues of the matrix $G_n^{(p)}$ by the deterministic roots of matrix valued orthogonal polynomials, and these results are applied to study the asymptotic behaviour of the spectrum of the matrix $G_n^{(p)}$. In particular, we derive a new class of limiting spectral distributions which generalize the classical Wigner semi circle law in the one-dimensional case. Roughly speaking, the limit distribution has a density, which can be represented as the trace of an integral of densities of matrix measures corresponding to the Chebyshev matrix polynomials of the first kind. Finally some examples are presented in Section 4 which illustrate the theoretical results.

2 Matrix orthogonal polynomials

In the following discussion we consider for each $k \in \mathbb{N}$ a sequence of $p \times p$ matrix polynomials $(R_{n,k}(x))_{n \geq 0}$, which are defined recursively by

$$xR_{n,k}(x) = A_{n+1,k}R_{n+1,k}(x) + B_{n,k}R_{n,k}(x) + A_{n,k}^T R_{n-1,k}(x), \quad (2.1)$$

with initial conditions $R_{-1,k}(x) = 0$, $R_{0,k}(x) = I_p$, where $B_{i,k} \in \mathbb{R}^{p\times p}$ denote symmetric and $A_{i,k} \in \mathbb{R}^{p\times p}$ denote non-singular matrices. Here and throughout this paper I_p denotes the $p \times p$ identity matrix and 0 a $p \times p$ matrix with all entries equal to 0. A matrix measure Σ is a $p \times p$ matrix of (real) Borel measures such that for each Borel set $A \subset \mathbb{R}$ the matrix $\Sigma(A) \in \mathbb{R}^{p\times p}$ is nonnegative definite. It follows from Sinap and van Assche (1996) that there exists a positive definite matrix measure $\Sigma_k \in \mathbb{R}^{p\times p}$ such that the polynomials $(R_{n,k}(x))_{n\geq 0}$ are orthonormal with respect to the inner product induced by the measure $d\Sigma_k(x)$, that is

$$\int R_{n,k}(x) d\Sigma_k(x) R_{m,k}^T(x) = \delta_{nm} I_p. \tag{2.2}$$

The roots of the matrix polynomial $R_{n,k}(x)$ are defined by the roots of the polynomial (of degree np)

$$\det R_{n,k}(x),$$

and it can be shown that the orthonormal matrix polynomial $R_{n,k}(x)$ has precisely np real roots, where each root has at most multiplicity p. Throughout this paper let $x_{n,k,j}$ $(j = 1, \ldots, m)$ denote the different roots of the matrix orthogonal polynomial $R_{n,k}(x)$ with multiplicities ℓ_j, and consider the empirical distribution of the roots defined by

$$\delta_{n,k} := \frac{1}{np} \sum_{j=1}^{m} \ell_j \delta_{x_{n,k,j}}, \;\; n, k \geq 1, \tag{2.3}$$

where δ_z denotes the Dirac measure at point $z \in \mathbb{R}$. In the following we are interested in the asymptotic properties of this measure if $n, k \to \infty$. For these investigations we consider sequences $(n_j)_{j\in\mathbb{N}}$, $(k_j)_{j\in\mathbb{N}}$ of positive integers such that $\frac{n_j}{k_j} \to u$ for some $u \in [0, \infty)$, as $j \to \infty$ and denote the corresponding limit as $\lim_{n/k\to u}$ (if it exists). Our main result in this section establishes the weak convergence of the sequence of measures $\delta_{n,k}$ under certain conditions on the matrices $A_{n,k}$ and $B_{n,k}$ if $n/k \to u$ (as $n \to \infty, k \to \infty$) in the above sense. In the following sections we will approximate the eigenvalues of the random block matrix $G_n^{(p)}$ by the roots of a specific sequence of matrix orthogonal polynomials and use this result to derive the asymptotic spectral distribution of the random block matrix $G_n^{(p)}$.

The theory of matrix orthogonal polynomials is substantially richer than the corresponding theory for the one-dimensional case. Even the case, where all coefficients in the recurrence relation (2.1) are constant, has not been studied in full detail. We refer the interested reader Aptekarev and Nikishin (1983), Geronimo (1982), Dette and Studden (2002) and the references cited in the introduction among many others. Before we state our main result regarding the asymptotic zero distribution of the measure $\delta_{n,k}$ we mention some facts which are required for the formulation of the following theorem. Following Duran, Lopez-Rodriguez and Saff (1999) the matrix Chebyshev polynomials of the first kind are defined recursively by

$$\begin{aligned} tT_0^{A,B}(t) &= \sqrt{2}AT_1^{A,B}(t) + BT_0^{A,B}(t), \\ tT_1^{A,B}(t) &= AT_2^{A,B}(t) + BT_1^{A,B}(t) + \sqrt{2}AT_0^{A,B}(t), \\ tT_n^{A,B}(t) &= AT_{n+1}^{A,B}(t) + BT_n^{A,B}(t) + AT_{n-1}^{A,B}(t),\ n \geq 2, \end{aligned} \tag{2.4}$$

where A is a symmetric and non-singular $p \times p$ matrix and B is a symmetric $p \times p$ matrix, and $T_0^{A,B}(t) = I_p$. Similarly, the Chebyshev polynomials of the second kind are defined by the recursion

$$tU_n^{A,B}(t) \;=\; A^T U_{n+1}^{A,B}(t) + BU_n^{A,B}(t) + AU_{n-1}^{A,B}(t),\ n \geq 0, \tag{2.5}$$

with initial conditions $U_{-1}^{A,B}(t) = 0$, $U_0^{A,B}(t) = I_p$ [see Duran (1999)]. Note that in the case $p = 1$, $A = 1, B = 0$ the polynomials $T_n^{A,B}(t)$ and $U_n^{A,B}(t)$ are proportional to the classical Chebyshev polynomials of the first and second kind, that is

$$T_n^{1,0}(t) = \sqrt{2}\cos(n\arccos\frac{t}{2});\ U_n^{1,0}(t) = \frac{\sin\left((n+1)\arccos\frac{t}{2}\right)}{\sin(\arccos\frac{t}{2})}\ .$$

In the following discussion the $p \times p$ matrices A and B will depend on a real parameter u, i.e. $A = A(u), B = B(u)$, and corresponding matrix measures of orthogonality are denoted by $X_{A(u),B(u)}$ (for the polynomials of the first kind) and $W_{A(u),B(u)}$ (for the polynomials of the second kind). These measures will be normalized such that

$$\int dX_{A(u),B(u)}(t) = \int dW_{A(u),B(u)}(t) = I_p.$$

If $\beta_1(u) \leq \cdots \leq \beta_p(u)$ are the eigenvalues of the matrix $B(u)$, then it follows from Lemma 2.4 in Duran (1999) that the matrix

$$K_{A(u),B(u)}(z) := (B(u) - zI_p)^{1/2}A^{-1}(u)(B(u) - zI_p)^{1/2} \tag{2.6}$$

can be diagonalized except for finitely many $z \in \mathbb{C}\backslash[\beta_1(u), \beta_p(u)]$. Throughout this paper the union of the set of these finitely many complex numbers and the set $[\beta_1(u), \beta_p(u)]$ will be denoted by Δ (note that the set Δ depends on the parameter $u > 0$ although this is not reflected in our notation). In this case we have

$$K_{A(u),B(u)}(z) = U(z)D(z)U^{-1}(z),\ z \in \mathbb{C}\backslash\Delta, \tag{2.7}$$

where

$$D(z) = \text{diag}\left(\lambda_1^{A(u),B(u)}(z), \ldots, \lambda_p^{A(u),B(u)}(z)\right)$$

denotes the diagonal matrix of eigenvalues of $K_{A(u),B(u)}(z)$ and $U(z)$ is a unitary $p \times p$ matrix (note that the matrices $U(z)$ and $D(z)$ depend on the parameter u, although this is not reflected by our notation). It is shown in Duran, Lopez-Rodriguez and Saff (1999) that under the assumption of a positive definite matrix $A(u)$ and a symmetric matrix $B(u)$ the measure $X_{A(u),B(u)}$ is absolute continuous with respect to the Lebesgue measure multiplied with the identity matrix and has density

$$dX_{A(u),B(u)}(x) \;=\; A^{-1/2}(u)U(x)\tilde{T}(x)U^T(x)A^{-1/2}(u)dx \tag{2.8}$$

where $\tilde{T}(x) := \operatorname{diag}(\tilde{t}_{11}(x), \ldots, \tilde{t}_{pp}(x))$ denotes a diagonal matrix with elements

$$\tilde{t}_{ii}(x) = \begin{cases} \frac{1}{\pi\sqrt{4-\left(\lambda_i^{A(u),B(u)}(x)\right)^2}}, & \text{if } \lambda_i^{A(u),B(u)}(x) \in (-2,2) \\ \\ 0, & \text{else} \end{cases} \qquad i = 1, \ldots, p. \tag{2.9}$$

For the sake of a simple notation this density is also denoted by $X_{A(u),B(u)}(x)$.

Theorem 2.1. *Consider the sequence of matrix orthonormal polynomials defined by the three term recurrence relation (2.1), where for all* $\ell \in \mathbb{N}_0$ *and a given* $u > 0$

$$\lim_{\frac{n}{k} \to u} A_{n-\ell,k} = A(u), \tag{2.10}$$

$$\lim_{\frac{n}{k} \to u} B_{n-\ell,k} = B(u), \tag{2.11}$$

with non-singular and symmetric matrices $\{A(u)|u > 0\}$ *and symmetric matrices* $\{B(u)|u > 0\}$. *If there exists a number* $M > 0$ *such that*

$$\bigcup_{k=1}^{\infty} \bigcup_{n=0}^{\infty} \Big\{ z \in \mathbb{C} | \det R_{n,k}(z) = 0 \Big\} \subset \; [-M, M] \,, \tag{2.12}$$

then the empirical measure $\delta_{n,k}$ *defined by (2.3) converges weakly to a matrix measure which is absolute continuous with respect to the Lebesgue measure multiplied with the identity matrix. The density of the limiting distribution is given by*

$$\frac{1}{u} \int_0^u tr \left[\frac{1}{p} X_{A(s),B(s)} \right] ds, \tag{2.13}$$

where $X_{A(s),B(s)}$ *denotes the density of the matrix measure corresponding to the matrix Chebyshev polynomials of the first kind.*

Proof. To be precise, let $(R_{n,k}(x))_{n \geq 0}$ $(k \in \mathbb{N})$ denote the sequence of matrix orthonormal polynomials defined by the recursive relation (2.1) and denote by $x_{n+1,k,1}, \ldots, x_{n+1,k,m}$ the different roots of the $(n+1)$th polynomial $R_{n+1,k}(x)$. At the end of the proof we will show the following auxiliary result, which generalizes a corresponding statement for the case $p = 1$ proved by Kuijlaars and van Assche (1999).

Lemma A.1. *If all roots of the matrix orthogonal polynomial $R_{n+1,k}(x)$ defined by (2.1) are located in the interval $[-M, M]$, then the inequality*

$$\left|v^T R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k} v\right| \le \frac{1}{dist(z, [-M, M])} v^T v \tag{2.14}$$

holds for all vectors $v \in \mathbb{C}^p$, and all complex numbers $z \in \mathbb{C}\backslash[-M, M]$. Moreover, we have

$$\left|v^T R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k} v\right| > \frac{1}{2|z|} v^T v \tag{2.15}$$

for all vectors $v \in \mathbb{C}^p$, $z \in \mathbb{C}$ with $|z| > M$.

We now normalize the orthonormal polynomials $R_{n,k}(x)$ such that their leading coefficients are equal to the identity matrix, that is

$$\underline{R}_{0,k}(x) := I_p\ ; \qquad \underline{R}_{n,k}(x) := A_{1,k} \cdots A_{n,k} R_{n,k}(x),\ n \ge 1. \tag{2.16}$$

Then a straightforward calculation shows that $\underline{R}^{-1}_{j,k}(x)\underline{R}_{j+1,k}(x) = R^{-1}_{j,k}(x) A_{j+1,k} R_{j+1,k}(x)$ $(j \ge 0)$ and we obtain

$$\underline{R}_{n,k}(x) = \prod_{j=0}^{n-1} R^{-1}_{j,k}(x) A_{j+1,k} R_{j+1,k}(x),\ n \ge 0. \tag{2.17}$$

This yields

$$\begin{aligned}\frac{1}{np} \log\left|\det \underline{R}_{n,k}(z)\right| &= \frac{1}{np} \sum_{j=0}^{n-1} \log\left|\det\left(R^{-1}_{j,k}(z) A_{j+1,k} R_{j+1,k}(z)\right)\right| \\ &= \frac{1}{p}\int_0^1 \log\left|\det\left(A_{[sn]+1,k} R_{[sn]+1,k}(z) R^{-1}_{[sn],k}(z)\right)\right| ds.\end{aligned} \tag{2.18}$$

If $\eta_{n,k,1}(z), \ldots, \eta_{n,k,p}(z)$ denote the eigenvalues of the $p \times p$ matrix $R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k}$, then it follows

$$\min_{v \ne 0}\left|\frac{v^T R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k} v}{v^T v}\right| \le \left|\eta_{n,k,i}(z)\right| \le \max_{v \ne 0}\left|\frac{v^T R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k} v}{v^T v}\right|$$

for all $i = 1, \ldots, p$ [see Horn and Johnson (1985), p. 181]. With these inequalities and Lemma A.1. we have

$$\begin{aligned}\left|\det(R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k})\right| &= \left|\prod_{i=1}^{p} \eta_{n,k,i}(z)\right| \ge \left(\min_{i=1}^{p} |\eta_{n,k,i}(z)|\right)^p \\ &\ge \left(\min_{v^T v = 1}\left|v^T R_{n,k}(z) R^{-1}_{n+1,k}(z) A^{-1}_{n+1,k} v\right|\right)^p > \left(\frac{1}{2|z|}\right)^p,\end{aligned} \tag{2.19}$$

whenever $|z| > M$, and

$$
\begin{aligned}
\left|\det(R_{n,k}(z)R^{-1}_{n+1,k}(z)A^{-1}_{n+1,k})\right| &\le \left(\max_{i=1}^{p} |\eta_{n,k,i}(z)|\right)^p \qquad (2.20)\\
&\le \left(\max_{v^T v=1} \left|v^T R_{n,k}(z)R^{-1}_{n+1,k}(z)A^{-1}_{n+1,k} v\right|\right)^p \le \frac{1}{\mathrm{dist}(z,[-M,M])^p}.
\end{aligned}
$$

From (2.19) and (2.20) we therefore obtain the estimate

$$
\left|\log\left|\det\left(A_{n+1,k}R_{n+1,k}(z)R^{-1}_{n,k}(z)\right)\right|\right| \le \max\left\{p\left|\log\left(\mathrm{dist}(z,[-M,M])\right)\right|, p\left|\log(2|z|)\right|\right\}, \qquad (2.21)
$$

whenever $|z| > M$. Now the representation (2.18) and Lebesgue's theorem yield for $|z| > M$

$$
\begin{aligned}
\lim_{\frac{n}{k}\to u} \frac{1}{np} \log\left|\det \underline{R}_{n,k}(z)\right| &= \frac{1}{p}\int_0^1 \log\left|\det(\lim_{\frac{n}{k}\to u} A_{[sn]+1,k}R_{[sn]+1,k}(z)R^{-1}_{[sn],k}(z))\right| ds \qquad (2.22)\\
&= -\frac{1}{p}\int_0^1 \log\left|\det\left(\Phi_{A(su),B(su)}(z)\right)\right| ds,\\
&= -\frac{1}{pu}\int_0^u \log\left|\det\left(\Phi_{A(s),B(s)}(z)\right)\right| ds,
\end{aligned}
$$

where

$$
\Phi_{A(u),B(u)}(z) := \int \frac{dW_{A(u),B(u)}(t)}{z-t} \qquad (2.23)
$$

denotes the (matrix valued) Stieltjes transform of the matrix measure $W_{A(u),B(u)}$ corresponding to the matrix Chebyshev polynomials of the second kind, and we have used Theorem 2.1 of Duran and Daneri-Vias (2001) for the second equality, which describes the "ratio asymptotics" of the matrices $R_{n+1,k}(z)\ R^{-1}_{n,k}(z)$. In (2.22) the convergence is uniform on compact subsets of $\mathbb{C}\setminus\Gamma$, where the set Γ is defined by

$$
\Gamma = \bigcap_{m\ge 0}\overline{\bigcup_{j\ge m}\Delta_j},
$$

$\Delta_j = \{x\in\mathbb{R} \mid \det\ R_{n_j,k_j}(x) = 0\}$ denotes the set of roots of the matrix polynomial $R_{n_j,k_j}(x)$, and $(n_j,k_j)_{j\ge 0}$ is the sequence along which the convergence is considered. The Stieltjes transform of the matrix measure $W_{A(u),B(u)}$ has been determined by Duran (1999) as

$$
\begin{aligned}
\Phi_{A(u),B(u)}(z) &= \frac{1}{2}A^{-1}(u)(B(u)-zI_p)^{1/2}\left(-I_p - \sqrt{I_p - 4K^{-2}_{A(u),B(u)}(z)}\right)(B(u)-zI_p)^{1/2}A^{-1}(u)\\
&= A^{-1}(u)(B(u)-zI_p)^{1/2}U(z)T(z)U^{-1}(z)(B(u)-zI_p)^{1/2}A^{-1}(u), \qquad (2.24)
\end{aligned}
$$

where the matrix $K_{A(u),B(u)}(z)$ is defined in (2.6), $T(z) := \mathrm{diag}\,(t_{11}(z),\ldots,t_{pp}(z))$ is a diagonal matrix with elements given by

$$
t_{ii}(z) = \frac{-\lambda_i^{A(u),B(u)}(z) - \sqrt{(\lambda_i^{A(u),B(u)}(z))^2 - 4}}{2\lambda_i^{A(u),B(u)}(z)}
$$

$(i = 1, \ldots, p)$ and $z \in \mathbb{C} \setminus \{\text{supp } (W_{A(u),B(u)}) \cup \Delta\}$. Here and throughout this paper we take the square root $\sqrt{w}$ such that $|w - \sqrt{w^2-4}| < 2$ for $w \in \mathbb{C} \setminus [-2,2]$, and consequently the function $w - \sqrt{w^2-4}$ is analytic on $\mathbb{C} \setminus [-2,2]$. Note that for $z \in \mathbb{C} \setminus \text{supp } (W_{A(u),B(u)})$ it follows that $|\lambda_i^{A(u),B(u)}(z)| > 2$. Observing (2.24) this implies for the Stieltjes transform

$$\begin{aligned}
\log|\det(\Phi_{A(u),B(u)}(z))| &= \log|\det(A^{-2}(u)T(z)(B(u)-zI_p)| \qquad (2.25)\\
&= \log|\det(A^{-1}(u)T(z)D(z)| = \log\Big|\prod_{j=1}^{p} \frac{t_{jj}(z)\lambda_j^{A(u),B(u)}(z)}{\alpha_j(u)}\Big| \\
&= \sum_{j=1}^{p} \log\Big|\frac{-\lambda_j^{A(u),B(u)}(z) - \sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}{2\alpha_j(u)}\Big| \\
&= \text{Re}\Big(\sum_{j=1}^{p} \log \frac{-\lambda_j^{A(u),B(u)}(z) - \sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}{2\alpha_j(u)}\Big),
\end{aligned}$$

where $\alpha_1(u), \ldots, \alpha_p(u)$ denote the eigenvalues of the matrix $A(u)$. Now define

$$f(z) := \sum_{j=1}^{p} \log \frac{-\lambda_j^{A(u),B(u)}(z) - \sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}{2\alpha_j(u)},$$

then it follows from Kato (1976), p. 64, that the eigenvalues $\lambda_j^{A(u),B(u)}(z)$ are holomorphic functions on $\mathbb{C} \setminus \Delta$ and we obtain for any

$$z \in G_0 := \mathbb{C}\setminus \big\{\text{supp } \big(W_{A(u),B(u)}\big) \cup \Delta\big\}$$

that

$$\begin{aligned}
\frac{d}{d\bar{z}}f &= \sum_{j=1}^{p} \frac{1}{2f_j(z)\alpha_j(u)}\Big(-\frac{d\lambda_j^{A(u),B(u)}(z)}{dx} - \frac{\lambda_j^{A(u),B(u)}(z)\frac{d\lambda_j^{A(u),B(u)}(z)}{dx}}{\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}\Big) \\
&+ \sum_{j=1}^{p} i\frac{1}{2f_j(z)\alpha_j(u)}\Big(-\frac{d\lambda_j^{A(u),B(u)}(z)}{dy} - \frac{\lambda_j^{A(u),B(u)}(z)\frac{d\lambda_j^{A(u),B(u)}(z)}{dy}}{\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}\Big) \\
&= \sum_{j=1}^{p} \frac{1}{2\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}\left(\frac{d}{dx}\lambda_j^{A(u),B(u)}(z) + i\frac{d}{dy}\lambda_j^{A(u),B(u)}(z)\right) = 0,
\end{aligned}$$

where the function f_j is defined by

$$f_j(z) = \frac{-\lambda_j^{A(u),B(u)}(z) - \sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}{2\alpha_j(u)}, \; j = 1, \ldots, p.$$

This implies that the function $f(z)$ is holomorphic on G_0, and it follows for $z \in G_0$ that

$$\begin{aligned}
\frac{d}{dz}\log(\det(\Phi_{A(u),B(u)}(z)) &= \frac{1}{2}\sum_{j=1}^{p}\frac{1}{\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}\left(\frac{d}{dx}\lambda_j^{A(u),B(u)}(z) - i\frac{d}{dy}\lambda_j^{A(u),B(u)}(z)\right) \\
&= \sum_{j=1}^{p}\frac{\frac{d}{dz}\lambda_j^{A(u),B(u)}(z)}{\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}}. \qquad (2.26)
\end{aligned}$$

In the following discussion define

$$U^X(z) := \int \log\frac{1}{|z-t|}\operatorname{tr}\left[X_{A(u),B(u)}(t)\right] dt \qquad (2.27)$$

as the logarithmic potential of the measure whose density with respect to the Lebesgue measure is given by $\operatorname{tr}[X_{A(u),B(u)}(t)]$. Then $U^X(z)$ is harmonic on

$$G_1 := \mathbb{C} \setminus \operatorname{supp}\left(\operatorname{tr}[X_{A(u),B(u)}]\right)$$

[see e.g. Saff and Totik (1997)]. In the following we show that the function f satisfies Re $f = U^X + c_1$, where $c_1 \in \mathbb{C}$ is a constant. For this purpose we note that the function

$$g(z) := \frac{d}{dx}U^X(z) - i\frac{d}{dy}U^X(z) \qquad (2.28)$$

is holomorphic on G_1 (note that g satisfies the Cauchy-Riemann differential equations because the logarithmic potential U^X is harmonic) and satisfies for all $z \in G_1$

$$\begin{aligned}
g(z) &= -\int \frac{\overline{z-t}}{|z-t|^2}\operatorname{tr}\left[X_{A(u),B(u)}(t)\right] dt \qquad (2.29) \\
&= -\int \frac{\operatorname{tr}\left[X_{A(u),B(u)}(t)\right]}{z-t} dt = -\operatorname{tr}\left[G_{A(u),B(u)}(z)\right] ,
\end{aligned}$$

where $G_{A(u),B(u)}$ denotes the Stieltjes transform of the matrix measure corresponding to the density $X_{A(u),B(u)}(t)$. In order to find a representation for the right hand side we note that the function $K_{A(u),B(u)}(z)$ defined in (2.6) is holomorphic on $\mathbb{C}\setminus\Delta$, and observing the representation (2.7) we obtain for all $z \in \mathbb{C}\setminus\Delta$

$$\begin{aligned}
&-\frac{1}{2}(B(u)-zI_p)^{-1/2}A^{-1}(u)(B(u)-zI_p)^{1/2} - \frac{1}{2}(B(u)-zI_p)^{1/2}A^{-1}(u)(B(u)-zI_p)^{-1/2} \\
&= K'_{A(u),B(u)}(z) = U'(z)D(z)U^{-1}(z) + U(z)D'(z)U^{-1}(z) + U(z)D(z)(U^{-1}(z))' .
\end{aligned}$$

From the identities

$$\begin{aligned}
A^{-1}(u)(B(u)-zI_p)^{1/2} &= (B(u)-zI_p)^{-1/2}U(z)D(z)U^{-1}(z), \\
(B(u)-zI_p)^{1/2}A^{-1}(u) &= U(z)D(z)U^{-1}(z)(B(u)-zI_p)^{-1/2}
\end{aligned}$$

and $(U^{-1}(z))'U(z) = -U^{-1}(z)U'(z)$ it follows that

$$
\begin{aligned}
&-\frac{1}{2}U^{-1}(z)(B(u)-zI_p)^{-1}U(z)D(z)-\frac{1}{2}D(z)U^{-1}(z)(B(u)-zI_p)^{-1}U(z) \\
&= U^{-1}(z)U'(z)D(z)+D'(z)-D(z)U^{-1}(z)U'(z),
\end{aligned} \tag{2.30}
$$

which yields for the diagonal elements of the matrix $U^{-1}(z)(B(u)-zI_p)^{-1}U(z)$

$$
\begin{aligned}
[U^{-1}(z)(B(u)-zI_p)^{-1}U(z)]_{jj} &= \frac{1}{\lambda_j^{A(u),B(u)}(z)}\Big(-[U^{-1}(z)U'(z)]_{jj}\lambda_j^{A(u),B(u)}(z)-\frac{d}{dz}\lambda_j^{A(u),B(u)}(z) \\
&\qquad + \lambda_j^{A(u),B(u)}(z)[U^{-1}(z)U'(z)]_{jj}\Big) \\
&= -\frac{\frac{d}{dz}\lambda_j^{A(u),B(u)}(z)}{\lambda_j^{A(u),B(u)}(z)},
\end{aligned}
$$

for all $z \in \mathbb{C}\backslash\Delta,\ j = 1,\dots,p$. From Zygmunt (2002) and (2.24) we have for the Stieltjes transforms $\Phi_{A(u),B(u)}$ and $G_{A(u),B(u)}$ corresponding to the matrix measures $W_{A(u),B(u)}$ and $X_{A(u),B(u)}$ the continued fraction expansion

$$
\begin{aligned}
G_{A(u),B(u)}(z) &= \Big\{zI_p - B(u) - \sqrt{2}A(u)\,\{zI - B(u) - A(u)\,\{zI_p - B(u) - \dots \\
&\qquad \dots\ \}^{-1}A(u)\}^{-1}\sqrt{2}A(u)\Big\}^{-1} \\
&= \big\{zI_p - B(u) - 2A(u)\Phi_{A(u),B(u)}(z)A(u)\big\}^{-1} \\
&= (B(u)-zI_p)^{-1/2}U(z)\hat{T}(z)U^{-1}(z)(B(u)-zI_p)^{-1/2},
\end{aligned}
$$

where $z \in G$, the set G is given by

$$
G := \mathbb{C}\backslash\{\mathrm{supp}\left(W_{A(u),B(u)}\right) \cup \mathrm{supp}\left(X_{A(u),B(u)}\right) \cup \Delta\},
$$

and the diagonal matrix $\hat{T}(z)$ is defined by

$$
\hat{T}(z) := \mathrm{diag}(\hat{t}_{11}(z),\dots,\hat{t}_{pp}(z)) = \mathrm{diag}\Big(\frac{\lambda_1^{A(u),B(u)}(z)}{\sqrt{(\lambda_1^{A(u),B(u)}(z))^2-4}},\dots,\frac{\lambda_p^{A(u),B(u)}(z)}{\sqrt{(\lambda_p^{A(u),B(u)}(z))^2-4}}\Big).
$$

This yields

$$
\begin{aligned}
g(z) \;=\; -\mathrm{tr}\left[G_{A(u),B(u)}(z)\right] &= -\sum_{j=1}^{p}[U^{-1}(z)(B(u)-zI_p)^{-1}U(z)]_{jj}\hat{t}_{jj}(z) \\
&= \sum_{j=1}^{p}\frac{\frac{d}{dz}\lambda_j^{A(u),B(u)}(z)}{\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}},
\end{aligned} \tag{2.31}
$$

for all $z \in G$. Consequently, we have for all $z \in G$

$$
f'(z) = \sum_{j=1}^{p}\frac{\frac{d}{dz}\lambda_j^{A(u),B(u)}(z)}{\sqrt{(\lambda_j^{A(u),B(u)}(z))^2-4}} = g(z) = \frac{d}{dx}U^X(z) - i\frac{d}{dy}U^X(z).
$$

For $h := \text{Re } f$ it follows that

$$f' = \frac{d}{dx}h - i\frac{d}{dy}h = \frac{d}{dx}U^X - i\frac{d}{dy}U^X,$$

so that $\frac{d}{dx}(h - U^X) \equiv 0$ and $\frac{d}{dy}(h - U^X) \equiv 0$, which implies for all $z \in G$ the identity $\text{Re } f(z) = U^X(z) + c_1$ for some constant $c_1 \in \mathbb{C}$. Therefore it follows that

$$\text{Re } (f(z)) = \text{Re } \big(\log(\det(\Phi_{A(u),B(u)}(z)))\big) = \log|\det(\Phi_{A(u),B(u)}(z))| = U^X(z) + c_1 \tag{2.32}$$

for any $z \in G$. Observing (2.22) we finally obtain for all $z \in G$

$$\begin{aligned}
\lim_{\frac{n}{k}\to u} \frac{1}{np} \log\left|\det \underline{R}_{n,k}(z)\right| &= -\frac{1}{pu}\int_0^u \int \log\frac{1}{|z-t|} \text{tr}\left[X_{A(s),B(s)}(t)\right] dtds - c \qquad (2.33)\\
&= -\int \log\frac{1}{|z-t|}\frac{1}{u}\int_0^u \text{tr}\left[\frac{1}{p}X_{A(s),B(s)}(t)\right] dsdt - c\\
&= -U^\sigma(z) - c,
\end{aligned}$$

where U^σ denotes the logarithmic potential of the measure σ with Lebesgue density defined in (2.13) and $c \in \mathbb{C}$ is a constant. Let

$$U^{\delta_{n,k}}(z) := \int \log\frac{1}{|z-t|}\ \delta_{n,k}(dt)$$

denote the logarithmic potential of the measure $\delta_{n,k}$ defined in (2.3). From (2.33) we obtain for all $z \in G$

$$\begin{aligned}
\lim_{\frac{n}{k}\to u} U^{\delta_{n,k}}(z) = \lim_{\frac{n}{k}\to u} \frac{1}{np}\sum_{j=1}^m \log\frac{1}{|z - x_{n,k,j}|^{\ell_j}} &= \lim_{\frac{n}{k}\to u}\frac{1}{np}\log\frac{1}{|\det(R_{n,k}(z))|}\\
&= U^\sigma(z) + c. \qquad (2.34)
\end{aligned}$$

The measures in the sequence $(\delta_{n_j,k_j})_{j\in\mathbb{N}}$ have compact support in $[-M, M]$, consequently, $(\delta_{n_j,k_j})_{j\in\mathbb{N}}$ contains a subsequence which converges weakly to a limit μ with supp $(\mu) \subset [-M, M]$. Therefore we obtain from (2.34)

$$U^\mu(z) = U^\sigma(z) + c,\ \ z \in G, |z| > M,$$

and the assertion of the theorem follows from the fact that the logarithmic potentials are unique [see Saff and Totik (1997)].

Proof of Lemma A.1. Let $x_{n+1,k,1}, \dots, x_{n+1,k,m}$ denote the different roots of the matrix orthogonal polynomial $R_{n+1,k}(x)$ with multiplicities $\ell_1, \dots, \ell_m$. Then we obtain from Duran (1996), p. 1184, the representation

$$R_{n,k}(z)R^{-1}_{n+1,k}(z)A^{-1}_{n+1,k} = \sum_{j=1}^m \frac{C_{n+1,k,j}A^{-1}_{n+1,k}}{z - x_{n+1,k,j}}, \tag{2.35}$$

where the weights are given by

$$
\begin{aligned}
C_{n+1,k,j}A_{n+1,k}^{-1} &= R_{n,k}(x_{n+1,k,j})\Gamma_{n+1,k,j}R_{n,k}^{T}(x_{n+1,k,j}), \qquad (2.36)\\
\Gamma_{n+1,k,j} &= \frac{\ell_k}{(\det(R_{n+1,k}(t)))^{(\ell_j)}(x_{n+1,k,j})}\left(\mathrm{Adj}(R_{n+1,k}(t)))^{(\ell_j-1)}(x_{n+1,k,j})Q_{n+1,k}(x_{n+1,k,j}\right),
\end{aligned}
$$

and $\mathrm{Adj}(A)$ denotes the adjoint of the $p \times p$ matrix A, that is

$$A\ \mathrm{Adj}(A) = \mathrm{Adj}(A)A = \det(A)I_p\,.$$

In (2.36) the matrix polynomial

$$Q_{n,k}(x) = \int \frac{R_{n,k}(x) - R_{n,k}(t)}{x-t} d\Sigma_k(t)$$

denotes the first associated matrix orthogonal polynomial and the matrices $\Gamma_{n+1,k,j}$ are nonnegative definite and have rank ℓ_j [see Duran (1996), Theorem 3.1 (2)]. From Duran and Daneri-Vias (2001) we have

$$
\begin{aligned}
\sum_{j=1}^{m} C_{n+1,k,j}A_{n+1,k}^{-1} &= \sum_{j=1}^{m} R_{n,k}(x_{n+1,k,j})\Gamma_{n+1,k,j}R_{n,k}^{T}(x_{n+1,k,j}) \qquad (2.37)\\
&= \int R_{n,k}(t)\Sigma_k(t)R_{n,k}^{T}(t) = I_p.
\end{aligned}
$$

For any $z \in \mathbb{C}\backslash[-M,M]$ we obtain the estimate $|z - x_{n+1,k,j}| \geq \mathrm{dist}(z,[-M,M])$ for all $j = 1,\dots,m$, because all roots of the matrix polynomial $R_{n+1,k}(z)$ are located in the interval $[-M,M]$. Note that by the representation (2.36) the matrix $C_{n+1,k}A_{n+1,k}^{-1}$ is nonnegative definite and the representations (2.35) and (2.37) yield for any $|z| > M$ and $v \in \mathbb{C}^p$

$$
\begin{aligned}
\left|v^T R_{n,k}(z)R_{n+1,k}^{-1}(z)A_{n+1,k}^{-1}v\right| &= \left|v^T \sum_{j=1}^{m} \frac{C_{n+1,k,j}A_{n+1,k}^{-1}}{z - x_{n+1,k,j}} v\right|\\
&\leq \sum_{j=1}^{m} \frac{v^T C_{n+1,k,j}A_{n+1,k}^{-1}v}{|z - x_{n+1,k,j}|}\\
&\leq \frac{1}{\mathrm{dist}(z,[-M,M])}\sum_{j=1}^{m} v^T C_{n+1,k,j}A_{n+1,k}^{-1}v = \frac{1}{\mathrm{dist}(z,[-M,M])}v^T v,
\end{aligned}
$$

which proves the first inequality of the Lemma. For a proof of the second part we note that for $|z| > M$ we have $|\frac{x_{n+1,k,j}}{z}| < 1$ for all $j = 1,\dots,m$, which gives

$$\mathrm{Re}\Big(\frac{1}{1 - \frac{x_{n+1,k,j}}{z}}\Big) > \frac{1}{2}$$

for all $j = 1, \dots, m$. With this inequality we obtain

$$\begin{aligned}
\left|v^T R_{n,k}(z) R_{n+1,k}^{-1}(z) A_{n+1,k}^{-1} v\right| &= \left|v^T \sum_{j=1}^{m} \frac{C_{n+1,k,j} A_{n+1,k}^{-1}}{z - x_{n+1,k,j}} v\right| \\
&= \frac{1}{|z|}\Big|\sum_{j=1}^{m} \frac{v^T C_{n+1,k,j} A_{n+1,k}^{-1} v}{1 - \frac{x_{n+1,k,j}}{z}}\Big| \\
&\geq \frac{1}{|z|}\mathrm{Re}\Big(\sum_{j=1}^{m} \frac{v^T C_{n+1,k,j} A_{n+1,k}^{-1} v}{1 - \frac{x_{n+1,k,j}}{z}}\Big) \\
&> \frac{1}{2|z|} \sum_{j=1}^{m} v^T C_{n+1,k,j} A_{n+1,k}^{-1} v \;=\; \frac{1}{2|z|} v^T v,
\end{aligned}$$

which proves the second inequality of the Lemma. □

3 Strong and weak asymptotics for eigenvalues of random block matrices

We now consider the random block matrix defined in equation (1.3) of the introduction and denote $\tilde{\lambda}_1^{(n,p)} \leq \cdots \leq \tilde{\lambda}_n^{(n,p)}$ as its (random) eigenvalues, where $n = mp$; $m, p \in \mathbb{N}$. Similarly, define $\tilde{x}_1^{(n,p)} \leq \cdots \leq \tilde{x}_n^{(n,p)}$ as (deterministic) eigenvalues of the matrix orthonormal polynomial $\tilde{R}_{m,n}^{(p)}(x) \in \mathbb{R}^{p \times p}$, which is defined recursively by $\tilde{R}_{0,n}^{(p)}(x) = I_p$, $\tilde{R}_{-1,n}^{(p)}(x) = 0$,

$$x\ \tilde{R}_{m,n}^{(p)}(x) = \tilde{A}_{m+1,n}^{(p)}\ \tilde{R}_{m+1,n}^{(p)}(x) + \tilde{B}_{m,n}^{(p)}\ \tilde{R}_{m,n}^{(p)}(x) + \tilde{A}_{m,n}^{(p)T}\ \tilde{R}_{m-1,n}^{(p)}(x) \tag{3.1}$$

($m \geq 0$), where the $p \times p$ block matrices $\tilde{A}_{i,n}^{(p)}$ and $\tilde{B}_{i,n}^{(p)}$ are given by

$$\tilde{A}_{i,n}^{(p)} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{((i-1)p+1)\gamma_p} & \sqrt{((i-1)p+2)\gamma_{p-1}} & \sqrt{((i-1)p+3)\gamma_{p-2}} & \cdots & \sqrt{ip\gamma_1} \\
\sqrt{((i-1)p+2)\gamma_{p-1}} & \sqrt{((i-1)p+2)\gamma_p} & \sqrt{((i-1)p+3)\gamma_{p-1}} & \cdots & \sqrt{ip\gamma_2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\sqrt{(ip-1)\gamma_2} & \cdots & \sqrt{(ip-1)\gamma_{p-1}} & \sqrt{(ip-1)\gamma_p} & \sqrt{ip\gamma_{p-1}} \\
\sqrt{ip\gamma_1} & \cdots & \sqrt{ip\gamma_{p-2}} & \sqrt{ip\gamma_{p-1}} & \sqrt{ip\gamma_p}
\end{pmatrix}, \tag{3.2}$$

$i \geq 1$, and

$$\tilde{B}_{i,n}^{(p)} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & \sqrt{(ip+1)\gamma_1} & \sqrt{(ip+1)\gamma_2} & \cdots & \sqrt{(ip+1)\gamma_{p-1}} \\
\sqrt{(ip+1)\gamma_1} & 0 & \sqrt{(ip+2)\gamma_1} & \cdots & \sqrt{(ip+2)\gamma_{p-2}} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\sqrt{(ip+1)\gamma_{p-2}} & \cdots & \sqrt{((i+1)p-2)\gamma_1} & 0 & \sqrt{((i+1)p-2)\gamma_1} \\
\sqrt{(ip+1)\gamma_{p-1}} & \cdots & \sqrt{((i+1)p-2)\gamma_2} & \sqrt{((i+1)p-1)\gamma_1} & 0
\end{pmatrix}, \tag{3.3}$$

$i \geq 0$, respectively, and $\gamma_1, \ldots, \gamma_p > 0$ are given constants. Our first result provides a strong uniform approximation of the ordered random eigenvalues of the matrices $G_n^{(p)}$ by the ordered deterministic roots of the matrix orthogonal polynomials $R_{m,n}^{(p)}(x)$.

Theorem 3.1. *Let $\tilde{\lambda}_1^{(n,p)} \leq \cdots \leq \tilde{\lambda}_n^{(n,p)}$ denote the eigenvalues of the random matrix $G_n^{(p)}$ defined by (1.3) and $\tilde{x}_1^{(n,p)} \leq \cdots \leq \tilde{x}_n^{(n,p)}$ denote the roots of the matrix orthonormal polynomial $\tilde{R}_{m,n}^{(p)}(x)$ defined by the recurrence relation (3.1), then*

$$\max_{1 \leq j \leq n} \left| \tilde{\lambda}_j^{(n,p)} - \tilde{x}_j^{(n,p)} \right| \leq [\log n]^{1/2} S \quad \forall\ n \geq 2, \tag{3.4}$$

where S is a random variable such that $S < \infty$ a.s.

Proof. For the proof we establish the bound

$$P\left\{ \max_{1 \leq j \leq n} \left| \tilde{\lambda}_j^{(n,p)} - \tilde{x}_j^{(n,p)} \right| \geq \varepsilon \right\} \leq 2n(p+1) \exp\left(\frac{-\varepsilon^2}{18p^2} \right), \tag{3.5}$$

then the assertion follows along the lines of the proof of Theorem 2.2 in Dette and Imhof (2007). First we note that the roots of the mth matrix orthonormal polynomial $\tilde{R}_{m,n}^{(p)}(x)$ are the eigenvalues of the tridiagonal block matrix

$$\tilde{F}_n^{(p)} := \begin{pmatrix} \tilde{B}_{0,n} & \tilde{A}_{1,n} & & & \\ \tilde{A}_{1,n} & \tilde{B}_{1,n} & \tilde{A}_{2,n} & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{A}_{\frac{n}{p}-2,n} & \tilde{B}_{\frac{n}{p}-2,n} & \tilde{A}_{\frac{n}{p}-1,n} \\ & & & \tilde{A}_{\frac{n}{p}-1,n} & \tilde{B}_{\frac{n}{p}-1,n} \end{pmatrix} \in \mathbb{R}^{n \times n}, \tag{3.6}$$

where the blocks $\tilde{A}_{i,n}^{(p)}$, $i = 1, \ldots, n/p - 1$, and $\tilde{B}_{i,n}^{(p)}$, $i = 0, \ldots, n/p - 1$, are defined by (3.2) and (3.3), respectively. Moreover, by interchanging rows and columns of the matrix $\tilde{F}_n^{(p)}$ it is easy to see that the matrix

$$F_n^{(p)} = \begin{pmatrix} E_{0,n}^{(p)} & D_{1,n}^{(p)} & & & \\ D_{1,n}^{(p)} & E_{1,n}^{(p)} & D_{2,n}^{(p)} & & \\ & \ddots & \ddots & \ddots & \\ & & D_{\frac{n}{p}-2,n}^{(p)} & E_{\frac{n}{p}-2,n}^{(p)} & D_{\frac{n}{p}-1,n}^{(p)} \\ & & & D_{\frac{n}{p}-1,n}^{(p)} & E_{\frac{n}{p}-1,n}^{(p)} \end{pmatrix} \tag{3.7}$$

with symmetric blocks

$$E_{i,n}^{(p)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{\gamma_1(n-ip-1)} & \sqrt{\gamma_2(n-ip-2)} & \cdots & \sqrt{\gamma_{p-1}(n-(i+1)p+1)} \\ \sqrt{\gamma_1(n-ip-1)} & 0 & \sqrt{\gamma_1(n-ip-2)} & \cdots & \sqrt{\gamma_{p-2}(n-(i+1)p+1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sqrt{\gamma_{p-2}(n-(i+1)p+2)} & \cdots & \sqrt{\gamma_1(n-(i+1)p+1)} & 0 & \sqrt{\gamma_1(n-(i+1)p+1)} \\ \sqrt{\gamma_{p-1}(n-(i+1)p+1)} & \cdots & \sqrt{\gamma_2(n-(i+1)p+1)} & \sqrt{\gamma_1(n-(i+1)p+1)} & 0 \end{pmatrix},$$

$i = 0, \ldots, \frac{n}{p} - 1$, and

$$D_{i,n}^{(p)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\gamma_p(n-ip)} & \sqrt{\gamma_{p-1}(n-ip)} & \sqrt{\gamma_{p-2}(n-ip)} & \cdots & \sqrt{\gamma_1(n-ip)} \\ \sqrt{\gamma_{p-1}(n-ip)} & \sqrt{\gamma_p(n-ip-1)} & \sqrt{\gamma_{p-1}(n-ip-1)} & \cdots & \sqrt{\gamma_2(n-ip-1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sqrt{\gamma_2(n-ip)} & \cdots & \sqrt{\gamma_{p-1}(n-(i+1)p+3)} & \sqrt{\gamma_p(n-(i+1)p+2)} & \sqrt{\gamma_{p-1}(n-(i+1)p+2)} \\ \sqrt{\gamma_1(n-ip)} & \cdots & \sqrt{\gamma_{p-2}(n-(i+1)p+3)} & \sqrt{\gamma_{p-1}(n-(i+1)p+2)} & \sqrt{\gamma_p(n-(i+1)p+1)} \end{pmatrix},$$

$(i = 1, \ldots, \frac{n}{p} - 1)$ also has eigenvalues $\tilde{x}_1^{(n,p)} \le \cdots \le \tilde{x}_n^{(n,p)}$. From Horn and Johnson (1985) we therefore obtain the estimate

$$\max_{1 \le j \le n} \left| \tilde{\lambda}_j^{(n,p)} - \tilde{x}_j^{(n,p)} \right| \le \left\| G_n^{(p)} - F_n^{(p)} \right\|_\infty , \tag{3.8}$$

where for a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |a_{ij}|$$

denotes the maximum row sum norm. If

$$Z_n := \max \left\{ \max_{1 \le j \le n} |N_j| \, , \max_{1 \le j \le n-1} \frac{1}{\sqrt{2}} \left| X_{j\gamma_1} - \sqrt{j\gamma_1} \right| , \ldots , \max_{1 \le j \le n-p} \frac{1}{\sqrt{2}} \left| X_{j\gamma_p} - \sqrt{j\gamma_p} \right| \right\} ,$$

then it follows for $i = 1, \ldots, p, n-p+1, \ldots, n$ by a straightforward but tedious calculation

$$\sum_{j=1}^n \left| \{ G_n^{(p)} - F_n^{(p)} \}_{ij} \right| \le 2pZ_n,$$

and for $i = p+1, \ldots, n-p$

$$\sum_{j=1}^n \left| \{ G_n^{(p)} - F_n^{(p)} \}_{ij} \right| \le 3pZ_n,$$

which yields observing (3.8)

$$\max_{1 \le j \le n} \left| \tilde{\lambda}_j^{(n,p)} - \tilde{x}_j^{(n,p)} \right| \le 3pZ_n$$

and

$$P \left\{ \max_{1 \le j \le n} \left| \tilde{\lambda}_j^{(n,p)} - \tilde{x}_j^{(n,p)} \right| \ge \varepsilon \right\} \le P \left\{ Z_n \ge \frac{\varepsilon}{3p} \right\} . \tag{3.9}$$

From Dette and Imhof (2007) we obtain the estimates

$$P \Big\{ \frac{ |X_{j\gamma_k} - \sqrt{j\gamma_k}| }{\sqrt{2}} \ge \frac{\varepsilon}{3p} \Big\} \le 2e^{-\varepsilon^2/9p^2} \ , \ P \Big\{ \max_{1 \le j \le n} |N_j| \ge \frac{\varepsilon}{3p} \Big\} \le 2ne^{-\varepsilon^2/18p^2} \ ,$$

and a combination of these inequalities with (3.9) yields

$$P\left\{\max_{1\le j\le n}|\tilde{\lambda}_j^{(n,p)}-\tilde{x}_j^{(n,p)}|\ge\varepsilon\right\}\le P\left\{Z_n\ge\frac{\varepsilon}{3p}\right\}\le 2n(p+1)e^{-\varepsilon^2/18p^2},$$

which completes the proof of the theorem. □

Theorem 3.2. *Let $\lambda_1^{(n,p)}\le\cdots\le\lambda_n^{(n,p)}$ denote the eigenvalues of the random matrix $\frac{1}{\sqrt{n}}G_n^{(p)}$, where $G_n^{(p)}$ is defined in (1.3) and $\gamma_1,\dots,\gamma_p>0$ are chosen such that all blocks $D_i^{(p)}$ in the matrix $F_n^{(p)}$ defined by (3.7) are non-singular. If*

$$\sigma_n=\frac{1}{n}\sum_{j=1}^n\delta_{\lambda_j^{(n,p)}}\tag{3.10}$$

denotes the empirical distribution of the eigenvalues of the matrix $\frac{1}{\sqrt{n}}G_n^{(p)}$, then σ_n converges a.s. weakly to a measure, which is absolute continuous with respect to the Lebesgue measure. The density of this measure is given by

$$f(t)=\int_0^{\frac{1}{p}}tr\left[X_{A^{(p)}(s),B^{(p)}(s)}(t)\right]ds,\tag{3.11}$$

where $X_{A^{(p)}(s),B^{(p)}(s)}(t)$ denotes the Lebesgue density of the matrix measure corresponding to the Chebyshev polynomials of the first kind defined in (2.4) with matrices

$$A^{(p)}(s) := \sqrt{\frac{sp}{2}}\begin{pmatrix}\sqrt{\gamma_p} & \sqrt{\gamma_{p-1}} & \sqrt{\gamma_{p-2}} & \cdots & \sqrt{\gamma_1}\\ \sqrt{\gamma_{p-1}} & \sqrt{\gamma_p} & \sqrt{\gamma_{p-1}} & \cdots & \sqrt{\gamma_2}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \sqrt{\gamma_2} & \cdots & \sqrt{\gamma_{p-1}} & \sqrt{\gamma_p} & \sqrt{\gamma_{p-1}}\\ \sqrt{\gamma_1} & \cdots & \sqrt{\gamma_{p-2}} & \sqrt{\gamma_{p-1}} & \sqrt{\gamma_p}\end{pmatrix}\in\mathbb{R}^{p\times p},$$

$$B^{(p)}(s) := \sqrt{\frac{sp}{2}}\begin{pmatrix}0 & \sqrt{\gamma_1} & \sqrt{\gamma_2} & \cdots & \sqrt{\gamma_{p-1}}\\ \sqrt{\gamma_1} & 0 & \sqrt{\gamma_1} & \cdots & \sqrt{\gamma_{p-2}}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \sqrt{\gamma_{p-2}} & \cdots & \sqrt{\gamma_1} & 0 & \sqrt{\gamma_1}\\ \sqrt{\gamma_{p-1}} & \cdots & \sqrt{\gamma_2} & \sqrt{\gamma_1} & 0\end{pmatrix}\in\mathbb{R}^{p\times p}.\tag{3.12}$$

Proof. Let $R_{m,n}^{(p)}(x)$ $(m=n/p)$ denote the orthonormal polynomials satisfying the recurrence relation (2.1) with coefficients $A_{i,n}^{(p)}=\frac{1}{\sqrt{n}}\tilde{A}_{i,n}^{(p)}$ and $B_{i,n}^{(p)}=\frac{1}{\sqrt{n}}\tilde{B}_{i,n}^{(p)}$, where $\tilde{A}_{i,n}^{(p)}$ and $\tilde{B}_{i,n}^{(p)}$ are defined by (3.2) and (3.3), respectively, then we have

$$\begin{aligned} R^{(p)}_{m,n}(x) &= \tilde{R}^{(p)}_{m,n}(\sqrt{n}x), \\ x_j^{(n,p)} &= \frac{\tilde{x}_j^{(n,p)}}{\sqrt{n}}, \qquad j = 1, \ldots, n, \end{aligned}$$

where the matrix polynomial $\tilde{R}^{(p)}_{m,n}(x)$ is defined by (3.1) with corresponding roots $\tilde{x}_j^{(n,p)}$, and $x_j^{(n,p)}$ denotes the jth root of the matrix polynomial $R^{(p)}_{m,n}(x)$. From the definition of the matrices $\tilde{A}^{(p)}_{i,n}$ and $\tilde{B}^{(p)}_{i,n}$ in (3.2) and (3.3) we have

$$\lim_{n\to\infty} A^{(p)}_{\frac{n}{p}-1,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\gamma_p} & \sqrt{\gamma_{p-1}} & \sqrt{\gamma_{p-2}} & \cdots & \sqrt{\gamma_1} \\ \sqrt{\gamma_{p-1}} & \sqrt{\gamma_p} & \sqrt{\gamma_{p-1}} & \cdots & \sqrt{\gamma_2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sqrt{\gamma_2} & \cdots & \sqrt{\gamma_{p-1}} & \sqrt{\gamma_p} & \sqrt{\gamma_{p-1}} \\ \sqrt{\gamma_1} & \cdots & \sqrt{\gamma_{p-2}} & \sqrt{\gamma_{p-1}} & \sqrt{\gamma_p} \end{pmatrix} =: A^{(p)},$$

$$\lim_{n\to\infty} B^{(p)}_{\frac{n}{p}-1,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{\gamma_1} & \sqrt{\gamma_2} & \cdots & \sqrt{\gamma_{p-1}} \\ \sqrt{\gamma_1} & 0 & \sqrt{\gamma_1} & \cdots & \sqrt{\gamma_{p-2}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sqrt{\gamma_{p-2}} & \cdots & \sqrt{\gamma_1} & 0 & \sqrt{\gamma_1} \\ \sqrt{\gamma_{p-1}} & \cdots & \sqrt{\gamma_2} & \sqrt{\gamma_1} & 0 \end{pmatrix} =: B^{(p)} ,$$

and Gerschgorin's disc theorem implies that all roots of the polynomials $R^{(p)}_{m,n}(x)$ are located in a compact interval, say $[-M, M]$ [see also the proof of Lemma 2.1 in Duran (1999)]. Moreover, we have for any $\ell \in \mathbb{N}_0$

$$\begin{aligned} \lim_{\frac{i}{n}\to u} B^{(p)}_{i-\ell,n} &= \sqrt{up}B^{(p)} =: B^{(p)}(u), \\ \lim_{\frac{i}{n}\to u} A^{(p)}_{i-\ell,n} &= \sqrt{up}A^{(p)} =: A^{(p)}(u), \end{aligned}$$

where $u > 0$ and the matrix $A^{(p)}(u)$ is non-singular by assumption. Consequently, Theorem 2.1 is applicable and yields (note that $\lim_{n\to\infty}(n/p)/n = 1/p$) that the empirical distribution of the roots of the matrix polynomials $R^{(p)}_{m,n}(x)$ $(m = n/p)$

$$\delta_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j^{(n,p)}} \tag{3.13}$$

converges weakly to the measure with Lebesgue density f defined in (3.11). Next we use Theorem 3.1 which shows that

$$\max_{j=1}^{n} \mid \lambda_j^{(n,p)} - x_j^{(n,p)} \mid \; = \frac{1}{\sqrt{n}} \max_{j=1}^{n} \mid \tilde{\lambda}_j^{(n,p)} - \tilde{x}_j^{(n,p)} \mid \; = O\Big(\Big(\frac{\log n}{n}\Big)^{1/2}\Big) \tag{3.14}$$

almost surely. Therefore we obtain for the Levy distance L between the distribution functions F_{σ_n} and F_{δ_n} of the measures σ_n and δ_n

$$L^3(F_{\sigma_n}, F_{\delta_n}) \leq \frac{1}{n} \sum_{j=1}^{n} \mid \lambda_j^{(n,p)} - x_j^{(n,p)} \mid^2 = O\left(\frac{\log n}{n}\right) \tag{3.15}$$

almost surely [for the inequality see Bai (1999), p. 615]. Consequently, it follows that the spectral measure σ_n of the matrix $\frac{1}{\sqrt{n}} G_n^{(p)}$ converges also weakly with the same limit as δ_n, that is the measure with Lebesgue density f defined by (3.11).

□

4 Examples

We conclude this paper with a discussion of a few examples. First note that in the case $p = 1$ Theorem 3.2 yields for the limiting distribution of eigenvalues of the matrix $\frac{1}{\sqrt{n}} G_n^{(1)}$ the Wigner's semicircle law with density

$$f(x) = \frac{1}{\pi\gamma_1} \sqrt{2\gamma_1 - x^2} I_{\{\sqrt{-2\gamma_1} < x < \sqrt{2\gamma_1}\}},$$

which have been considered by numerous authors.

Next we concentrate on the case $p = 2$, for which it is easily seen that the matrix $D_i^{(2)}$ in (3.7) is non-singular whenever $\gamma_2 \neq \gamma_1$. In this case the density of the limit of the spectral measure is a mixture of two arcsine densities and given by

$$f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^{(n,2)}} \; = \sum_{j=1}^{2} \int_0^{1/2} \frac{1}{\pi \sqrt{4\alpha_j^2(s) - (x - \beta_j(s))^2}} I_{\{-2\alpha_j(s) + \beta_j(s) < x < 2\alpha_j(s) + \beta_j(s)\}} ds$$

where

$$\alpha_1(s) = \sqrt{s}\,(\sqrt{\gamma_2} + \sqrt{\gamma_1}), \;\; \alpha_2(s) = \sqrt{s}\,(\sqrt{\gamma_2} - \sqrt{\gamma_1}), \;\; \beta_1(s) = \sqrt{s\gamma_1}, \;\; \beta_2(s) = -\sqrt{s\gamma_1}\,.$$

In Figure 1 and 2 we display the limiting spectral density corresponding to the case $\gamma_1 = 2, \gamma_2 = 8$ and $\gamma_1 = 1, \gamma_2 = 100$, respectively. The left part of the figures shows a simulated histogram of the eigenvalues of the matrix $\frac{1}{\sqrt{n}} G_n^{(2)}$ for $n = 5000$, while the right part of the figures shows the corresponding limiting distribution obtained from Theorem 3.2.

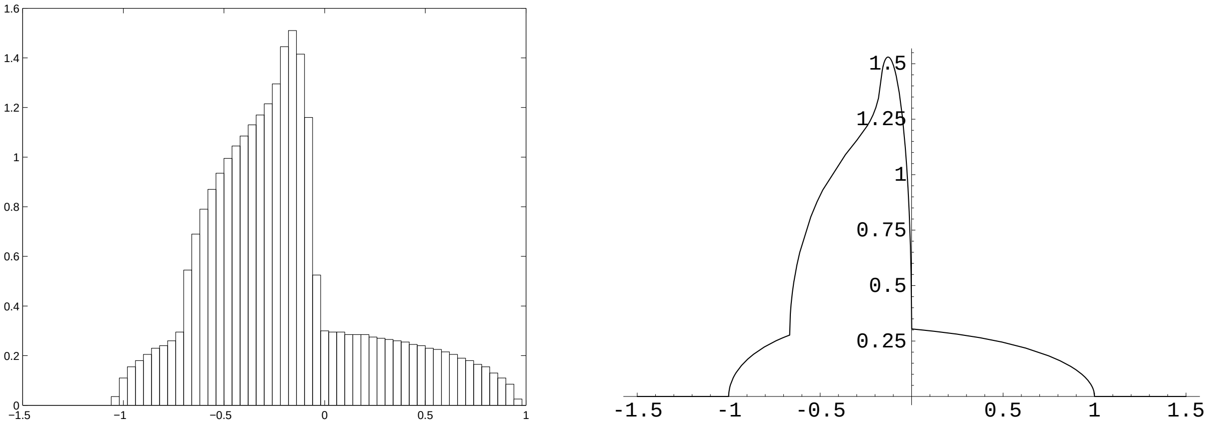

Figure 1: *Simulated and limiting spectral density of the random block matrix* $G_n^{(p)}/\sqrt{n}$ *in the case* $p = 2$*,* $\gamma_1 = 2$*,* $\gamma_2 = 8$*. In the simulation the eigenvalue distribution of a* 5000×5000 *matrix was calculated (i.e.* $m = n/p = 2500$*).*

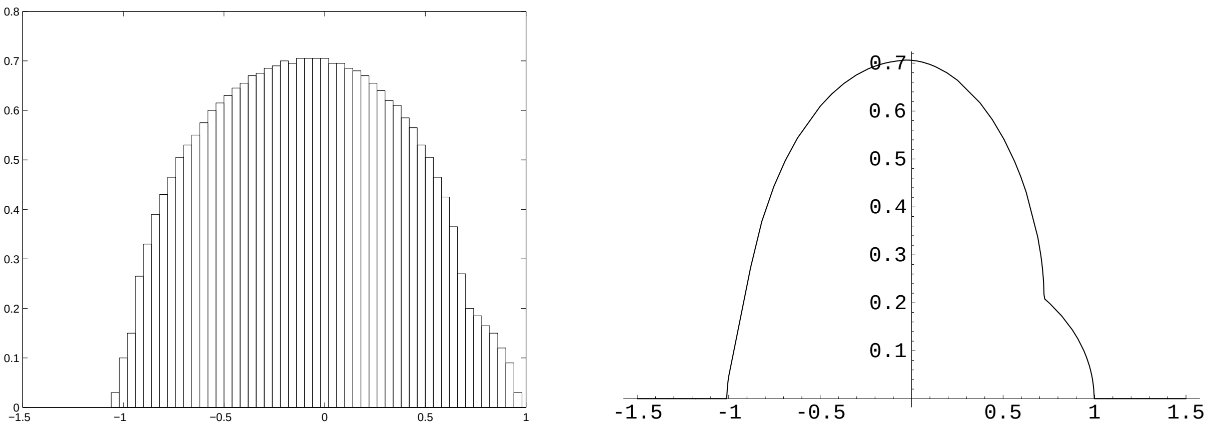

Figure 2: *Simulated and limiting spectral density of the random block matrix* $G_n^{(p)}/\sqrt{n}$ *in the case* $p = 2$*,* $\gamma_1 = 1$*,* $\gamma_2 = 100$*. In the simulation the eigenvalue distribution of a* 5000×5000 *matrix was calculated (i.e.* $m = n/p = 2500$*).*

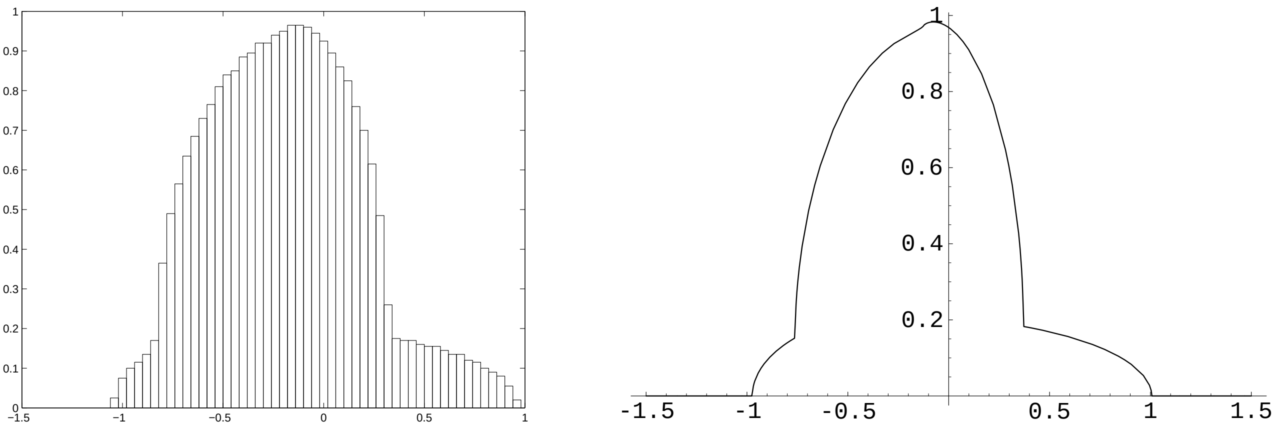

Figure 3: *Simulated and limiting spectral density of the random block matrix* $G_n^{(p)}/\sqrt{n}$ *in the case* $p = 3,\ \gamma_1 = \gamma_2 = 4,\ \gamma_3 = 100$. *In the simulation the eigenvalue distribution of a* 5001×5001 *matrix was calculated (i.e.* $m = n/p = 1667$*).*

If $p \geq 3$ the general formulas for the density of the limit distribution are too complicated to be displayed here, but it can be shown that

$$\operatorname{tr}[X_{A^{(p)}(u),B^{(p)}(u)}(t)] = \sum_{j=1}^{p} \frac{-\frac{d}{dt}\lambda_j^{A^{(p)}(u),B^{(p)}(u)}(t)}{\pi\sqrt{4-(\lambda_j^{A^{(p)}(u),B^{(p)}(u)}(t))^2}} I_{\{-2<\lambda_j^{A^{(p)}(u),B^{(p)}(u)}(t)<2\}},$$

if the matrix $A^{(p)}(u)$ is positive definite. This identity follows in a similiar way as (2.31). In Figure 3, 4 and 5 we show a simulated histogram of the eigenvalues of $G_n^{(3)}/\sqrt{n}$ and the corresponding density of the limit distribution obtained from Theorem 3.2 in the case $\gamma_1 = \gamma_2 = 4,\ \gamma_3 = 100$; $\gamma_1 = 1,\ \gamma_2 = 4, \gamma_3 = 25$ and $\gamma_1 = 1,\ \gamma_2 = 100, \gamma_3 = 200$, respectively.

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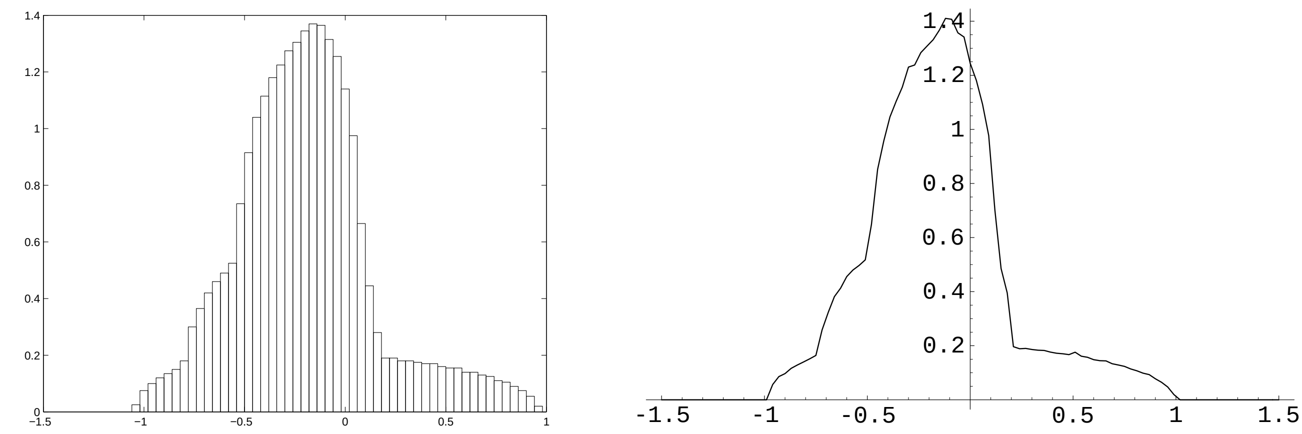

Figure 4: *Simulated and limiting spectral density of the random block matrix* $G_n^{(p)}/\sqrt{n}$ *in the case* $p = 3,\ \gamma_1 = 1,\ \gamma_2 = 4,\ \gamma_3 = 25$*. In the simulation the eigenvalue distribution of a* 5001×5001 *matrix was calculated (i.e.* $m = n/p = 1667$*).*

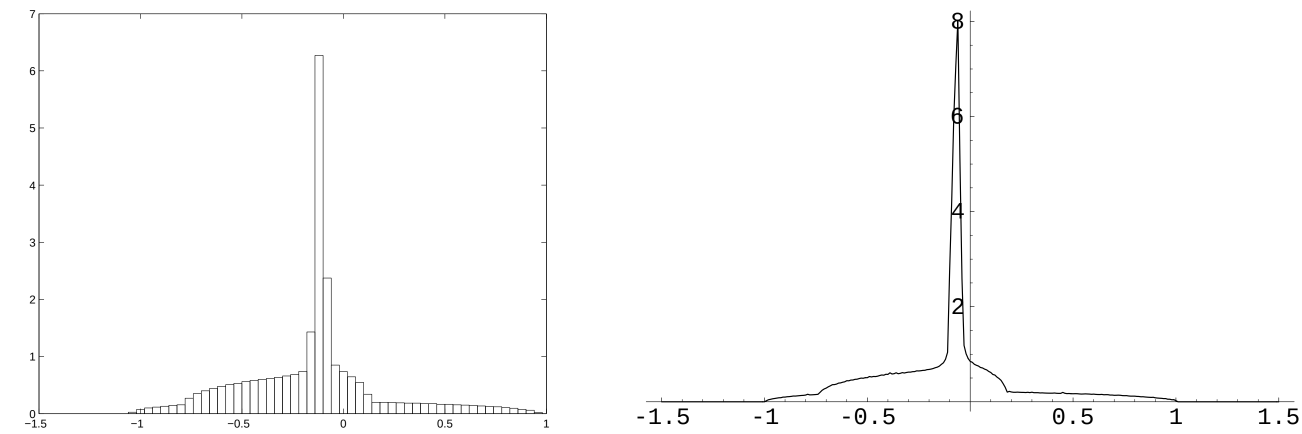

Figure 5: *Simulated and limiting spectral density of the random block matrix* $G_n^{(p)}/\sqrt{n}$ *in the case* $p = 3,\ \gamma_1 = 1,\ \gamma_2 = 100,\ \gamma_3 = 200$*. In the simulation the eigenvalue distribution of a* 5001×5001 *matrix was calculated (i.e.* $m = n/p = 1667$*).*

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