Testing for symmetries in multivariate inverse problems

Melanie Birke
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
e-mail: melanie.birke@rub.de

Nicolai Bissantz
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
e-mail: nicolai.bissantz@rub.de

April 19, 2011

Abstract

We propose a test for shape constraints which can be expressed by transformations of the coordinates of multivariate regression functions. The method is motivated by the constraint of symmetry with respect to some unknown hyperplane but can easily be generalized to other shape constraints of this type or other semi-parametric settings. In a first step, the unknown parameters are estimated and in a second step, this estimator is used in the L_2-type test statistic for the shape constraint. We consider the asymptotic behavior of the estimated parameter and show, that it converges with parametric rate if the shape constraint is true. Moreover we derive the asymptotic distribution of the test statistic under the null hypothesis and furthermore propose a bootstrap test based on the residual bootstrap. In a simulation study we investigate the finite sample performance of the estimator as well as the bootstrap test.

Keywords: Deconvolution, Goodness-of-Fit, Inverse Problems, Semi-Parametric Regression, Symmetry

1 Introduction

Several kinds of symmetry play an important role in many areas of research. For example, many objects or parts of objects are symmetric with respect to reflection or rotation. Symmetry can be used in image compression and also in image analysis to detect certain objects. If symmetry of a certain object is violated one can sometimes deduce some results from it. Usually, parts of the human body are (nearly) symmetric, e.g. the left hand is symmetric to the right hand, the left part of the face to the right part and so on. This is usually also true for the thermographic distribution of those parts. If in a thermographic image of both hands this symmetry is severely violated, this can be a hint to some inflammation in this part. Problems of this and similar type make testing for symmetry to a problem of considerable interest. Technically, modeling the object of interest

as a multivariate function, we end up with the problem of testing for symmetry of a multivariate function.
Whereas several results exist which discuss the symmetry of density functions (see e.g. Ahmad and Li (1997), Cabaña and Cabaña (2000) and Dette, Kusi-Appiah and Neumeyer (2002) among many others) only few authors have considered testing for symmetry of a regression function so far. Recent results have been presented in Bissantz, Holzmann and Pawlak (2009) and Birke, Dette and Stahljans (2011), where both are for the case of bivariate functions in direkt regression models and for symmetry with respect to some known axis.
In some cases it is not possible to observe the object of interest directly. This leads to an inverse problem. Testing for symmetry in inverse regression problems can be of even higher interest than testing for symmetry in direct regression models. The reason is as follows. Whereas, at least in bivariate settings, symmetry in direct regression models can approximately be recognized by simply looking at the data, symmetrical structures in the true object can lack any symmetry in the observed (indirect) data. Consider, for example, the well known convolution problem which commonly appears in image analysis where the true object is distorted by a so called point-spread function we can easily find situations (e.g. for asymmetric point-spread functions or if the point-spread function has a different axis of symmetry than the true object) where the symmetry is not visible in the image. To the best of our knowledge there are no methods for testing for symmetry in inverse regression problems so far.
In the following we will develop a testing procedure for reflection symmetry of d-variate functions with respect to some hyperplane of dimension $d-1$. The method can, however, easily be generalized to rotational symmetry or other shape constraints of similar type. Therefore, whereas we motivate the problem by the case of a symmetry constraint, the theoretical results and their proofs will be formulated as general as possible. Since the symmetry hyperplane is unknown we estimate it in a first step by minimizing an L_2-criterion function. If the true function is really symmetric with respect to this hyperplane, we derive, under some regularity conditions, consistency with parametric rate of the estimator and show that it is asymptotically normally distributed. In a second step, we use the minimized criterion function as test statistic for symmetry and show that it is asymptotically normal. Since the problem under consideration is closely related to certain semi-parametric problems we will use similar techniques as Härdle and Marron (1990). However note the important differences, that our problem is inverse and our regression function is multivariate.
In nonparametric regression tests based on such asymptotic distributions usually do not perform satisfactorily in finite samples because the convergence is very slow and there is the problem of dealing with a bias term. To avoid this problem we propose a bootstrap test based on residual bootstrap and investigate the finite sample performance of this test in a simulation study.
The rest of the paper is organized as follows. In section 2 we describe the model and define the estimator for the hyperplane as well as the test statistic. The asymptotic behavior of both is considered in section 3 while we show the finite sample performance in section 4. Finally all proofs are defered to the appendix

2 The model and test statistic

We consider the nonparametric inverse regression model

$$Y_{\mathbf{r}} = \Psi m(\mathbf{x_r}) + \sigma\varepsilon_{\mathbf{r}} \tag{1}$$

with $\mathbf{x_r} = (r_1/(n_1 a_{n_1}), \ldots, r_d/(n_d a_{n_d}))^T$, $r_j = -n_j, \ldots, n_j$ and $a_{n_j} \to 0$, $j = 1, \ldots, d$ such that with increasing sample size we have observations on the whole $\mathbb{R}^d$. For the sake of simplicity we assume in the following that $n_j = n$ and $a_{n_j} = a_n$ such that $\mathbf{x_r} = (r_1, \ldots, r_d)^T/(na_n)$ and for fixed n we have observations on the compact set $I_n = [-1/a_n, 1/a_n]^d$. In (1) m is a two times continuously differentiable regression function, and Ψ is an operator which maps m to the convolution $m * \psi$ with a known convolution function ψ. Finally, with $\mathbf{r} = (r_1, \ldots, r_d)$, $\{\varepsilon_{\mathbf{r}}\}^n_{\mathbf{r}\in\{-n\ldots,n\}^d}$ are independent identically distributed errors with $\mathrm{E}[\varepsilon_{\mathbf{r}}] = 0$, $\mathrm{E}[\varepsilon_{\mathbf{r}}^2] = 1$ and $\mathrm{E}[\varepsilon_{\mathbf{r}}^4] < \infty$. If m is j times continuously differentiable according to Bissantz and Birke (2009)

$$\hat{m}^{(\mathbf{j})}(\mathbf{x}) = \sum_{\mathbf{r}\in\{-n,\ldots,n\}^d} w_{\mathbf{r},\mathbf{j}}(\mathbf{x}) Y_{\mathbf{r}} \tag{2}$$

with

$$w_{\mathbf{r},\mathbf{j}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}(nh^j a_n)^d} \int_{[-1,1]^d} \frac{(-i\omega)^j e^{-i\omega^T(\mathbf{x}-\mathbf{x_r})/h}}{\Phi_\psi(\omega/h)} d\omega \tag{3}$$

with $\mathbf{j} = (j_1, \ldots, j_d)$, $j = j_1 + \ldots + j_d$ is an appropriate estimate for $\frac{\partial^{j_1+\ldots+j_d}}{\partial x_1^{j_1}\ldots\partial x_d^{j_d}} m$. If $j = 0$ we write $\hat{m}^{(\mathbf{0})}(\mathbf{x}) = \hat{m}(\mathbf{x})$ and $w_{\mathbf{r},\mathbf{0}}(\mathbf{x}) = w_{\mathbf{r}}(\mathbf{x})$. As an abbreviation we write in the following $\Psi m = g$. In (3) Φ_f denotes the Fourier transform of a function f.
We consider the case of reflection symmetry with respect to some hyperplane in $\mathbb{R}^d$ parameterized by $\theta \in \mathbb{R}^d$. Then, for every fixed $\theta \in \mathbb{R}^d$ mirrowing m at the corresponding hyperplane can be realized by some linear functional $T_\theta S_\theta^{-1}$ where T_θ contains the shift of the hyperplane and the rotation and S_θ^{-1} is mainly the inverse of T_θ concatenated with the mirrowing at the $(x_2, \ldots, x_d)$-hyperplane $\}_\theta$. The condition of symmetry of m with respect to that hyperplane in some area A_θ around that hyperplane is

$$m(\mathbf{z}) = m(T_\theta S_\theta^{-1}\mathbf{z}) \text{ for all } \mathbf{z} \in A_\theta \tag{4}$$

or equivalently

$$m(T_\theta\mathbf{x}) = m(S_\theta\mathbf{x}) \text{ for all } \mathbf{x} \in A = T_\theta^{-1}A_\theta. \tag{5}$$

To this end we will use

$$L(\theta) \; = \; \int_A \left(m(T_\theta\mathbf{x}) - m(S_\theta\mathbf{x})\right)^2 d\mathbf{x}. \tag{6}$$

to check whether m exhibits such a symmetry on A_θ. In the following we will assume without loss of generality that $A = T_\theta^{-1}A_\theta$ is independent of θ. The parameter ϑ of the true hyperplane minimizes this criterion function. Since m is not known, we estimate the criterion function as

$$\hat{L}_n(\theta) \; = \; \int_A \left(\hat{m}(T_\theta\mathbf{x}) - \hat{m}(S_\theta\mathbf{x})\right)^2 d\mathbf{x} \tag{7}$$

and find the estimator of ϑ by minimizing $\hat{L}_n(\theta)$

$$\hat{\vartheta} \;=\; \arg\min_{\theta \in B_0 \times B_1} \hat{L}_n(\theta),$$

where $B_0 \subset \mathbb{R}^{d-1}$ is the compact set of all possible rotation angles and $B_1 \subset \mathbb{R}$ the compact set of all possible shifts. If $\hat{m}$ is continuously differentiable, we can equivalently solve

$$\hat{l}_n(\theta) \;=\; \text{grad} \;\; \hat{L}_n(\theta) = 0 \tag{8}$$

to find $\hat{\vartheta}$.

Example. For illustrational purposes we discuss the case $d = 2$. Here, the hyperplane reduces to a straight line parameterized by $\}_\theta = \left\{ (\cos\theta_1, \sin\theta_1)^T \lambda + \theta_2 \left(-\sin(\theta_1), \cos(\theta_1)\right)^T | \lambda \in \mathbb{R} \right\}$, $\theta = (\theta_1, \theta_2)^T \in \mathbb{R}^2$ unknown such that mirrowing $\mathbf{z} \in \mathbb{R}^2$ at that straight line can be obtained by transforming $\mathbf{z}$ to

$$T_\theta^{-1}\mathbf{z} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \mathbf{z} - \begin{pmatrix} 0 \\ \theta_2 \end{pmatrix},$$

mirrowing at $\}_0 = \left\{ (0,1)^T \lambda | \lambda \in \mathbb{R} \right\}$ which gives

$$S_\theta^{-1}\mathbf{z} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T_\theta^{-1}\mathbf{z}$$

and transforming back, which finally yields

$$T_\theta S_\theta^{-1}\mathbf{z}.$$

3 Asymptotic inference

To consider asymptotic theory, we further assume that Ψ is ordinary smooth, i.e. we consider mildly ill-posed problems in model (1). This can be summarized in the following assumption.

Assumption 1. The Fourier transform Φ_ψ satisfies

$$|\Phi_\psi(\omega)| \; |\omega|^\beta \to \kappa, \qquad \omega \to \infty$$

for some $\beta > 0$ and $\kappa \in \mathbb{R} \setminus \{0\}$.

Assumption 2. The Fourier transform Φ_m of m satisfies $\int_{\mathbb{R}} |\Phi_m(\omega)||\omega|^{\mathbf{k}} d\omega < \infty$ for any multiindex $\mathbf{k}$ with $k_1 + \ldots + k_d \leq r$ for some $r > \beta + 1$ and m is two times continuously differentiable

Assumption 3. The bandwidth h fulfills $h \to 0$, $n^{d/2} a_n^{d/2} h^{\beta+d} \to \infty$, $(\log n)^{1/4}/n^d h^d a_n^d = o(1)$, $n^d h^{2\beta+2s+d/2-1} a_n^{3d/2} \to 0$ and $a_n^r = o(h^{\beta+s+d-1})$

Assumption 2 is, for example fulfilled, if for grad (m) (and hence also for the products and sums in the integral) the $\mathbf{k}$-th derivative exists for all $||\mathbf{k}|| \leq \beta$. Note also, that in Assumption 3 a_n cannot be seen as regularization parameter since it is determined by the underlying design. Therefore, all conditions have to be read as conditions on h_n, s, β, j and r dependent on the rate of a_n.

Under the above conditions we can now discuss the asymptotic properties. We first consider the consistency and the asymptotic distribution of the estimator $\hat{\vartheta}$

Theorem 1. *Let $L(\theta)$ be locally convex near the true parameter ϑ. Then, under Assumptions 1 $\hat{\vartheta}_n \xrightarrow{P} \vartheta$ for $n \to \infty$.*

Theorem 2. *If $\hat{m}$ is continuously differentiable, $\hat{\vartheta}$ is defined by (8) and $\}_\vartheta$ is the true symmetry hyperplane, we have*

$$\sqrt{n^d a_n^d}\left(\hat{\vartheta} - \vartheta\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2 h^{-1}(\vartheta)\Sigma(\vartheta)(h^{-1}(\vartheta))^T)$$

with

$$\begin{aligned}
\Sigma(\theta) &= \frac{\sigma^2}{(2\pi^2\kappa)^d}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}||\omega||^\beta I_{[-1,1]^d}(\omega)e^{-i\omega^T\mathbf{y}}d\mathbf{y}d\omega\right|^2\int_{\mathbb{R}^d}\sigma_\theta(\mathbf{u})\sigma_\theta(\mathbf{u})^T d\mathbf{u} \\
\sigma_\theta(\mathbf{u}) &= \left(\left(\frac{\partial}{\partial\theta}T_\theta\right)(T_\theta^{-1}(\mathbf{u})) - M_\theta N_\theta^{-1}\left(\frac{\partial}{\partial\theta}S_\theta\right)(T_\theta^{-1}(\mathbf{u})) - N_\theta M_\theta^{-1}\left(\frac{\partial}{\partial\theta}T_\theta\right)(S_\theta^{-1}(\mathbf{u}))\right. \\
&\qquad \left. - \left(\frac{\partial}{\partial\theta}S_\theta\right)(S_\theta^{-1}(\mathbf{u}))\right)^T (\text{grad } m(\mathbf{u}))^T \\
h(\theta) &= 2\int_A\left(\text{grad } m(T_\theta\mathbf{x})\frac{\partial}{\partial\theta}T_\theta\mathbf{x} - \text{grad } m(S_\theta\mathbf{x})\frac{\partial}{\partial\theta}S_\theta\mathbf{x}\right)\left(\text{grad } m(T_\theta\mathbf{x})\frac{\partial}{\partial\theta}T_\theta\mathbf{x} - \text{grad } m(S_\theta\mathbf{x})\frac{\partial}{\partial\theta}S_\theta\mathbf{x}\right)^T d\mathbf{x}
\end{aligned}$$

The second point of interest is to test whether the image obeys a symmetry of some kind. We use the test statistic

$$\hat{L}_n(\hat{\vartheta}) = \int_A \left(\hat{m}(T_{\hat{\vartheta}}\mathbf{x}) - \hat{m}(S_{\hat{\vartheta}}\mathbf{x})\right)^2 d\mathbf{x} \tag{9}$$

which has the following asymptotic distribution.

Theorem 3. *Under the above assumptions, if ϑ parametrizes the true symmetry hyperplane, we have*

$$\sigma_n^{-1/2}\left(\hat{L}_n(\hat{\vartheta}) - \frac{2\sigma^2}{(2\pi)^d n^d h^{2\beta+d} a_n^d}\int_A\int_{[-1,1]^d}|\omega|^{2\beta}\left|\sin\left(\frac{\omega^T S_\vartheta\mathbf{x}}{h}\right)\right|^2 d\omega d\mathbf{x}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

with

$$\sigma_n = \frac{32\sigma^4}{\kappa^4(2\pi)^{2d}n^{2d}h^{2d+4\beta}a_n^{2d}}\int_{\mathbb{R}^{2d}}|\omega|^{2\beta}|\eta|^{2\beta}\left|\int_A \sin\left(\frac{\omega^T S_\vartheta\mathbf{x}}{h}\right)\sin\left(\frac{\eta^T S_\vartheta\mathbf{x}}{h}\right)d\mathbf{x}\right|^2 d(\omega,\eta)$$

It can be shown similarly as in the proof of Theorem 4 in the Appendix, that the effective rate of convergence is $n^d h^{2\beta+d/2} a_n^{3d/2}$.

4 Simulations

4.1 Simulation framework

In this section we present the results of a simulation study. To this end we generate observations according to model (1), i.e.

$$Y_{(r,s)} = \Psi m(\mathbf{x}_{(r,s)}) + \sigma\varepsilon_{(r,s)}.$$

In our simulations, the noise terms are i.i.d. normally distributed with variance 1 and $\mathbf{x}_{(r,s)} = \left(\frac{r}{n}, \frac{s}{n}\right)$, $(r,s) \in \{-n, -n+1, \ldots, n-1, n\}^2$ are the coordinates of a grid with equidistant stepsize in both coordinates and with $a_n = 1$. In the following we use the parameter values $n = 50$ and σ (in dependence of the underlying function m) such that σ makes up for 1/10-th and 1/25-th of the maximum of the signal Ψm, which amounts to signal-to-noise ratios - defined as the mean signal of the image divided by σ - of ≈ 10 and ≈ 4, respectively. These values amount to rather poor signal-to-noise ratios, and in a practical application, S/N will frequently be larger and our simulations be expected to be conservative with respect to the performance of our method.
We consider two different "true" images m_1 and m_2 from which the data is generated. These images represent the cases of having a unique axis of symmetry (image m_1) and of not having any axis of symmetry at all (image m_2). The images are generated from the following bivariate functions (with $(x_t, y_t) \in \mathbb{R}^2$).

$$\begin{aligned} m_1(x,\, y) &= \exp(-3 \cdot (4 \cdot x_t^2 + (y_t + 0.1)^2)) + 0.5 \cdot \exp(-3 \cdot (x_t^2 + 3 \cdot (y_t - 0.4)^2)) \\ m_2(x,\, y) &= 0.5 \cdot \exp(-5 \cdot ((x_t - 0.3)^2 + 5 \cdot (y_t + 0.3)^2 +)) \\ &+ 0.5 \cdot \exp(-5 \cdot ((x_t + 0.2)^2 + 5 \cdot (y_t - 0.3)^2)) \\ &+ 0.5 \cdot \exp(-5 \cdot ((x_t + 0.5)^2 + 5 \cdot (y_t + 0.6)^2)), \end{aligned}$$

where

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\delta \\ 0 \end{pmatrix}$$

are the coordinates of a coordinate system which is rotated by an angle $\alpha = -0.3$ with respect to the original coordinate system of y in counterclockwise direction and shifted (along the transformed y_t-axis) by $\delta = 0.1$. Hence, image m_1 is symmetric with respect to an axis of symmetry which passes the x-axis at $x = 0.1$ and is tilted away to the right from the y-axis by an angle of -0.3 rad., that is $\vartheta = (\alpha, \delta)^T = (-0.3, 0.1)^T$
In accordance with model 1 for the observations, we do not assume to be able to observe m_i directly, but that at our disposal are only observations of the convolution of m_i, $i = 1, 2$ with a convolution function ψ given by

$$\psi(x,y) = \frac{\lambda}{2} \cdot \exp\left(-\lambda \cdot \sqrt{x^2 + 0.25 \cdot y^2}\right)$$

(with $\lambda = 5$). Figure 1 shows the images of m_1 and m_2, their convolutions with Ψ and typical examples for estimates $\hat{m}_1$ and $\hat{m}_2$.
The convolution function ψ is symmetric with respect to the x- and y-axis of the (original) coordinate system, that is symmetric with respect to axes which are different to the axes of symmetry of m_1. In consequence, the convolved (observed) image Ψm_1 does not have any axis of axial symmetry. Note

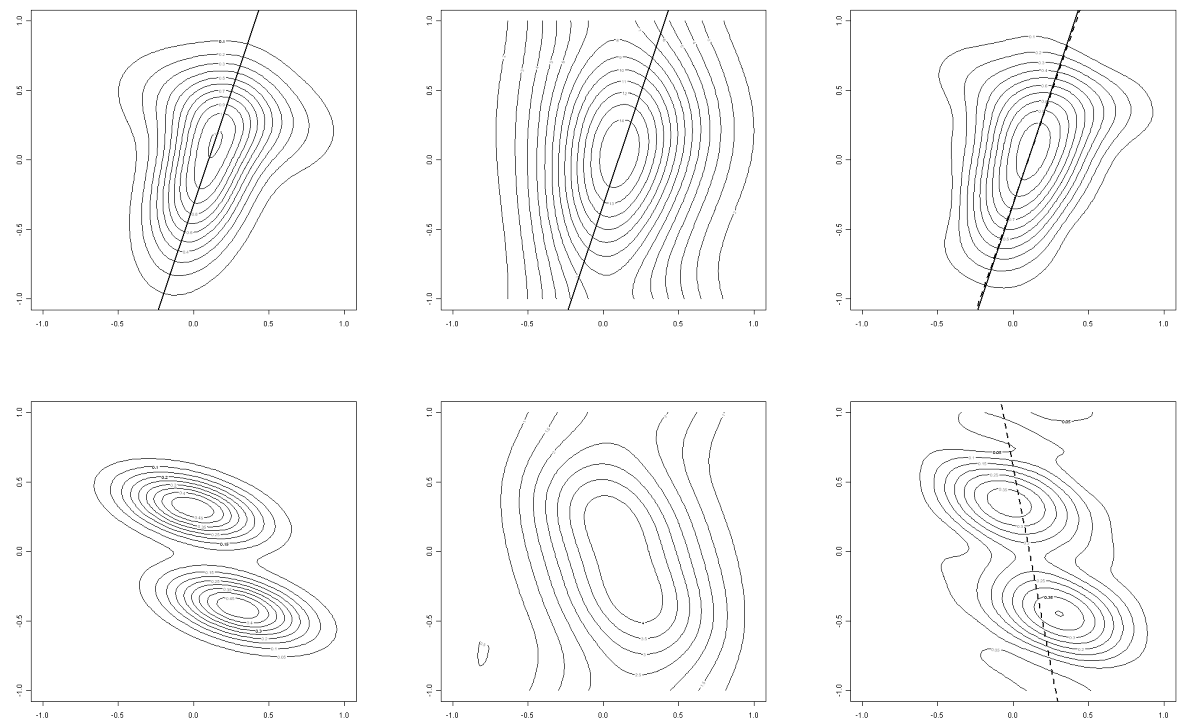


Figure 1: True images and typical examples for the observed image and associated selected axis for m_1 (top panels)and m_2 (bottom panels). Left column: true functions, middle column: true function convolved with Ψ, right column: reconstructions from data with $n = 50$, $S/N = 25$. The full line indicates the true axis of symmetry and the dashed line the estimated symmetry axis. Note that m_2 is not symmetric to any axis, hence the full line is missing.

that this implies that testing for symmetry of m can in general not be substituted by testing for symmetry of Ψm, except under specific, strong assumptions on the symmetry properties of m and ψ. Instead, it is required that the observed image is deconvolved in a first step, with the symmetry test being performed in a subsequent second step.
In our simulations we use the spectral cut-off estimator (2) with equal bandwidths in both coordinate axes. From a visual inspection of 5 randomly selected noisy images and the associated estimates $\hat{m}$ we chose $h \approx 0.05$. This bandwidth was kept fixed in all subsequent simulations.

4.2 Critical functions and the distribution of estimated parameters and test statistics

In this section we describe the performance of the estimators for the symmetry axis parameters δ and α, and the properties of the underlying criterion function (7), which can, as already pointed out in Section 3, be used as test statistic for symmery of the regression function, for the two different images considered here.

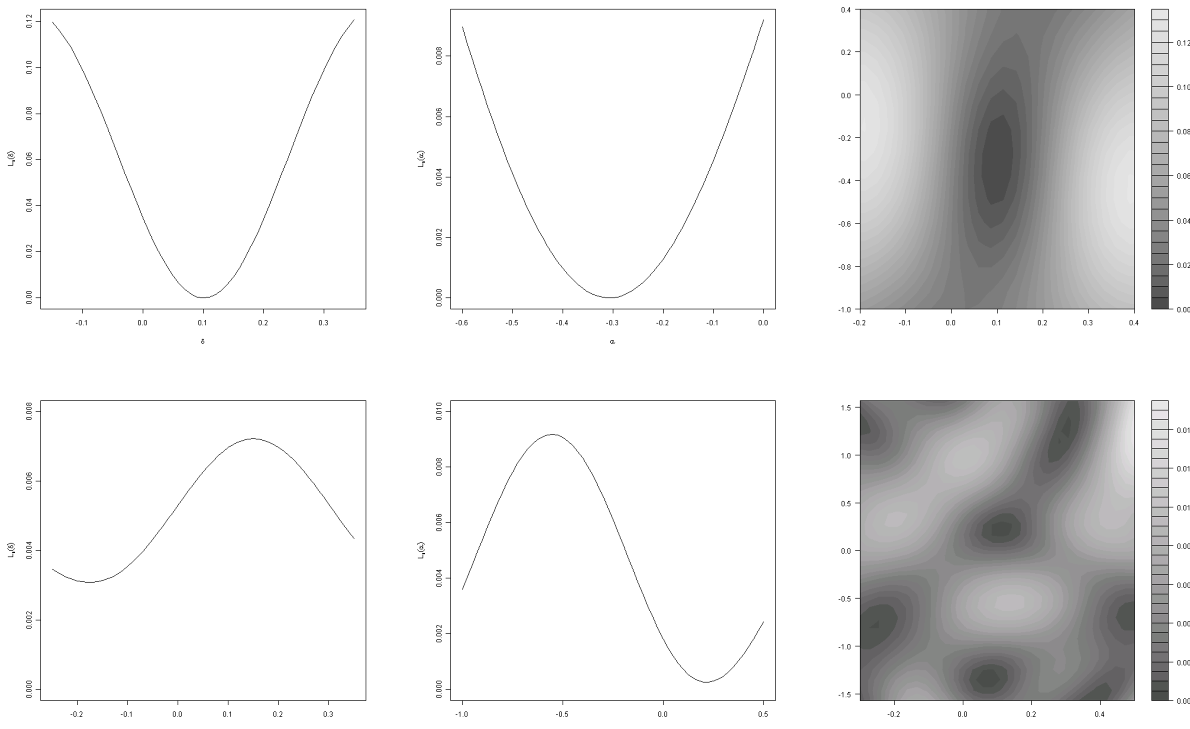
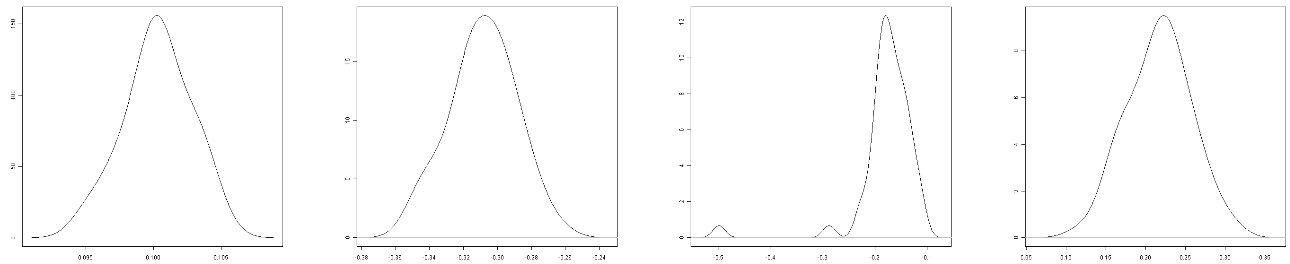


Figure 2: True (noiseless) criterion function L_n for the translation axis for m_1 (top panels) and m_2 (bottom panels) for $n = 50$ and signal-to-noise ration $S/N = 25$. Left column: $L_n(\delta)$ for $\alpha = -0.3$ assumed to be known, middle column: $L_n(\alpha)$ for $\delta = 0.1$ assumed to be known, right column: $L_n(\delta, \alpha)$.



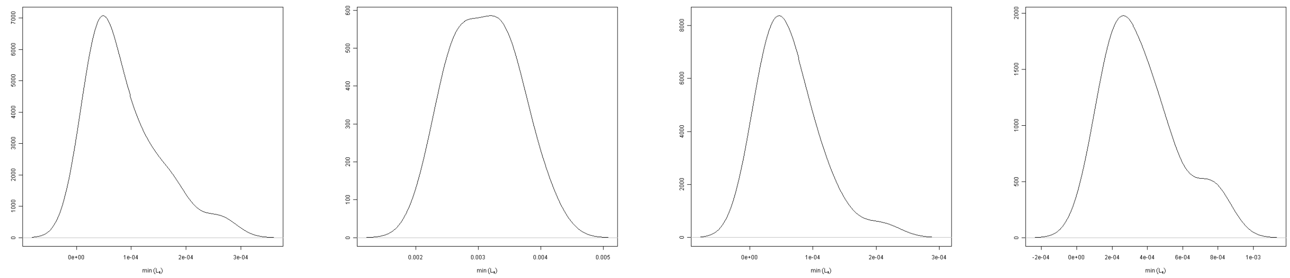
(a) (b) (c) (d)

Figure 3: Distribution of the estimated symmetry parameters for m_1 ((a) and (c)) and m_2 ((b) and (d)). (a) and (b): only shift estimated, (c) and (d): only rotation angle estimated for sample size parameter $n = 50$, and signal-to-noise ratio $S/N = 25$.

Figure 2 shows the critical function $L_n(\delta, \alpha)$ both for the case of univariate estimation of the shift δ resp. the angle α (where the other parameter is assumed to be known) and for bivariate estimation of the pair (δ, α). For m_2 the criterion function for the selection of the shift only (top right panel) does not come close to the minimal value it attains for the symmetric function m_1 at all, but the situation is different for the estimation of the rotation angle, where the minimal values differ less strongly. Now consider the bivariate estimation of shift and rotation angle. For m_2, a complicated pattern appears without a distinct minimum.
Next, Figure 3 shows the simulated distribution of the estimated parameters for rotation and shift for the various simulation setups. For m_2, which does not have an axis of symmetry at all, the critical function still showes clear minima of the criterion function if only one of the parameters was estimated. This is reflected in the right column of Figure 3 for the estimated parameter, that is the value where the minimum is attained.
Finally, consider Figure 4, which compares the simulated distributions of the test statistic for the case of one parameter estimated under H_0 (i.e. for m_1) with the results under H_1 (i.e. for m_2). In the latter case the distributions are shifted to significantly larger mean values, which reflects the fact that there exists no axis of symmetry. Moreover, their shape appears more symmetric than under H_0, where it is (much) more skewed to the right, similar to other L_2-based test statistics (e.g. Dette (1999), Bissantz et al. (2010) and Birke, Dette and Stahljans (2011)).

4.3 Testing for symmetry

In the final part of our simulations let us now turn to a more precise analysis of the performance of our proposed test for symmetry. Since the convergence of L_2-tests is known to be slow and the asymptotic distribution apparently depends on unknown parameters we use bootstrap quantiles as critical values for the test.
Hence, our testing procedure consists of two main parts. In the first *bootstrap* part we determine a bootstrap approximation to the distribution of the test statistics. In more detail, this consists of three steps: (1) to estimate the distribution of residuals, (2) to determine a "true image" $\hat{m}_B$ from



(a) (b) (c) (d)

Figure 4: Distribution of the test statistics under $H_0 : m = m_1$ ((a) and (c)) resp. $m = m_2$ ((b) and (d)). (a) and (b): only shift estimated, (c) and (d): only rotation angle estimated for sample size parameter $n = 50$, and signal-to-noise ratio $S/N = 25$.

which the bootstrap data are generated, and (3) to perform the bootstrap replications of the test statistic. The subsequent, second *test decision* part of the procedure is performed by computation of the test statistic for the original (observed) data and a decision based on this test statistic and the bootstrap approximation to its distribution. We now describe all steps in detail.

A. Bootstrap part of the testing procedure:

1. Estimation of the distribution of residuals: In our simulations we use a residual bootstrap as follows. In the first step we determine the empirical distribution of the residuals as the centered distribution of differences between the observations and an estimate $\Psi\hat{m}$ of Ψm. Then, in each of the bootstrap replications, we draw residuals from this distribution and generate bootstrap data as the sum of a suitable "true bootstrap image" $\hat{m}_B$ and these residuals.

2. Determination of a "true image" $\mathbf{\hat{m}_B}$: The "true bootstrap image" $\hat{m}_B$ is generated as follows such that it obeys a known axis of symmetry and closely resembles the true (unknown) function m, assuming H_0 to be true.

Step 2.1 - Estimating m: Determination of an estimate $\hat{m}$ of m as described above.

Step 2.2 - Estimation of symmetry axis parameter: Minimization of the criterion function yields estimates $\hat{\delta}$ and/or $\hat{\alpha}$ of the symmetry axis parameter(s) of $\hat{m}$.

Step 2.3 - Backshift and rotation of $\hat{m}$: We shift and rotate $\hat{m}$ back by the estimated parameters $\hat{\delta}$ and/or $\hat{\alpha}$ (and, if applicable, the known true values of the other parameter). Under H_0, and if no noise would be present in the observed data, the new image $\check{m}$ would now be symmetric with respect to the y-axis.

Step 2.4 - Symmetrization: To ensure symmetry, we average the image over both sides of the y-axis, that is according to the scheme $\tilde{m}(x, y) = \frac{1}{2}\left(\check{m}(x, y) + \check{m}(-x, y)\right)$ for all (x, y).

Step 2.5 - Backrotation and -shifting of the image to the estimated symmetry axis: The image $\tilde{m}$ is rotated and shifted such that it is symmetric with respect to the axis

Hypothesis/Nominal level	$S/N = 10$ 5%	10%	20%	$S/N = 25$ 5%	10%	20%
$H_0: m = m_1$	5.5%	10.5%	21.5%	6.5%	11.0%	20.5%
H_1, $\kappa = 0.1$	8.0%	12.0%	23.5%	8.5%	17.0%	27.0%
H_1, $\kappa = 0.2$	10.5%	20.0%	33.0%	54.0%	70.5%	81.5%
H_1, $\kappa = 0.4$	57.0%	71.5%	82.0%	100%	100%	100%

Table 1: Estimated rejection probabilities of the test for axial symmetry from 200 simulations each in case of estimating the axis-shift δ (with α known), under $H_0: m = m_1$, and under an alternative $m = \kappa \cdot m_2 + (1-\kappa) \cdot m_1$, respectively.

Hypothesis/Nominal level	$S/N = 10$ 5%	10%	20%	$S/N = 25$ 5%	10%	20%
$H_0: m = m_1$	0%	2%	7%	6%	12%	20%
H_1, $\kappa = 0.4$	3%	5%	15%	8%	19%	39%
H_1, $\kappa = 1.0$	9%	19%	50%	78%	87%	96%

Table 2: Estimated rejection probabilities of the test for axial symmetry from 100 simulations each in case of estimating both the axis-shift δ and the angle of rotation α, and under an alternative $m = \kappa \cdot m_2 + (1-\kappa) \cdot m_1$, respectively.

with the estimated parameters $\hat{\delta}$ and/or $\hat{\alpha}$, or - if applicable - the known values of shift and rotation, respectively. We call the resulting image $\hat{m}_B$.

3. Bootstrap replications: In the final step of the bootstrap part of the testing procedure we generate bootstrap data from the model $Y^*_{\mathbf{r}} = \Psi\hat{m}_B(\mathbf{x_r}) + \varepsilon^*_{\mathbf{r}}$, where $\varepsilon^*_{\mathbf{r}}$ are drawn independently from the empirical distribution of the residuals $\hat{\varepsilon}_{\mathbf{r}} = Y_{\mathbf{r}} - \Psi\hat{m}(\mathbf{x_r})$. From each set of bootstrap data the image is estimated and the minimal value of the criterion function, that is the test statistics, determined. In our simulations we always use $B = 200$ bootstrap replications. The $\lfloor B(1-\alpha)\rfloor$-th order statistic of all those bootstrap test statistics gives the critical value for the test.

Test decision part of the testing procedure:
In the second part of the testing procedure we use once more the estimate $\hat{m}$ of m described above. From this estimate we determine the test statistics $\hat{L}_n(\hat{\alpha}, \hat{\delta})$, that is the minimal value of the criterion function (9). The test decision by itself is then to reject the null hypothesis of m obeying an axial symmetry to level α, if the test statistics for the original set of data is larger than the $(1-\alpha)$-quantile of the bootstrap distribution of the test statistics.

In the following, we consider the functions

$$m_\kappa(x, y) = \kappa m_2(x, y) + (1-\kappa) m_1(x, y), \;\; \kappa = 0, 0.1, 0.2, 0.4, 1$$

to analyse the sensitivity of our test to small deviations from symmetry. Tables 1 and 2 summarize the simulated levels and power of the test for axial symmetry for the case of an unknown shift parameter δ only (with α known), and for the case that both parameters are unknown. The results demonstrate the substantial additional difficulty of disproving the existence of *any* axis of symmetry if both δ and α are unknown. Slightly acceptable results for the moderate sample size of $n = 50$ only appear for a comparable large deviation from symmetry (i.e. $\kappa = 1$). This effect is to a large part due to the complicated shape of the critical function in this case (cf. Fig. 2) with several local minima. If only the shift parameter is unknown, the test already performes well for small deviations from symmetry (e.g. $\kappa = 0.2$ for a signal-to-noise ratio of $S/N = 25$ or $\kappa = 0.4$ for $S/N = 10$).

Acknowledgements. This work has been supported in part by the Collaborative Research Center "Statistical modeling of nonlinear dynamic processes" (SFB 823 project C4) of the German Research Foundation and the BMBF project INVERS.

References

I.A. Ahmad and Q. Li (1997). *Testing symmetry of an unknown density function by kernel method.* Nonparam. Statist. 7, 279-293.

M. Birke, H. Dette, K. Stahljans (2011). *Testing symmetry of a nonparametric bivariate regression function.* Nonparam. Statist., to appear

N. Bissantz and M. Birke (2009). *Asymptotic normality and confidence intervals for inverse regression models with convolution-type operators.* J. Multivariate Anal. 100, 2364 - 2375.

N. Bissantz, H. Dette, K. Proksch (2010). *Model checks in inverse regression models with convolution-type operators.* Technical report SFB 823.

Bissantz, Holzmann and Pawlak (2009). *Testing for Image Symmetries - with Application to Confocal Microscopy.* IEEE Trans. Information Theory 55, 1841-1855.

A. Cabaña and M. Cabaña (2000). *Tests of symmetry based on transformed empirical processes.* Canad. J. Statist. 28, 829-839.

H. Dette (1999). *A consistent test for the functional form of a regression based on a difference of variance estimators.* Ann. Statist. 27, 1012 - 1040.

H. Dette, S. Kusi-Appiah and N. Neumeyer (2002). *Testing symmetry in nonparametric regression models.* Nonparam. Statist. 14 (5), 477-494.

Eubank, R.L. (1999). *Nonparametric regression and spline smoothing.* Second edition. Statistics: Textbooks and Monographs, 157. Marcel Dekker, Inc., New York

W. Härdle and J.S. Marron (1990). *Semiparametric Comparison of Regression Curves.* Ann. Statist. 18, 63-89

P. de Jong (1987). *A Central Limit Theorem for Generalized Quadratic Forms.* Probab. Th. Rel. Fields 75, 261-277

A Proofs

Theorem 4.

$$n^d h^{2j+2\beta+d/2} a_n^{3d/2} \left(\int_B \left[\hat{m}^{(\mathbf{j})}(x) - m^{(\mathbf{j})}(x) \right]^2 dx - \frac{2^d \sigma^2 \prod_{k=1}^d (2(j_k+\beta_k)+1)^{-1}}{\kappa \pi^d n^d h^{2j+2\beta+d} a_n^{2d}} \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, s^{(\mathbf{j})})$$

for $\mathbf{j} = (j_1, \ldots, j_k)$ *with* $j_1 + \ldots + j_k \leq 2$ *and*

$$s^{(\mathbf{j})} = \frac{2\sigma^4}{\kappa^2 (2\pi)^{2d}} \lim_{n\to\infty} \prod_{l=1}^d a_n h^{4\beta_l + 4j_l + 1} \int\int I_{[-1,1]}(\omega_l) I_{[-1,1]}(\eta_l) |\omega_l \eta_l|^{2j_l + 2\beta_l} \frac{\sin^2(\frac{\omega_l - \eta_l}{a_n})}{(\omega_l - \eta_l)^2} \, d\omega_l \, d\eta_l$$

Proof. In the following we write the L^2-distance as a quadratic form and some bias terms and apply a central limit theorem by de Jong (1987). There is

$$\begin{aligned}
\int_B \left[\hat{m}^{(\mathbf{j})}(x) - m^{(\mathbf{j})}(x) \right]^2 dx &= \int_B \left(\sum_r w_{r,\mathbf{j}}(x) \varepsilon_r \right)^2 dx + 2 \int_B \left(\sum_r w_{r,\mathbf{j}}(x) \varepsilon_r \right) (\mathrm{E}[\hat{m}^{(\mathbf{j})}(x)] - m^{(\mathbf{j})}(x)) dx \\
&\quad + \int_B (\mathrm{E}[\hat{m}^{(\mathbf{j})}(x)] - m^{(\mathbf{j})}(x))^2 dx \\
&= I_1^{(\mathbf{j})} + I_2^{(\mathbf{j})} + I_3^{(\mathbf{j})}.
\end{aligned}$$

Using the definition of $w_{r,\mathbf{j}}(x)$ and Parseval's equality we obtain

$$\begin{aligned}
I_1^{(\mathbf{j})} &= \frac{1}{(2\pi)^d n^{2d} h^{2j+d} a_n^{2d}} \int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} \left| \sum_r e^{i\omega^T x_r/h} \varepsilon_r \right|^2 d\omega \\
&\quad - \frac{1}{(2\pi)^d n^{2d} h^{2j+d} a_n^{2d}} \int_{(B/h)^c} \left| \int_{\mathbb{R}^d} e^{-i\omega^T x} (-i\omega)^j \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} \sum_r e^{i\omega^T x_r/h} \varepsilon_r d\omega \right|^2 dx \\
&= I_{1.1}^{(\mathbf{j})} - I_{1.2}^{(\mathbf{j})}.
\end{aligned}$$

We write

$$I_{1.1}^{(\mathbf{j})} = \sum_u a_{u,u}^{(\mathbf{j})} \tilde{\varepsilon}_u^2 + \tilde{\varepsilon}^T \tilde{A}^{(\mathbf{j})} \tilde{\varepsilon} = I_{1.1.1}^{(\mathbf{j})} + I_{1.1.2}^{(\mathbf{j})}$$

with

$$\begin{aligned}
a_{u,v}^{(\mathbf{j})} &= \frac{1}{(2\pi)^d n^{2d} h^{2j+d} a_n^{2d}} \int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} e^{i\omega^T \tilde{x}_u/h} e^{-i\omega^T \tilde{x}_v/h} d\omega \\
\tilde{A}^{(\mathbf{j})} &= (\tilde{a}_{u,v}^{(\mathbf{j})})_{1 \leq u,v \leq (2n+1)^d}, \ \tilde{a}_{u,v}^{(\mathbf{j})} = a_{u,v}^{(\mathbf{j})} \text{ for } u \neq v, \ \tilde{a}_{u,u}^{(\mathbf{j})} = 0 \\
\tilde{x}_1 &= x_{(-n,\ldots,-n)}, \ldots, \tilde{x}_{(2n+1)^d} = x_{(n,\ldots,n)} \\
\tilde{\varepsilon}^T &= (\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{(2n+1)^d}) = (\varepsilon_{(-n,\ldots,-n)}, \ldots, \varepsilon_{(n,\ldots,n)}) \in \mathbb{R}^{(2n+1)^d}.
\end{aligned}$$

For $I_{1.1.1}^{(\mathbf{j})}$ we obtain

$$\begin{aligned}
\mathrm{E}[I_{1.1.1}^{(\mathbf{j})}] &= \sigma^2 \sum_u a_{u,u}^{(\mathbf{j})} = \frac{\sigma^2}{(2\pi)^d n^{2d} h^{2j+d} a_n^{2d}} \sum_r \int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} d\omega \\
&= \frac{\sigma^2 (2n+1)^d}{(2\pi)^d n^{2d} h^{2j+d} a_n^{2d}} \int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} d\omega \\
&\sim \frac{\sigma^2 (2n+1)^d}{\kappa^2 (2\pi)^d n^{2d} h^{2j+2\beta+d} a_n^{2d}} \int_{\mathbb{R}^d} |\omega|^{2j+2\beta} I_{[-1,1]^d}(\omega) d\omega \\
&= \frac{\sigma^2 (2n+1)^d}{\kappa^2 \pi^d n^{2d} h^{2j+2\beta+d} a_n^{2d}} \prod_{k=1}^d \frac{1}{2(j_k+\beta_k)+1} = O\left(\frac{1}{n^d h^{2j+2\beta+d} a_n^{2d}}\right) \\
\mathrm{Var}(I_{1.1.1}) &= \sum_u \left(a_{u,u}^{(\mathbf{j})}\right)^2 \mu_4(\varepsilon) = \frac{\mu_4(\varepsilon)}{(2\pi)^{2d} n^{4d} h^{4j+2d} a_n^{4d}} \sum_r \left(\int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} d\omega\right)^2 \\
&= \frac{\mu_4(\varepsilon)(2n+1)^d}{(2\pi)^{2d} n^{4d} h^{4j+2d} a_n^{4d}} \left(\int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2}\right)^2 d\omega \\
&\sim \frac{\mu_4(\varepsilon)(2n+1)^d}{\kappa^4 (2\pi)^{2d} n^{4d} h^{4j+4\beta+2d} a_n^{4d}} \left(\int_{\mathbb{R}^d} |\omega|^{2j+2\beta} |I_{[-1,1]^d}(\omega)|^2 d\omega\right)^2 = O\left(\frac{1}{n^{3d} h^{4j+4\beta+2d} a_n^{4d}}\right) \\
&= o\left(\frac{1}{n^{2d} h^{4j+4\beta+d} a_n^{3d}}\right).
\end{aligned}$$

We now check the assumptions of Theorem 5.2 in de Jong (1987) for $I_{1.1.2}$. First of all we calculate the variance

$$\begin{aligned}
\sigma(n)^2 &= \mathrm{Var}(\tilde{\varepsilon}^T \tilde{A}^{(\mathbf{j})} \tilde{\varepsilon}) = 2\sigma^4 \operatorname{tr}(\tilde{A}^{(\mathbf{j})})^2 = 2\sigma^4 \sum_{u \neq v} (a_{u,v}^{(\mathbf{j})})^2 \\
&= \frac{2\sigma^4}{(2\pi)^{2d} n^{4d} h^{4j+2d} a_n^{4d}} \sum_{r \neq s} \left(\int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} e^{i\omega^T x_r/h} e^{-i\omega^T x_s/h} d\omega\right)^2 \\
&\sim \frac{2\sigma^4}{(2\pi)^{2d} n^{2d} h^{4j} a_n^{2d}} \int_{I_n/h} \int_{I_n/h} \left(\int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} e^{i\omega^T y/h} e^{-i\omega^T z/h} d\omega\right)^2 dy dz \\
&= \frac{2\sigma^4}{(2\pi)^{2d} n^{2d} h^{4j} a_n^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\omega|^{2j} |\eta|^{2j} \frac{I_{[-1,1]^d}(\omega) I_{[-1,1]^d}(\eta)}{|\Phi_\psi(\omega/h)|^2 |\Phi_\psi(\eta/h)|^2} \left| \int_{I_n/h} e^{i(\omega-\eta)^T u} du \right|^2 d\omega d\eta \\
&= \frac{2\sigma^4}{(2\pi)^{2d} n^{2d} h^{4j} a_n^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\omega|^{2j} |\eta|^{2j} \frac{I_{[-1,1]^d}(\omega) I_{[-1,1]^d}(\eta)}{|\Phi_\psi(\omega/h)|^2 |\Phi_\psi(\eta/h)|^2} \prod_{l=1}^d \frac{|e^{i(\omega_l-\eta_l)/(ha_n)} - e^{-i(\omega_l-\eta_l)/(ha_n)}|^2}{|\omega_l - \eta_l|^2} d\omega d\eta \\
&= \frac{2\sigma^4}{\kappa^4 (2\pi)^{2d} n^{2d} h^{4j+4\beta} a_n^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} I_{[-1,1]^d}(\omega) I_{[-1,1]^d}(\eta) \prod_{l=1}^d |\omega_l|^{2j_l+2\beta_l} |\eta_l|^{2j_l+2\beta_l} \frac{|\sin(\frac{\omega_l-\eta_l}{ha_n})|^2}{|\omega_l-\eta_l|^2} d\omega d\eta \\
&= \frac{2\sigma^4}{\kappa^4 (2\pi)^{2d} n^{2d} h^{4j+4\beta} a_n^{2d}} \prod_{l=1}^d \int_{\mathbb{R}} \int_{\mathbb{R}} I_{[-1,1]}(\omega_l) I_{[-1,1]}(\eta_l) |\omega_l \eta_l|^{2j_l+2\beta_l} \frac{\sin^2(\frac{\omega_l-\eta_l}{ha_n})}{(\omega_l-\eta_l)^2} \, d\omega_l \, d\eta_l
\end{aligned}$$

$$= \frac{2\sigma^4 h^{\sum_{l=1}^d (4j_l+4\beta_l+2)}}{\kappa^4(2\pi)^{2d} n^{2d} h^{4j+\beta+2d} a_n^{2d}} \prod_{l=1}^d \int_{-1/h}^{1/h}\int_{-1/h}^{1/h} |\omega_l \eta_l|^{2j_l+2\beta_l} \frac{\sin^2(\frac{\omega_l-\eta_l}{a_n})}{(\omega_l-\eta_l)^2}\, d\omega_l\, d\eta_l$$
$$= \frac{2\prod_{l=1}^d C_l \sigma^4}{\kappa^4(2\pi)^{2d} n^{2d} h^{4j+4\beta+d} a_n^{3d}}$$

using that

$$\lim_{n\to\infty} a_n h^{4\beta_l+4j_l+1} \int_{-1/h}^{1/h}\int_{-1/h}^{1/h} |\omega_l\eta_l|^{2j_l+2\beta_l} \frac{\sin^2(\frac{\omega_l-\eta_l}{a_n})}{(\omega_l-\eta_l)^2}\, d\omega_l\, d\eta_l = C_l,$$

following from the integrability of sinc^2 by some slightly tedious algebra. In the following, we check the assumptions (1) - (3) of Theorem 5.2 in de Jong (1987) to show the asymptotic normality of $I_{1.1.2}^{(\mathbf{j})}$.

(1) We have uniformly over all $\mathbf{s} \in \{-n, \dots, n\}^d$

$$\sum_{\mathbf{r}\in\{-n,\dots,n\}^d} |a_{\mathbf{r},\mathbf{s}}^{(\mathbf{j})}|^2$$
$$= \frac{1}{(2\pi)^{4d} n^{4d} h^{4j+2d} a_n^{4d}} \sum_{\mathbf{r}\in\{-n,\dots,n\}^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\omega\eta|^{2j} \frac{I_{[-1,1]^d}(\omega) I_{[-1,1]^d}(\eta)}{|\Phi_\psi(\omega/h)|^2 |\Phi_\psi(\eta/h)|^2} e^{i(\omega-\eta)^T x_{\mathbf{r}}/h} e^{-i(\omega-\eta)^T x_{\mathbf{s}}/h} d\omega d\eta$$
$$\sim \frac{1}{(2\pi)^{4d} n^{3d} h^{4j+d} a_n^{3d}} \int_{A_n}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\omega\eta|^{2j} \frac{I_{[-1,1]^d}(\omega) I_{[-1,1]^d}(\eta)}{|\Phi_\psi(\omega/h)|^2 |\Phi_\psi(\eta/h)|^2} e^{i(\omega-\eta)^T u} e^{-i(\omega-\eta)^T x_{\mathbf{s}}/h} d\omega d\eta du$$
$$= \frac{1}{(2\pi)^{4d} n^{3d} h^{4j+d} a_n^{3d}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\omega\eta|^{2j} \frac{I_{[-1,1]^d}(\omega) I_{[-1,1]^d}(\eta)}{|\Phi_\psi(\omega/h)|^2 |\Phi_\psi(\eta/h)|^2} \left(\prod_{\nu=1}^d \frac{\sin\left(\frac{\omega_\nu-\eta_\nu}{ha_n}\right)}{\omega_\nu-\eta_\nu}\right) e^{-i(\omega-\eta)^T x_{\mathbf{s}}/h} d\omega d\eta$$
$$= \frac{1}{(2\pi)^{4d} n^{3d} h^{4j+4\beta+d} a_n^{3d}} \prod_{\nu=1}^d \int_{\mathbb{R}}\int_{\mathbb{R}} |\omega_\nu\eta_\nu|^{2j} I_{[-1,1]^d}(\omega_\nu) I_{[-1,1]^d}(\eta_\nu) \left(\frac{\sin\left(\frac{\omega_\nu-\eta_\nu}{ha_n}\right)}{\omega_\nu-\eta_\nu}\right) e^{-i(\omega_\nu-\eta_\nu)^T x_{\mathbf{s},\nu}/h} d\omega_\nu d\eta_\nu.$$

Since $|\sin((\omega_\nu-\eta_\nu)/(ha_n))/(\omega_\nu-\eta_\nu)| \le (ha_n)^{-1}$ we obtain

$$\sum_{\mathbf{r}\in\{-n,\dots,n\}^d} |a_{\mathbf{r},\mathbf{s}}^{(\mathbf{j})}|^2 \;=\; O\left(\frac{1}{n^{3d} h^{4j+4\beta+2d} a_n^{4d}}\right)$$

and therefore with $\kappa(n) = (\log n)^{1/4}$

$$\frac{\kappa(n)}{\sigma(n)^2} \sum_{\mathbf{r}\in\{-n,\dots,n\}^d} |a_{\mathbf{r},\mathbf{s}}|^2 \;=\; O\left(\frac{(\log n)^{1/4}}{n^d h^d a_n^d}\right) = o(1)$$

(2) Since $\kappa(n) \to \infty$ and $\varepsilon_{\mathbf{r}}$ are independent identically distributed with $\mathrm{E}[\varepsilon_{\mathbf{r}}^2] = \sigma^2 < \infty$, it immediately follows that

$$\mathrm{E}[\varepsilon_{\mathbf{r}}^2 I\{|\varepsilon_{\mathbf{r}}| > \kappa(n)\}] = o(1).$$

(3) For estimating the eigenvalues $\mu_{\mathbf{r}}$ of $\tilde{A}^{(\mathbf{j})}$ we use Gerschgorin's Theorem and obtain uniformly over all $\mathbf{s} \in \{-n, \ldots, n\}^d$

$$
\begin{aligned}
\mu_{\mathbf{s}} &\leq \sum_{\mathbf{r}\in\{-n,\ldots,n\}^d} |a^{(\mathbf{j})}_{\mathbf{r},\mathbf{s}}| \\
&\sim \frac{1}{(2\pi)^{2d} n^d h^{2j} a_n^d} \int_{A_n} \left| \int_{\mathbb{R}^d} |\omega|^{2j} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|^2} e^{i\omega^T u} e^{-i\omega^T x_{\mathbf{s}}/h} d\omega \right| du \\
&= \frac{1}{(2\pi)^{2d} n^d h^{2j+2\beta+d} a_n^{2d}} \prod_{\nu=1}^{d} \int_{-1/(ha_n)}^{1/(ha_n)} \int_{\mathbb{R}} |\omega_\nu|^{2j_\nu+2\beta_\nu} I_{[-1,1]^d}(\omega_\nu) e^{i\omega_\nu u_\nu} e^{-i\omega_\nu x_{\mathbf{s},\nu}/h} d\omega_\nu du_\nu
\end{aligned}
$$

It now follows by similar but tedious calculations as above, that this term is of order $O(\log n / n^d a_n^d h^{2j+2\beta})$ and

$$
\frac{1}{\sigma(n)^2} \max_{\mathbf{s}\in\{-n,\ldots,n\}^d} \mu_{\mathbf{s}}^2 \; = \; O\left(h a_n \log n\right) = o(1).
$$

It now remains to discuss the remainder terms For $I_{1.2}$ we get

$$
I_{1.2} \; = \; o_P(I_{1.1})
$$

since it consists of the tails of the integral in $I_{1.1}$, before Parseval's equality was used, and the upper respective lower bound of the integral tails asymptotically diverge to $\pm\infty$. This means, that $I_{1.2}$ is asymptotically negligible.

Since the bias of $\hat{m}^{(\mathbf{j})}$ is uniformly of order $o(h^{s-j-1})$ on B (see e.g. Bissantz and Birke, 2009) we have with condition (3)

$$
I_3 \; = \; O(h^{2s-2j-2}) = o\left(\frac{1}{n^d h^{2\beta+2j+d/2} a_n^{3d/2}}\right)
$$

and by applying the Cauchy-Schwarz inequality also

$$
I_2 = O\left(\frac{1}{n^{d/2} h^{\beta+j+d/4} a_n^{3d/4}}\right) o(h^{s-j-1}) = o\left(\frac{1}{n^d h^{2\beta+2j+d/2} a_n^{3d/2}}\right).
$$

A.1 Proof of Theorem 1.

Since $L(\theta)$ is locally convex near ϑ, for every $\varepsilon > 0$ exists a constant $K_\varepsilon > 0$ with

$$
\mathrm{P}(|\hat{\vartheta}_n - \vartheta_n| > \varepsilon) \; \leq \; \mathrm{P}(L(\hat{\vartheta}_n) - L(\vartheta) > K_\varepsilon) \leq \mathrm{P}(|\hat{L}(\hat{\vartheta}_n) - L(\hat{\vartheta}_n)| > K_\varepsilon/2) + \mathrm{P}(|\hat{L}(\vartheta) - L(\vartheta)| > K_\varepsilon/2)
$$

since $\hat{\vartheta}_n$ minimizes $\hat{L}(\theta)$ and the assertion follows if we show that $\hat{L}(\theta) - L(\theta)$ stochastically converges to 0 uniformly in θ. To this end note that

$$
\begin{aligned}
|\hat{L}(\theta) - L(\theta)| &= \left| \int_A (\hat{m}(T_\theta x) - \hat{m}(S_\theta x))^2 dx - \int_A (m(T_\theta x) - m(S_\theta x))^2 dx \right| \\
&\leq C \left(\int_A (\hat{m}(T_\theta x) - m(T_\theta x))^2 dx + \int_A (\hat{m}(S_\theta x) - m(S_\theta x))^2 dx \right) \\
&\leq 2C \int_{A_\theta} (\hat{m}(z) - m(z))^2 dz \leq 2C \int_B (\hat{m}(z) - m(z))^2 dz.
\end{aligned}
$$

Therefore we have for any $\tilde{\delta} > 0$ and $\delta = \tilde{\delta}/(2C)$

$$\mathrm{P}(\sup_{\theta} |\hat{L}(\theta) - L(\theta)| > \tilde{\delta}) \ \leq \ \mathrm{P}\left(\int_B (\hat{m}(z) - m(z))^2 dz > \delta \right).$$

But the right probability converges to 0 because of Theorem 4. □

A.2 Proof of Theorem 2.

Note, that $\hat{l}_n(\hat{\vartheta}) = 0$. With this and a first order Taylor expansion of $\hat{l}_n$ in ϑ we write

$$-\hat{h}(\xi_n)(\hat{\vartheta} - \vartheta) = \hat{l}_n(\vartheta) \tag{10}$$

for some ξ_n between $\hat{\vartheta}$ and ϑ Theorem 2 now follows after we have shown the following two Lemmata

Lemma 1. *Under the assumptions of Theorem 2 we have*

$$\sqrt{n^d a_n^d}\,\hat{l}_n(\vartheta) \stackrel{\mathcal{D}}{\to} \mathcal{N}(0, \Sigma(\vartheta))$$

with $\Sigma(\theta)$ *and* $\sigma_\theta(u)$ *defined as in Theorem 2.*

Lemma 2. *Under the assumptions of Theorem 2 we have*

$$\hat{h}(\xi_n) \stackrel{P}{\to} h(\vartheta).$$

Proof of Lemma 1. We write

$$\Delta_{m,\theta}(\mathbf{x}) \ = \ \left(\text{grad } m(T_\theta \mathbf{x}) \frac{\partial}{\partial \theta} T_\theta \mathbf{x} - \text{grad } m(S_\theta \mathbf{x}) \frac{\partial}{\partial \theta} S_\theta \mathbf{x} \right)^T.$$

and

$$\begin{aligned}
\hat{l}_n(\vartheta) \ &= \ 2 \int_A \left[\hat{m}(T_\vartheta \mathbf{x}) - \hat{m}(S_\vartheta \mathbf{x}) \right] \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} + R_{n,1} \\
&= \sum_{\mathbf{r} \in \{-n,\ldots,n\}^d} \left(2 \int_A \left(w_{\mathbf{r}}(T_\vartheta \mathbf{x}) - w_{\mathbf{r}}(S_\vartheta \mathbf{x}) \right) \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \right) Z_{\mathbf{r}} + R_{n,1} \\
&= \sum_{\mathbf{r} \in \{-\mathbf{n},\ldots,\mathbf{n}\}^{\mathbf{d}}} v_{\mathbf{r}}(\vartheta) \varepsilon_{\mathbf{r}} + R_{n,1} + R_{n,2} = \tilde{l}_n(\vartheta) + 2R_{n,1} + 2R_{n,2}.
\end{aligned}$$

with

$$\begin{aligned}
v_{\mathbf{r}}(\vartheta) \ &= \ 2 \int_A \left(w_{\mathbf{r}}(T_\vartheta \mathbf{x}) - w_{\mathbf{r}}(S_\vartheta \mathbf{x}) \right) \left(\text{grad } m(T_\vartheta \mathbf{x}) \frac{\partial}{\partial \theta} T_\theta \Big|_{\theta=\vartheta} \mathbf{x} - \text{grad } m(S_\vartheta \mathbf{x}) \frac{\partial}{\partial \theta} S_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right)^T d\mathbf{x} \in \mathbb{R}^d \\
R_{n,1} \ &= \ \int_A \left[\hat{m}(T_\vartheta \mathbf{x}) - \hat{m}(S_\vartheta \mathbf{x}) \right] \left(\text{grad } (\hat{m} - m)(T_\vartheta \mathbf{x}) \frac{\partial}{\partial \theta} T_\theta \Big|_{\theta=\vartheta} \mathbf{x} - \text{grad } (\hat{m} - m)(S_\vartheta \mathbf{x}) \frac{\partial}{\partial \theta} S_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right) d\mathbf{x} \\
R_{n,2} \ &= \ \int_A \left(\mathrm{E}[\hat{m}(T_\vartheta \mathbf{x})] - \mathrm{E}[\hat{m}(S_\vartheta \mathbf{x})] \right) \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x}.
\end{aligned}$$

This means, that $\hat{l}_n(\vartheta)$ consists of a sum of weighted independently distributed random variables for which we determine the asymptotic distribution by using a central limit theorem (see e.g. Eubank, 1999) and remainders $R_{n,1}$ and $R_{n,2}$ for which we show that they are asymptotically negligible. We will first consider the asymptotic distribution of $\tilde{l}_n$. To this end we have to check the condition

$$\frac{\max_{\mathbf{r}\in\{-\mathbf{n},\ldots,\mathbf{n}\}^{\mathbf{d}}} |c^T v_{\mathbf{r}}(\vartheta)|}{\left(\sum_{\mathbf{r}\in\{-\mathbf{n},\ldots,\mathbf{n}\}^{\mathbf{d}}} c^T v_{\mathbf{r}}(\vartheta) v_{\mathbf{r}}^T c(\vartheta)\right)^{1/2}} \;=\; o(1) \tag{11}$$

for every $c \in \mathbb{R}^2$. Note that from (4) we have

$$\begin{aligned}
\text{grad } m(S_\vartheta \mathbf{x}) \;&=\; \text{grad } m(T_\vartheta \mathbf{x}) M_\vartheta N_\vartheta^{-1} \\
\text{grad } m(T_\vartheta \mathbf{x}) \;&=\; \text{grad } m(S_\vartheta \mathbf{x}) N_\vartheta M_\vartheta^{-1}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
|c^T v_{\mathbf{r}}(\vartheta)| \;&=\; \left| 2\int_A \frac{1}{(nha_n)^d} \int_{\mathbb{R}^d} (e^{-i\omega(T_\vartheta \mathbf{x}-\mathbf{x}_{\mathbf{r}})/h} - e^{-i\omega(S_\vartheta \mathbf{x}-\mathbf{x}_{\mathbf{r}})/h}) \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega c^T \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \right| \\
&\le\; \frac{2}{(na_n)^d} \int\int_{\mathbb{R}^2} |e^{-i\omega T_\vartheta \mathbf{u}} - e^{-i\omega S_\vartheta \mathbf{u}}|\, |e^{i\omega \mathbf{x}_{\mathbf{r}}/h}| \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|} d\omega \left| c^T \Delta_{m,\vartheta}(h\mathbf{u}) \right| d\mathbf{u} \\
&\le\; \frac{4}{(na_n)^d} \int_{\mathbb{R}^2} \frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|} d\omega \int_A \left| c^T \Delta_{m,\vartheta}(h\mathbf{u}) \right| d\mathbf{u} \\
&=\; O\left(\frac{1}{n^d h^\beta a_n^d} \right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\mathbf{r}\in\{-\mathbf{n},\ldots,\mathbf{n}\}^{\mathbf{d}}} (c^T v_{\mathbf{r}}(\vartheta))^2 \\
&=\; \frac{4}{(2\pi nha_n)^{2d}} \sum_{\mathbf{r}} \left(\int_A \int_{\mathbb{R}^d} (e^{-i\omega^T(T_\vartheta \mathbf{x}-\mathbf{x}_{\mathbf{r}})/h} - e^{-i\omega^T(S_\vartheta \mathbf{x}-\mathbf{x}_{\mathbf{r}})/h}) \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega c^T \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \right)^2 \\
&=\; \frac{4}{(na_n)^d h^{2d}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} c^T \left(\int_A e^{-i\omega^T(T_\vartheta \mathbf{x}-\mathbf{u})/h} \Delta_{m,\vartheta}(\mathbf{x}) (\text{grad } m(T_\vartheta \mathbf{x}))^T d\mathbf{x} \right.\right. \\
&\quad \left.\left. - \int_A e^{-i\omega^T(S_\vartheta \mathbf{x}-\mathbf{u})/h} \left(N_\vartheta M_\vartheta^{-1} \frac{\partial}{\partial\theta} T_\theta \Big|_{\theta=\vartheta} \mathbf{x} - \frac{\partial}{\partial\theta} S_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right)^T (\text{grad } m(S_\vartheta \mathbf{x}))^T d\mathbf{x} \right) d\omega \right)^2 d\mathbf{u}(1+o(1)) \\
&=\; \frac{4}{(na_n)^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{I_{[-1,1]^d}(h\omega)}{\Phi_\psi(\omega)} c^T \left(\int_A e^{-i\omega^T(T_\vartheta \mathbf{x}-\mathbf{u})} \Delta_{m,\vartheta}(\mathbf{x}) (\text{grad } m(T_\vartheta \mathbf{x}))^T d\mathbf{x} \right.\right. \\
&\quad \left.\left. - \int_A e^{-i\omega^T(S_\vartheta \mathbf{x}-\mathbf{u})} \left(N_\vartheta M_\vartheta^{-1} \frac{\partial}{\partial\theta} T_\theta \Big|_{\theta=\vartheta} \mathbf{x} - \frac{\partial}{\partial\theta} S_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right)^T (\text{grad } m(S_\vartheta \mathbf{x}))^T d\mathbf{x} \right) d\omega \right)^2 d\mathbf{u}(1+o(1))
\end{aligned}$$

With assumption 2 the integral on the r.h.s. of the equation exists, and we have

$$\sum_{\mathbf{r}\in\{-\mathbf{n},\ldots,\mathbf{n}\}^{\mathbf{d}}} (c^T v_{\mathbf{r}}(\vartheta))^2 = \frac{4C_a}{(na_n)^d}$$

with

$$C_a = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{I_{[-1,1]^d}(h\omega)}{\Phi_\psi(\omega)} c^T \left(\int_A e^{-i\omega^T(T_\vartheta \mathbf{x}-\mathbf{u})} \Delta_{m,\vartheta}(\mathbf{x}) (\text{grad } m(T_\vartheta \mathbf{x}))^T d\mathbf{x} \right.\right.$$
$$\left.\left. - \int_A e^{-i\omega^T(S_\vartheta \mathbf{x}-\mathbf{u})} \left(N_\vartheta M_\vartheta^{-1} \frac{\partial}{\partial\theta} T_\theta \Big|_{\theta=\vartheta} \mathbf{x} - \frac{\partial}{\partial\theta} S_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right)^T (\text{grad } m(S_\vartheta \mathbf{x}))^T d\mathbf{x} \right) d\omega \right)^2 d\mathbf{u}.$$

This yields by

$$\frac{\max_{\mathbf{r}\in\{-\mathbf{n},\ldots,\mathbf{n}\}^{\mathbf{d}}} |c^T v_{\mathbf{r}}(\vartheta)|}{(\sum_{\mathbf{r}}^n c^T v_{\mathbf{r}}(\vartheta) v_{\mathbf{r}}^T c(\vartheta))^{1/2}} = O\left(\frac{1}{(na_n)^{d/2} h^\beta} \right) = o(1)$$

and the Cramér-Wold device the asymptotic normality of $\tilde{l}_n(\vartheta)$. We will now discuss the remainder terms. Using the Cauchy-Schwarz inequality we get

$$R_{n,1} \le \left(\int_A [\hat{m}(T_\vartheta \mathbf{x}) - \hat{m}(S_\vartheta \mathbf{x})]^2 d\mathbf{x} \right)^{1/2}$$
$$\times \left(\int_A \left(\left(\frac{\partial}{\partial\theta} T_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right)^T (\text{grad } (\hat{m}-m)(T_\vartheta \mathbf{x}))^T - \left(\frac{\partial}{\partial\theta} S_\theta \Big|_{\theta=\vartheta} \mathbf{x} \right)^T (\text{grad } (\hat{m}-m)(S_\vartheta \mathbf{x}))^T \right)^2 d\mathbf{x} \right)^{1/2}.$$

We apply Theorem 4 and obtain $R_{n,1} = O_P(1/n^d a_n^d h^{2\beta+d}) = o_p(1/n^{d/4} a_n^{d/4} h^{\beta/2})$ since $n^{d/2} a_n^{d/2} h^{\beta+d} \to \infty$ by assumption 3. Now it remains to estimate

$$\begin{aligned}
R_{n,2} &= \frac{1}{(2\pi n a_n h)^d} \sum_{\mathbf{r}} \int_A \int_{\mathbb{R}^d} (e^{-i\omega^T(T_\vartheta \mathbf{x}-\mathbf{x_r})/h} - e^{-i\omega^T(S_\vartheta \mathbf{x}-\mathbf{x_r})/h}) \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} g(\mathbf{x_r}) \\
&= \frac{1}{(2\pi h)^d} \int_{[-1/a_n,1/a_n]^d} \int_A \int_{\mathbb{R}^d} (e^{-i\omega^T T_\vartheta \mathbf{x}/h} - e^{-i\omega^T S_\vartheta \mathbf{x}/h}) e^{i\omega^T \mathbf{u}/h} \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} g(\mathbf{u}) d\mathbf{u} \\
&\quad + O\left(\frac{1}{n^d a_n^d} \right) \int_A \int_{\mathbb{R}^d} (e^{-i\omega^T T_\vartheta \mathbf{x}/h} - e^{-i\omega^T S_\vartheta \mathbf{x}/h}) e^{i\omega^T \mathbf{u}/h} \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \\
&= \frac{1}{(2\pi h)^2} \int_A \int_{\mathbb{R}^d} (e^{-i\omega^T T_\vartheta \mathbf{x}/h} - e^{-i\omega^T S_\vartheta \mathbf{x}/h}) \Phi_m(\omega/h) I_{[-1,1]^d}(\omega) d\omega \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \\
&\quad - \frac{1}{(2\pi h)^2} \int_A \int_{\mathbb{R}^d} (e^{-i\omega^T T_\vartheta \mathbf{x}/h} - e^{-i\omega^T S_\vartheta \mathbf{x}/h}) \left(\int_{([-1/a_n,1/a_n]^d)^c} e^{i\omega^T \mathbf{u}/h} g(\mathbf{u}) d\mathbf{u} \right) \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \\
&\quad + O\left(\frac{1}{n^d a_n^d} \right) \int_A \int_{\mathbb{R}^d} (e^{-i\omega^T T_\vartheta \mathbf{x}/h} - e^{-i\omega^T S_\vartheta \mathbf{x}/h}) e^{i\omega^T \mathbf{u}/h} \frac{I_{[-1,1]^d}(\omega)}{\Phi_\psi(\omega/h)} d\omega \Delta_{m,\vartheta}(\mathbf{x}) d\mathbf{x} \\
&= R_{n,2}^{[1]} + R_{n,2}^{[2]} + R_{n,2}^{[3]} O\left(\frac{1}{n^d a_n^d h^d} \right).
\end{aligned}$$

There is

$$R_{n,2}^{[1]} = R_{n,2}^{[1.1]} - R_{n,2}^{[1.2]}$$

with

$$
\begin{aligned}
R_{n,2}^{[1.1]} &= \frac{1}{(2\pi h)^d}\int_{A_\vartheta}\int_{\mathbb{R}^d} e^{-i\omega^T\mathbf{y}/h}\Phi_m\left(\frac{\omega}{h}\right) I_{[-1,1]^d}(\omega)d\omega\Delta_{m,\vartheta}(T_\vartheta^{-1}\mathbf{y})d\mathbf{y}\\
R_{n,2}^{[1.2]} &= \frac{1}{(2\pi h)^d}\int_{A_\vartheta}\int_{\mathbb{R}^d} e^{-i\omega^T(S_\vartheta T_\vartheta^{-1}\mathbf{y})/h}\Phi_m\left(\frac{\omega}{h}\right) I_{[-1,1]^d}(\omega)d\omega\Delta_{m,\vartheta}(T_\vartheta^{-1}\mathbf{y})d\mathbf{y}.
\end{aligned}
$$

Since $m(\mathbf{z}) = m(T_\vartheta S_\vartheta^{-1}\mathbf{z})$ it is easy to show that $\Phi_m = \Phi_{m(T_\vartheta S_\vartheta^{-1}\cdot)}$ and

$$
\begin{aligned}
\Phi_{m(T_\vartheta S_\vartheta^{-1}\cdot)}(\omega/h) &= \int_{\mathbb{R}^2} e^{i\omega^T\mathbf{v}/h}m(T_\vartheta S_\vartheta^{-1}\mathbf{v})d\mathbf{v}\\
&= \int_{\mathbb{R}^d} e^{i\omega^T(S_\vartheta T_\vartheta^{-1}\mathbf{u}/h}m(\mathbf{u})d\mathbf{u} = e^{-i\omega^T b_\vartheta(I-N_\vartheta M_\vartheta^{-1})/h}\int_{\mathbb{R}^d} e^{i\omega^T N_\vartheta M_\vartheta^{-1}\mathbf{u}/h}m(\mathbf{u})d\mathbf{u}\\
&= e^{-i\omega^T b_\vartheta(I-N_\vartheta M_\vartheta^{-1})/h}\Phi_m\left((N_\vartheta M_\vartheta^{-1})^T\omega)/h\right).
\end{aligned}
$$

Furthermore

$$
e^{-i\omega^T(S_\vartheta T_\vartheta^{-1}\mathbf{y})/h} = e^{i\omega^T b_\vartheta(I-N_\vartheta M_\vartheta^{-1})/h}e^{-i\omega^T N_\vartheta M_\vartheta^{-1}\mathbf{y}/h}.
$$

Substituting this in $R_{n,2}^{[1.2]}$ we obtain

$$
\begin{aligned}
R_{n,2}^{[1.2]} &= \frac{1}{(2\pi h)^d}\int_{A_\vartheta}\int_{\mathbb{R}^d} e^{-i\omega^T(S_\vartheta T_\vartheta^{-1}\mathbf{y})/h}\Phi_{m(T_\vartheta S_\vartheta^{-1}\cdot)}\left(\frac{\omega}{h}\right) I_{[-1,1]^d}(\omega)d\omega\Delta_{m,\vartheta}(T_\vartheta^{-1}\mathbf{y})d\mathbf{y}\\
&= \frac{1}{(2\pi h)^d}\int_{A_\vartheta}\int_{\mathbb{R}^d} e^{-i((N_\vartheta M_\vartheta^{-1})^T\omega)^T\mathbf{y}/h}\Phi_m\left(\frac{(N_\vartheta M_\vartheta^{-1})^T\omega}{h}\right) I_{[-1,1]^d}(\omega)d\omega\Delta_{m,\vartheta}(T_\vartheta^{-1}\mathbf{y})d\mathbf{y}\\
&= R_{n,2}^{[1.1]}
\end{aligned}
$$

with $(N_\vartheta M_\vartheta^{-1})^T\omega = \eta$. Therefore $R_{n,2}^{[1]} = 0$.

$$
\begin{aligned}
||R_{n,2}^{[2]}|| &\le \frac{1}{2\pi^d h^{d+\beta}}\int_{([-1/a_n,1/a_n]^d)^c}\frac{1}{||\mathbf{u}||^r}||\mathbf{u}||^r|g(\mathbf{u})|d\mathbf{u}\int_{\mathbb{R}^d}||\omega||^\beta\frac{1}{||\omega/h||^\beta}\frac{|I_{[-1,1]^d}(\omega)|}{|\Phi_\psi(\omega/h)|}d\omega\\
&\quad\times\int_A\left\|\text{grad } m(T_\vartheta\mathbf{x})\frac{\partial}{\partial\theta}T_\theta\Big|_{\theta=\vartheta}\mathbf{x} - \text{grad } m(S_\vartheta\mathbf{x})\frac{\partial}{\partial\theta}S_\theta\Big|_{\theta=\vartheta}\mathbf{x}\right\|d\mathbf{x}\\
&\le O\left(\frac{a_n^r}{h^{d+\beta}}\right)\int_{\mathbb{R}^d}||\mathbf{u}||^r|g(\mathbf{u})|d\mathbf{u}\int_{\mathbb{R}^d}||\omega||^\beta I_{[-1,1]^d}(\omega)\\
&\quad\times\int_A\left\|\text{grad } m(T_\vartheta\mathbf{x})\frac{\partial}{\partial\theta}T_\theta\Big|_{\theta=\vartheta}\mathbf{x} - \text{grad } m(S_\vartheta\mathbf{x})\frac{\partial}{\partial\theta}S_\theta\Big|_{\theta=\vartheta}\mathbf{x}\right\|d\mathbf{x}\\
&= O\left(\frac{a_n^r}{h^{d+\beta}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|R_{n,2}^{[3]}| &\le \frac{2}{h^\beta}\int_{\mathbb{R}^d}||\omega||^\beta\frac{1}{||\omega/h||^\beta}\frac{I_{[-1,1]^d}(\omega)}{|\Phi_\psi(\omega/h)|}d\omega\\
&\quad\times\int_A\left\|\text{grad } m(T_\vartheta\mathbf{x})\frac{\partial}{\partial\theta}T_\theta\Big|_{\theta=\vartheta}\mathbf{x} - \text{grad } m(S_\vartheta\mathbf{x})\frac{\partial}{\partial\theta}S_\theta\Big|_{\theta=\vartheta}\mathbf{x}\right\|d\mathbf{x}\\
&= O\left(\frac{1}{h^\beta}\right).
\end{aligned}
$$

Altogether this yields with the assumptions $n^{d/2}a_n^{r+d/2}/h^d \to 0$ and $n^{d/2}a_n^{d/2}h^d \to \infty$

$$|R_{n,2}| = 0 + O\left(\frac{a_n^r}{h^{d+\beta}}\right) + O\left(\frac{1}{h^\beta}\right) O\left(\frac{1}{n^d a_n^d}\right) = o\left(\frac{1}{n^{d/2}a_n^{d/2}h^\beta}\right).$$

Proof of Lemma 2. First of all note that $||\xi_n - \vartheta|| \leq ||\hat{\vartheta}_n - \vartheta||$ and therefore $\xi_n \stackrel{P}{\to} \vartheta$ for $n \to \infty$.

$$\hat{h}(\xi_n) - h(\vartheta) \;=\; (\hat{h}(\xi_n) - h(\xi_n)) + (h(\xi_n) - h(\vartheta))$$

With the above remark and the continuity of h it is immediatly clear that the second part stochastically converges to 0. For the first part it suffices to show that $\sup_\theta ||\hat{h}(\theta) - h(\theta)||_M$ stochastically converges to 0 where $||\cdot||_M$ denotes the maximum norm of a matrix. We have

$$\begin{aligned}
\frac{1}{2}(\hat{h}(\theta) - h(\theta)) \;&=\; \frac{1}{2}(\frac{\partial}{\partial\theta}\hat{l}_n(\theta) - \frac{\partial}{\partial\theta}l(\theta)) \\
&=\; \int_A (\Delta_{\hat{m},\theta}(\mathbf{x}) - \Delta_{m,\theta}(\mathbf{x}))^T(\Delta_{\hat{m},\theta}(\mathbf{x}) - \Delta_{m,\theta}(\mathbf{x}))d\mathbf{x} \\
&\quad + \int_A \Delta_{m,\theta}(\mathbf{x})^T(\Delta_{\hat{m},\theta}(\mathbf{x}) - \Delta_{m,\theta}(\mathbf{x}))d\mathbf{x} \\
&\quad + \int_A (\Delta_{\hat{m},\theta}(\mathbf{x}) - \Delta_{m,\theta}(\mathbf{x}))^T\Delta_{m,\theta}(\mathbf{x})d\mathbf{x} \\
&\quad + \int_A (\hat{m}(T_\theta\mathbf{x}) - m(T_\theta\mathbf{x}) - (\hat{m}(S_\theta\mathbf{x}) - m(S_\theta\mathbf{x})))\left(\frac{\partial}{\partial\theta}\Delta_{\hat{m},\theta}(\mathbf{x}) - \frac{\partial}{\partial\theta}\Delta_{m,\theta}(\mathbf{x})\right)d\mathbf{x} \\
&\quad + \int_A (m(T_\theta\mathbf{x}) - m(S_\theta\mathbf{x}))\left(\frac{\partial}{\partial\theta}\Delta_{\hat{m},\theta}(\mathbf{x}) - \frac{\partial}{\partial\theta}\Delta_{m,\theta}(\mathbf{x})\right)d\mathbf{x} \\
&\quad + \int_A (\hat{m}(T_\theta\mathbf{x}) - m(T_\theta\mathbf{x}) - (\hat{m}(S_\theta\mathbf{x}) - m(S_\theta\mathbf{x})))\frac{\partial}{\partial\theta}\Delta_{m,\theta}(\mathbf{x})d\mathbf{x}
\end{aligned}$$

There is

$$\begin{aligned}
\Delta_{m,\theta}(\mathbf{x})^T(\Delta_{\hat{m},\theta}(\mathbf{x}) - \Delta_{m,\theta}(\mathbf{x})) \;&=\; (a_{i,j}(\mathbf{x}))_{1\leq i,j\leq k} \\
\frac{\partial}{\partial\theta}\Delta_{\hat{m},\theta}(\mathbf{x}) - \frac{\partial}{\partial\theta}\Delta_{m,\theta}(\mathbf{x}) \;&=\; (h_{i,j}(\mathbf{x}))_{1\leq i,j\leq k}
\end{aligned}$$

with

$$\begin{aligned}
a_{i,j}(\mathbf{x}) \;&=\; \sum_{s=1}^d\sum_{t=1}^d \left(\frac{\partial}{\partial x_s}m(T_\theta\mathbf{x})\frac{\partial}{\partial\theta_i}(T_\theta\mathbf{x})_s - \frac{\partial}{\partial x_s}m(S_\theta\mathbf{x})\frac{\partial}{\partial\theta_i}(S_\theta\mathbf{x})_s\right)\frac{\partial}{\partial\theta_j}(T_\theta\mathbf{x})_t\left(\frac{\partial}{\partial x_s}\hat{m}(T_\theta\mathbf{x}) - \frac{\partial}{\partial x_s}m(T_\theta\mathbf{x})\right) \\
&\quad - \sum_{s=1}^d\sum_{t=1}^d \left(\frac{\partial}{\partial x_s}m(T_\theta\mathbf{x})\frac{\partial}{\partial\theta_i}(T_\theta\mathbf{x})_s - \frac{\partial}{\partial x_s}m(S_\theta\mathbf{x})\frac{\partial}{\partial\theta_i}(S_\theta\mathbf{x})_s\right)\frac{\partial}{\partial\theta_j}(S_\theta\mathbf{x})_t\left(\frac{\partial}{\partial x_s}\hat{m}(S_\theta\mathbf{x}) - \frac{\partial}{\partial x_s}m(S_\theta\mathbf{x})\right) \\
h_{i,j}(\mathbf{x}) \;&=\; \sum_{s=1}^d\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}(T_\theta\mathbf{x})_s\left(\frac{\partial}{\partial x_s}\hat{m}(T_\theta\mathbf{x}) - \frac{\partial}{\partial x_s}m(T_\theta\mathbf{x})\right) - \frac{\partial^2}{\partial\theta_i\partial\theta_j}(S_\theta\mathbf{x})_s\left(\frac{\partial}{\partial x_s}\hat{m}(S_\theta\mathbf{x}) - \frac{\partial}{\partial x_s}m(S_\theta\mathbf{x})\right)\right]
\end{aligned}$$

$$+\sum_{s=1}^{d}\sum_{t=1}^{d}\Big[\frac{\partial}{\partial\theta_i}(T_\theta\mathbf{x})_s\frac{\partial}{\partial\theta_j}(T_\theta\mathbf{x})_t\left(\frac{\partial^2}{\partial x_s\partial x_t}\hat{m}(T_\theta\mathbf{x})-\frac{\partial^2}{\partial x_s\partial x_t}m(T_\theta\mathbf{x})\right)$$
$$-\frac{\partial}{\partial\theta_i}(S_\theta\mathbf{x})_s\frac{\partial}{\partial\theta_j}(S_\theta\mathbf{x})_t\left(\frac{\partial^2}{\partial x_s\partial x_t}\hat{m}(S_\theta\mathbf{x})-\frac{\partial^2}{\partial x_s\partial x_t}m(S_\theta\mathbf{x})\right)\Big]$$
$$= \sum_{s=1}^{d} I_s^{[1]}(\mathbf{x},i,j)+\sum_{s=1}^{d}\sum_{t=1}^{d} I_{s,t}^{[2]}(\mathbf{x},i,j)$$

From the definition of T_θ and S_θ it is immediately clear, that terms like $||\partial/\partial\theta T_\theta\mathbf{x}||$ are uniformly bounded over θ and $\mathbf{x}\in B$. By applying the Cauchy-Schwarz inequality several times it therefore suffices to show that

$$\begin{aligned}
\int_A(\hat{m}(T_\theta\mathbf{x})-m(T_\theta\mathbf{x}))^2d\mathbf{x} &= o_P(1),\ \int_A(\hat{m}(S_\theta\mathbf{x})-m(S_\theta\mathbf{x}))^2d\mathbf{x}=o_P(1),\\
\int_A(\frac{\partial}{\partial x_i}\hat{m}(T_\theta\mathbf{x})-\frac{\partial}{\partial x_i}m(T_\theta\mathbf{x}))^2d\mathbf{x} &= o_P(1),\\
\int_A(\frac{\partial}{\partial x_i}\hat{m}(S_\theta\mathbf{x})-\frac{\partial}{\partial x_i}m(S_\theta\mathbf{x}))^2d\mathbf{x} &= o_P(1),\ 1\le i\le d\\
\int_A(\frac{\partial^2}{\partial x_i\partial x_j}\hat{m}(T_\theta\mathbf{x})-\frac{\partial^2}{\partial x_i\partial x_j}m(T_\theta\mathbf{x}))^2d\mathbf{x} &= o_P(1),\\
\int_A(\frac{\partial^2}{\partial x_i\partial x_j}\hat{m}(S_\theta\mathbf{x})-\frac{\partial^2}{\partial x_i\partial x_j}m(S_\theta\mathbf{x}))^2d\mathbf{x} &= o_P(1),\ 1\le i,j\le d
\end{aligned}$$

uniformly over θ. We obtain for example, if $\max\{|\partial^2/\partial\theta_i\partial\theta_j(T_\theta\mathbf{x})_s|,\partial^2/\partial\theta_i\partial\theta_j(S_\theta\mathbf{x})_s|\}\le C$ for some $C>0$

$$\begin{aligned}
&\int_A|(\hat{m}(T_\theta\mathbf{x})-m(T_\theta\mathbf{x})-(\hat{m}(S_\theta\mathbf{x})-m(S_\theta\mathbf{x})))I_s^{[1]}(\mathbf{x},i,j)|d\mathbf{x}\\
&\le C\int_A|\hat{m}(T_\theta\mathbf{x})-m(T_\theta\mathbf{x})-(\hat{m}(S_\theta\mathbf{x})-m(S_\theta\mathbf{x}))|\left|\frac{\partial}{\partial x_s}\hat{m}(T_\theta\mathbf{x})-\frac{\partial}{\partial x_s}m(T_\theta\mathbf{x})\right|d\mathbf{x}\\
&\quad +C\int_A|\hat{m}(T_\theta\mathbf{x})-m(T_\theta\mathbf{x})-(\hat{m}(S_\theta\mathbf{x})-m(S_\theta\mathbf{x}))|\left|\frac{\partial}{\partial x_s}\hat{m}(S_\theta\mathbf{x})-\frac{\partial}{\partial x_s}m(S_\theta\mathbf{x})\right|d\mathbf{x}\\
&= C_2\left(\int_{A_\theta}(\hat{m}(\mathbf{z})-m(\mathbf{z}))^2d\mathbf{z}\right)^{1/2}\left(\int_{A_\theta}\left(\frac{\partial}{\partial z_s}\hat{m}(\mathbf{z})-\frac{\partial}{\partial z_s}m(\mathbf{z})\right)^2d\mathbf{z}z\right)^{1/2}\\
&\le C_2\left(\int_B(\hat{m}(\mathbf{z})-m(\mathbf{z}))^2d\mathbf{z}\right)^{1/2}\left(\int_B\left(\frac{\partial}{\partial z_s}\hat{m}(\mathbf{z})-\frac{\partial}{\partial z_s}m(\mathbf{z})\right)^2d\mathbf{z}\right)^{1/2}=o_P(1)
\end{aligned}$$

by using Theorem 4. The other terms are estimated similarly.

A.3 Proof of Theorem 3

We use the decomposition

$$\hat{L}_n(\hat{\vartheta})=\hat{L}_n(\vartheta)-(\vartheta-\hat{\vartheta})^T\hat{l}_n(\hat{\vartheta})-(\vartheta-\hat{\vartheta})^T\hat{h}(\xi_n)(\vartheta-\hat{\vartheta})$$

and immediatly see from the previous proof that the second term on the right is 0 and the last term on the right is of order $O_P(n^{-1}a_n^{-1}) = o_P((n^d h^{2\beta+d/2} a_n^{3d/2})^{-1})$. Therefore it suffices to show the weak convergence of the first term to the desired distribution. It is

$$\begin{aligned} L_n(\vartheta) &= \int_A \left(\sum_{\mathbf{r}} (w_{\mathbf{r}}(S_\vartheta \mathbf{x}) - w_{\mathbf{r}}(T_\vartheta \mathbf{x}))\varepsilon_{\mathbf{r}} \right)^2 d\mathbf{x} \\ &+2 \int_A \left(\sum_{\mathbf{r}} (w_{\mathbf{r}}(S_\vartheta \mathbf{x}) - w_{\mathbf{r}}(T_\vartheta \mathbf{x}))\varepsilon_{\mathbf{r}} \right) \left(\sum_{\mathbf{s}} (w_{\mathbf{s}}(S_\vartheta \mathbf{x}) - w_{\mathbf{s}}(T_\vartheta \mathbf{x}))\Psi m(\mathbf{x}_{\mathbf{s}}) \right) d\mathbf{x} \end{aligned}$$

As in the proof of Theorem 4 one easily sees that the last two terms on the right are of order $o_P((n^d h^{2\beta+d/2} a_n^{3d/2})^{-1})$. We get

$$L_n(\vartheta) = \sum_{\mathbf{r}} \int_A (w_{\mathbf{r}}(S_\vartheta \mathbf{x}) - w_{\mathbf{r}}(T_\vartheta \mathbf{x}))^2 d\mathbf{x}\varepsilon_{\mathbf{r}} + \sum_{\mathbf{r}\neq\mathbf{s}} \int_A (w_{\mathbf{r}}(S_\vartheta \mathbf{x}) - w_{\mathbf{r}}(T_\vartheta \mathbf{x}))(w_{\mathbf{s}}(S_\vartheta \mathbf{x}) - w_{\mathbf{s}}(T_\vartheta \mathbf{x}))d\mathbf{x}\varepsilon_{\mathbf{r}}\varepsilon_{\mathbf{s}}.$$

The rest of the proof now follows along the lines of the proof of Theorem 4 when considering $I_1^{(\mathbf{j})}$.