

On the Containment Problem for Linear Sets

Hans U. Simon

Department of Computer Science, Ruhr University Bochum, Germany

E-Mail Address: `hans.simon@rub.de`

February 20, 2018

Abstract

It is well known that the containment problem (as well as the equivalence problem) for semilinear sets is log-complete in Π_2^P . It had been shown quite recently that already the containment problem for multi-dimensional linear sets is log-complete in Π_2^P (where hardness even holds for a unary encoding of the numerical input parameters). In this paper, we show that already the containment problem for 1-dimensional linear sets (with binary encoding of the numerical input parameters) is log-hard (and therefore also log-complete) in Π_2^P . However, combining both restrictions (dimension 1 and unary encoding), the problem becomes solvable in polynomial time.

1 Introduction

The containment problem for a family of sets consists in finding an answer to the following question: given two sets of the family, is the first one a subset of the second one?

It had been shown in a very early stage of complexity theory that the containment and the equivalence problem for semilinear sets are log-complete in Π_2^P (the second level of the polynomial hierarchy) [4]. This early investigation had been motivated by the fact that, first, the equivalence problem for contextfree languages is recursively undecidable and, second, the commutative images of contextfree languages happen to be semilinear sets according to Parikh's theorem [5]. Showing inequivalence of the commutative images of two given contextfree languages would therefore demonstrate their inequivalence.

Linear sets are the basic building blocks of semilinear sets. (The latter are finite unions of linear sets.) Moreover, 1-dimensional linear sets are the central object of research in the study of numerical semigroups [6]. It was shown quite recently that the containment problem for linear sets of variable dimension is log-complete in Π_2^P , where hardness even holds when numbers are encoded in unary [2]. In this paper, we extend the latter result as follows:

1. The containment problem for 1-dimensional linear sets (with a binary encoding of numbers) is log-hard (and therefore also log-complete) in Π_2^P .

2. On the other hand, the containment problem for 1-dimensional linear sets with a unary encoding of numbers is solvable in polynomial time.

Moreover, in order to prove these results, we show the following:

- The containment problem for so-called simple unary $(+, \cup)$ -expressions¹ is log-hard in Π_2^p .
- The containment problem for linear sets is still log-hard in Π_2^p under a relatively strong promise. See Sections 2.5 and 3 for details.

These results might be of independent interest.

As for semilinear sets, the containment and the inequivalence problem have the same inherent complexity: both are log-complete in Π_2^p . We briefly note that the situation is different for linear sets. The equivalence problem for linear sets is easily shown to be computationally equivalent to the word problem for linear sets, and the latter is easily shown to be NP-complete. Hence, for linear sets, verifying containment is much harder than verifying equivalence.

This paper is structured as follows. In Section 2 we present the basic definitions and notations, and we mention some facts. Our main results are stated and proved in Section 3. One of these proofs is however postponed to the final Section 4 because it is a suitable modification of a similar proof of Stockmeyer (and is given for the sake of completeness). In the final Section 5, an open problem is mentioned.

2 Definitions, Notations and Facts

We assume familiarity with basic concepts from complexity theory (e.g., logspace reductions, log-hardness or log-completeness, polynomial hierarchy etc.). The complexity classes of the polynomial hierarchy will be denoted, as usual, by Σ_k^p and Π_k^p for $k = 0, 1, 2, \dots$. We will mainly deal with the class Π_2^p on the second level of the hierarchy.

In Section 2.1, we briefly call into mind the definition of true quantified Boolean formulas which give rise to a hierarchy of problems with one log-complete problem at every level of the polynomial hierarchy. Section 2.2 contains the basic definitions that we need in connection with integer expressions. In Section 2.3, we briefly remind the reader to the definition of linear and semilinear sets. Some well known results on the inherent complexity of the containment problem for integer expressions resp. for semilinear sets are mentioned in Section 2.4. Section 2.5 briefly calls into mind the notion of promise problems.

2.1 Quantified Boolean Formulas

Definition 2.1 ([7]) *Let X_1, X_2, \dots, X_k with $X_i = \{x_{i1}, x_{i2}, \dots\}$ be disjoint collections of Boolean variables. Let $f(X_1, \dots, X_k)$ denote any Boolean formula over (finitely many of)*

¹a variant of a problem that has originally been analyzed by Stockmeyer [7]

the variables from $X_1 \cup \dots \cup X_k$. Let $Q_k = \exists$ if $k \geq 1$ is odd and $Q_k = \forall$ if $k \geq 1$ is even. The notation “ $\exists X_i : \dots$ ” means “there exists an assignment of the variables in X_i such that ...”. The analogous remark applies to the notation “ $\forall X_i : \dots$ ”. Given these notations, we define

$$\mathcal{B}_k = \{f(X_1, \dots, X_k) : (\exists X_1, \forall X_2, \dots, Q_k X_k : f(X_1, \dots, X_k) = 1)\} .$$

The set consisting of Boolean formulas $f(X_1, \dots, X_k)$ outside of \mathcal{B}_k is denoted as $\overline{\mathcal{B}_k}$. The subproblem of \mathcal{B}_k (resp. of $\overline{\mathcal{B}_k}$) with f being a formula in conjunctive normal form is denoted as \mathcal{B}_k^{CNF} (resp. as $\overline{\mathcal{B}_k}^{CNF}$). The corresponding subproblems with f being a formula in disjunctive normal form are denoted as \mathcal{B}_k^{DNF} and $\overline{\mathcal{B}_k}^{DNF}$, respectively.

Theorem 2.2 ([7]) *For any $k \geq 1$, \mathcal{B}_k is log-complete in Σ_k^p . The same is true for \mathcal{B}_k^{CNF} if k is odd and for \mathcal{B}_k^{DNF} if k is even.*

Corollary 2.3 *For any $k \geq 1$, $\overline{\mathcal{B}_k}$ is log-complete in Π_k^p . This even holds for the set $\overline{\mathcal{B}_k}^{CNF}$ if $k \geq 1$ is odd and for the set $\overline{\mathcal{B}_k}^{DNF}$ if $k \geq 1$ is even.*

Example 2.4 *The set $\overline{\mathcal{B}_2}^{DNF}$, which coincides with the set of all Boolean DNF-formulas $f(X_1, X_2)$ satisfying*

$$\forall X_1, \exists X_2 : f(X_1, X_2) = 0 .$$

is log-complete in Π_2^p .

2.2 Integer Expressions

Definition 2.5 *Let $m \geq 1$ be a positive integer. The set \mathcal{E}_m of m -dimensional unary integer expressions, simply called unary integer expressions if m is clear from context, is the smallest set with the following properties:*

1. $\{0, 1\}^m \subseteq \mathcal{E}_m$. The tuples $(b_1, \dots, b_m) \in \{0, 1\}^m$ are called atomic expressions.
2. For any $E_1, E_2 \in \mathcal{E}_m$: $(E_1 \cup E_2), (E_1 + E_2) \in \mathcal{E}_m$.

Every expression $E \in \mathcal{E}_m$ represents a set $L(E) \subseteq \mathbb{N}_0^m$ that is defined in the obvious manner.

We briefly note that the classical definition of integer expressions in [7] is different from ours: there the expressions define subsets of \mathbb{N}_0 , and an atomic expression is a binary representation of a single number in \mathbb{N}_0 . In other words, the classical definition deals with 1-dimensional binary expressions whereas we deal with multi-dimensional unary expressions.

Since “ \cup ” is an associative operation, we may simply write $(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_s)$ instead of $(\dots((E_1 \cup E_2) \cup E_3) \cup \dots \cup E_s)$. The analogous remark applies to the operation “ $+$ ”.

Definition 2.6 *An expression $E \in \mathcal{E}_m$ is said to be a $(+, \cup)$ -expression if it is a sum of unions of atomic expressions. A $(+, \cup)$ -expression is called simple if every union in the sum is the union of precisely two (not necessarily different) atomic expressions.*

Example 2.7 *The string*

$$E = ((1, 1, 0) \cup (0, 0, 0)) + ((1, 0, 0) \cup (1, 0, 0)) + ((1, 1, 1) \cup (0, 0, 0))$$

is a simple unary $(+, \cup)$ -expression. It represents the set

$$L(E) = \{(1, 0, 0), (2, 1, 0), (2, 1, 1), (3, 2, 1)\} .$$

2.3 Linear and Semilinear Sets

Definition 2.8 *The set $L(\mathbf{c}, P) \subseteq \mathbb{N}_0^m$ induced by $\mathbf{c} \in \mathbb{N}_0^m$ and a finite set $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\} \subset \mathbb{N}_0^m$ is defined as*

$$L(\mathbf{c}, P) = \mathbf{c} + \langle P \rangle \quad \text{where} \quad \langle P \rangle = \left\{ \sum_{i=1}^k a_i \cdot \mathbf{p}_i : a_i \in \mathbb{N}_0 \right\} .$$

The elements in P are called periods and \mathbf{c} is called the constant vector of $L(\mathbf{c}, P)$. A subset L of \mathbb{N}_0^m is called linear if $L = L(\mathbf{c}, P)$ for some $\mathbf{c} \in \mathbb{N}_0^m$ and some finite set $P \subset \mathbb{N}_0^m$. A semilinear set in \mathbb{N}_0^m is a finite union of linear sets in \mathbb{N}_0^m .

2.4 Containment Problems

As mentioned already in the introduction, the *containment problem* for a family of sets consists in finding an answer to the following question: given two sets of the family, is the first one a subset of the second one? We will be mainly concerned with the containment problem for integer expressions and with the containment problem for linear and semilinear sets. We will assume that the dimension m of sets in \mathbb{N}_0^m is part of the input unless we explicitly talk about an m -dimensional problem for some fixed constant m . The following is known:

1. The containment problem for 1-dimensional binary integer expressions is log-complete in Π_2^P [7].
2. The containment problem for semilinear sets is log-complete in Π_2^P [4]. The log-hardness in Π_2^P even holds either when numbers are encoded in unary or when the dimension is fixed to 1.
3. The containment problem for linear sets is log-complete in Π_2^P [2]. The log-hardness in Π_2^P even holds when numbers are encoded in unary.

The first two hardness results are shown by means of a logspace reduction from $\overline{\mathcal{B}}_2^{DNF}$ to the respective containment problem. A suitable modification of Stockmeyer's reduction from $\overline{\mathcal{B}}_2^{DNF}$ to the containment problem for 1-dimensional binary integer expressions leads to the following result:

Theorem 2.9 *The containment problem for simple unary $(+, \cup)$ -expressions is log-complete in Π_2^P .*

The proof of this theorem will be given in Section 4.

Notation for Vectors: The j -th component of a vector \mathbf{x} is denoted as x_j or, occasionally, as $\mathbf{x}[j]$. The latter notation is used, for instance, if there is a sequence of vectors, say $\mathbf{x}_1, \dots, \mathbf{x}_n$. The j -th component of \mathbf{x}_i is then denoted as $\mathbf{x}_i[j]$ (as opposed to $x_{i,j}$ or $(x_i)_j$). Throughout the paper, we use \mathbf{a}^m (with $a \in \mathbb{N}_0$) as a short notation for $(a, \dots, a) \in \mathbb{N}_0^m$. For instance $\mathbf{1}^m$ denotes the all-ones vector in \mathbb{N}_0^m . The vector with value 1 in the i -th component and zeros in the remaining $m - 1$ components is denoted as \mathbf{e}_i^m . The all-ones matrix is denoted as J .²

2.5 Promise Problems

A decision problem (without promise) is a problem with “yes”- and “no”-instances. A *promise problem* is a decision problem augmented by a promise that the input instances passed to an algorithm satisfy a certain condition. An algorithm needs to solve the promise problem only on the input instances that satisfy this condition. It may output anything on the remaining instances. Hence a promise problem has besides the “yes”- and the “no”-instances a third kind of instances: the ones that violate the promised condition. Decision problem can be viewed as promise problems with an empty promise. Reductions between promise problems should map “yes”-instances (resp. “no”-instances) of the first problem to “yes”-instances (resp. “no”-instances) of the second problem.

3 Main Results

The first result in this section will be concerned with the containment problem for linear sets when the latter is viewed as the following promise problem.

Instance: dimension m , finite sets $P, Q \subset \mathbb{N}_0^m$, vectors $\mathbf{c}, \mathbf{d} \in \mathbb{N}_0^m$ and $s \in \{1, \dots, |P|\}$.

Question: $L(\mathbf{c}, P) \subseteq L(\mathbf{d}, Q)$?

Promise: Let $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ and let $K_s = \left\{ \mathbf{a} \in \{0, 1\}^k \mid \sum_{i=1}^k a_i = s \right\}$. With this notation, the following holds:

$$\forall x \in \mathbb{N}_0^k \setminus K_s : \sum_{i=1}^k x_i \cdot \mathbf{p}_i \in L(\mathbf{d}, Q) . \quad (1)$$

In other words: we make the promise that the inclusion $L(\mathbf{c}, P) \subseteq L(\mathbf{d}, Q)$ can possibly fail only on linear combinations of $\mathbf{p}_1, \dots, \mathbf{p}_k$ with coefficient vectors taken from K_s .

²The order of the matrix will always be clear from context.

It is easy to see that we may set $\mathbf{d} = \mathbf{0}$ in this problem without any loss of generality.

In [2], it was shown that the containment problem for linear sets is log-hard in Π_2^P . We strengthen this result by showing that even the corresponding promise problem exhibits this kind of hardness. This slightly stronger result will later help us to prove the hardness of the containment problem for 1-dimensional linear sets.

Theorem 3.1 *The containment problem for linear sets is log-hard in Π_2^P even under the promise (1) and even when numbers are encoded in unary.*

Proof We will describe a logspace reduction from the containment problem for simple unary $(+, \cup)$ -expressions to the containment problem for linear sets. An instance of the former problem is of the form

$$E = \sum_{i=1}^s (\mathbf{B}_{i1} \cup \mathbf{B}_{i2}) \quad \text{and} \quad E' = \sum_{i=1}^{s'} (\mathbf{B}'_{i1} \cup \mathbf{B}'_{i2}) \quad (2)$$

where $\mathbf{B}_{i1}, \mathbf{B}_{i2}, \mathbf{B}'_{i1}, \mathbf{B}'_{i2} \in \{0, 1\}^m$. Note that we may set $s' = s$ because we could add sum-terms of the form $(\mathbf{0}^m \cup \mathbf{0}^m)$ to the expression which has fewer terms. Our goal is to design $(2m + 2s)$ -dimensional linear sets $\mathbf{c} + \langle P \rangle$ and $\langle P' \cup P'' \rangle$ such that

$$L(E) \subseteq L(E') \Leftrightarrow \mathbf{c} + \langle P \rangle \subseteq \langle P' \cup P'' \rangle . \quad (3)$$

Intuitively, we should think of vectors from \mathbb{N}_0^{2m+2s} as being decomposed into four sections of dimension m, s, s, m , respectively. The first section is called the “base section”; the latter three are called “control sections”. The constant vector \mathbf{c} and the periods in $P = \{\mathbf{p}_{ij} : i \in [s], j \in [2]\}$ are chosen as follows:

$$\mathbf{c} = (\mathbf{0}^m, \mathbf{2}^s, \mathbf{1}^s, \mathbf{1}^m) \quad \text{and} \quad \mathbf{p}_{ij} = (\mathbf{B}_{ij}, \mathbf{e}_i^s, \mathbf{0}^s, \mathbf{0}^m) . \quad (4)$$

Note that the base section of the periods in P contains the atomic sub-expressions of E . The vectors in \mathbb{N}_0^{2m+2s} having $(\mathbf{3}^s, \mathbf{1}^s, \mathbf{1}^m)$ in their control sections are said to be “essential”. It is evident that

$$L(E) \times \{3\}^s \times \{1\}^s \times \{1\}^m = (\mathbf{c} + \langle P \rangle) \cap (\mathbb{N}_0^m \times \{3\}^s \times \{1\}^s \times \{1\}^m) .$$

In other words: the set of base sections of the essential vectors in $\mathbf{c} + \langle P \rangle$ coincides with $L(E)$. The periods in $P' = \{\mathbf{p}'_{ij} : i \in [s], j \in [2]\}$ are similarly defined as the periods in P :

$$\mathbf{p}'_{ij} = \begin{cases} (\mathbf{B}'_{ij}, 3 \cdot \mathbf{e}_i^s, \mathbf{e}_i^s, \mathbf{0}^m) & \text{if } i \in [s-1] \\ (\mathbf{B}'_{sj}, 3 \cdot \mathbf{e}_s^s, \mathbf{e}_i^s, \mathbf{1}^m) & \text{if } i = s \end{cases} .$$

Clearly,

$$L(E') \times \{3\}^s \times \{1\}^s \times \{1\}^m = \langle P' \rangle \cap (\mathbb{N}_0^m \times \{3\}^s \times \{1\}^s \times \{1\}^m) .$$

Note that $L(E) \subseteq L(E')$ iff any essential vector in $\mathbf{c} + \langle P \rangle$ is contained in $\langle P' \rangle$. In order to get the desired equivalence (3), we will design P'' such that the following holds:

Claim 1: Any inessential vector from $\mathbf{c} + \langle P \rangle$ is contained in $\langle P'' \rangle$.

Claim 2: Any essential vector in $\mathbf{c} + \langle P \rangle$ is contained in $\langle P' \cup P'' \rangle$ only if it is already contained in $\langle P' \rangle$.

It is evident that (3) is valid if P'' can be defined in accordance with the two above claims. Let $n = 1 + \max\{x_i : \mathbf{x} \in L(E), i \in [m]\}$, i.e., $n - 1$ is the largest number that occurs in a component of some vector in $L(E)$. We now set $P'' = P''_1 \cup P''_2$ where

$$\begin{aligned} P''_1 &= \{(\mathbf{0}^m, 2 \cdot \mathbf{e}_i^s, \mathbf{1}^s, \mathbf{0}^m), (\mathbf{0}^m, 2 \cdot \mathbf{e}_i^s, \mathbf{0}^s, \mathbf{0}^m), (\mathbf{0}^m, 3 \cdot \mathbf{e}_i^s, \mathbf{0}^s, \mathbf{0}^m) : i \in [s]\} , \\ P''_2 &= \{(r \cdot \mathbf{e}_i^m, \mathbf{0}^s, \mathbf{0}^s, \mathbf{e}_i^m), (n \cdot \mathbf{e}_i^m, \mathbf{0}^s, \mathbf{0}^s, \mathbf{0}^m) : i \in [m], r \in \{0, 1, \dots, n-1\}\} . \end{aligned}$$

The proof of the theorem can now be accomplished by showing that the above two claims are valid for our definition of P'' (and by adding some easy observations).

Proof of Claim 1: Let $\mathbf{x} \in \mathbf{c} + \langle P \rangle$ be inessential. An inspection of (4) reveals that there must exist an index $i_0 \in [s]$ such that the i_0 -th component of the first control section of \mathbf{x} has a value that differs from 3. Since already the constant vector \mathbf{c} makes a contribution of 2 in this control section, the possible values for x_{m+i_0} are 2, 4, 5, 6, ... In order to cast \mathbf{x} as a member of $\langle P'' \rangle$, we first pick the vector $\mathbf{u} = (\mathbf{0}^m, 2 \cdot \mathbf{e}_{i_0}^s, \mathbf{1}^s, \mathbf{0}^m)$. Note that $\mathbf{u} \leq \mathbf{x}$ and \mathbf{u} already coincides with \mathbf{x} in the second control section. Adding to \mathbf{u} properly chosen multiples of vectors of the form $(\mathbf{0}^m, 2 \cdot \mathbf{e}_i^s, \mathbf{0}^s, \mathbf{0}^m)$ or $(\mathbf{0}^m, 3 \cdot \mathbf{e}_i^s, \mathbf{0}^s, \mathbf{0}^m)$, we obtain a vector $\mathbf{v} \leq \mathbf{x}$ that coincides with \mathbf{x} also in the first control section. Consider now the entries of \mathbf{v} and \mathbf{x} in the base section. For any $i \in [m]$, consider the decomposition $x_i - v_i = q_i n + r_i$ with $q_i \geq 0$ and $0 \leq r_i \leq n - 1$. Adding to \mathbf{v} the vector

$$\sum_{i=1}^m (q_i \cdot (n \cdot \mathbf{e}_i^m, \mathbf{0}^s, \mathbf{0}^s, \mathbf{0}^m) + (r_i \cdot \mathbf{e}_i^m, \mathbf{0}^s, \mathbf{0}^s, \mathbf{e}_i^m)) ,$$

we obtain a vector that coincides with \mathbf{x} (since, by now, it also coincides with \mathbf{x} in the base section and in the third control section).

Proof of Claim 2: Let $\mathbf{x} \in \mathbf{c} + \langle P \rangle$ be essential and suppose that $\mathbf{x} \in \langle P' \cup P'' \rangle$. A representation of \mathbf{x} as a member of $\langle P' \cup P'' \rangle$ cannot make use of a vector of the form $(\mathbf{0}^m, 2 \cdot \mathbf{e}_i^s, \mathbf{1}^s, \mathbf{0}^m)$ because there is no way to extend the value 2 in the i -th component of the first control section to 3 (since any period in $P' \cup P''$ adds either 0 or a value greater than 1 to this component). Given that we do not employ these vectors, it follows that any representation of \mathbf{x} as a member of $\langle P' \cup P'' \rangle$ must be of the form $\mathbf{x} = \mathbf{x}' + \mathbf{x}''$ for some essential vector $\mathbf{x}' \in \langle P' \rangle$ and some vector $\mathbf{x}'' \in \langle P' \cup P'' \rangle$ (because, without employing an essential vector from $\langle P' \rangle$, we wouldn't get $\mathbf{1}^s$ into the second control section). Since \mathbf{x}' is essential, it will already contribute $(\mathbf{3}^s, \mathbf{1}^s, \mathbf{1}^m)$ to the three control sections. It follows that $\mathbf{x}'' = \mathbf{0}^{2m+2s}$ because adding any period from $P' \cup P''$ to \mathbf{x}' will destroy the pattern $(\mathbf{3}^s, \mathbf{1}^s, \mathbf{1}^m)$ in the control sections or will induce a component of value at least n in the base section (which is larger than any component of \mathbf{x} in the base section). It follows that $\mathbf{x} = \mathbf{x}' \in \langle P' \rangle$.

It can be shown by standard arguments that the transformation $(E, E') \mapsto (\mathbf{c}, P, P', P'')$ is logspace-computable (even when numbers are encoded in unary). Finally observe that the above definition of essential vectors implies that every essential vector from $\mathbf{c} + \langle P \rangle$ employs a coefficient vector from $\{0, 1\}^{|P|}$ with precisely s ones. Since any inessential vector from $\mathbf{c} + \langle P \rangle$ also belongs to $\langle P' \rangle \subseteq \langle P' \cup P'' \rangle$, the promised condition (1) is satisfied (with $P' \cup P''$ at the place of Q). This concludes the proof. \bullet

We will show in the sequel that the containment problem for 1-dimensional linear sets (with numerical input parameters given in binary representation) is log-hard in Π_2^p . To this end, we will make use of the following result on the aggregation of diophantine equations:

Lemma 3.2 ([3]) *Let*

$$\sum_{j=1}^r a_{1j}x_j = b_1 \quad \text{and} \quad \sum_{j=1}^r a_{2j}x_j = b_2 \quad (5)$$

be a system of two linear diophantine equations where a_{1j}, a_{2j} are non-negative integers and b_1, b_2 are strictly positive integers. Let t_1, t_2 be positive integers satisfying the following conditions:

1. t_1 and t_2 are relatively prime.
2. t_1 does not divide b_2 and t_2 does not divide b_1 .
3. $t_1 > b_2 - a_2$ and $t_2 > b_1 - a_1$ where a_i denotes the smallest nonzero coefficient in $\{a_{i1}, \dots, a_{ir}\}$.

Then, restricting x_j to non-negative integers, the solution set of (5) is the same as the solution set of

$$t_1 \cdot \sum_{j=1}^r a_{1j}x_j + t_2 \cdot \sum_{j=1}^r a_{2j}x_j = t_1 \cdot b_1 + t_2 \cdot b_2 .$$

Note that

$$t_1 = 1 + \max\{b_1, b_2\} \quad \text{and} \quad t_2 = 1 + t_1 \quad (6)$$

is among the choices for t_1, t_2 such that the three conditions mentioned in Theorem 3.2 are satisfied. An iterative application of Lemma 3.2 leads to the following result:

Corollary 3.3 *Let $\mathbf{Ax} = \mathbf{b}$ with $A \in \mathbb{N}_0^{m \times r}$ and $\mathbf{b} \in \mathbb{N}^m$ be a system of linear diophantine equations. Let $A_1, \dots, A_m \in \mathbb{N}_0^r$ denote the rows of A . Then there exist “aggregation coefficients” $t_1, \dots, t_m \in \mathbb{N}$ such that the solution set for $\mathbf{Ax} = \mathbf{b}$ within \mathbb{N}_0^r is the same as the solution set for the single equation*

$$\left(\sum_{i=1}^m t_i A_i \right) \mathbf{x} = \sum_{i=1}^m t_i \mathbf{b}_i .$$

We are ready now for the next result:

Theorem 3.4 *The containment problem for 1-dimensional linear sets is log-hard in Π_2^P .*

Proof We will describe a logspace reduction from the containment problem for multidimensional linear sets (under the promise (1)) to the containment problem for 1-dimensional linear sets. The proof proceeds in stages. In Stage 1, we express the containment problem for multidimensional linear sets in terms of a system of diophantine equations. The latter arises naturally when the the containment problem is written in matrix notation. In Stage 2, we rewrite the system of diophantine equations so that Corollary 3.3 comes into play and an aggregation into a single equation takes place. Although the solution set of the single equation and the one of the original system of equations do not fully coincide, they are related sufficiently closely so that, in Stage 3, we finally obtain the desired logspace reduction.

Let us start with Stage 1. Let $\mathbf{c} \in \mathbb{N}_0^m$, $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\} \subseteq \mathbb{N}_0^m$ and $Q = \{\mathbf{q}_1, \dots, \mathbf{q}_\ell\} \subseteq \mathbb{N}_0^m$ form an instance of the containment problem. We set

$$A := [\mathbf{p}_1 \ \dots \ \mathbf{p}_k] \in \mathbb{N}_0^{m \times k} \text{ and } B := [\mathbf{q}_1 \ \dots \ \mathbf{q}_\ell] \in \mathbb{N}_0^{m \times \ell},$$

which yields two matrices with non-negative entries. With this notation:

$$L(\mathbf{c}, P) \subseteq L(\mathbf{0}, Q) \Leftrightarrow \forall \mathbf{x} \in \mathbb{N}_0^k, \exists \mathbf{y} \in \mathbb{N}_0^\ell : \mathbf{c} + A\mathbf{x} = B\mathbf{y}.$$

In matrix notation, the promise (1) can be written as follows:

$$\forall \mathbf{x} \in \mathbb{N}_0^k \setminus K_s, \exists \mathbf{y} \in \mathbb{N}_0^\ell : \mathbf{c} + A\mathbf{x} = B\mathbf{y}. \quad (7)$$

We now proceed to Stage 2 of the proof. We say that two systems of diophantine equations with the same collection (\mathbf{x}, \mathbf{y}) of variables are *fully equivalent* if their solution sets in $\mathbb{N}_0^k \times \mathbb{N}_0^\ell$ are the same. We say that they are *s-equivalent* if their solution sets in $K_s \times \mathbb{N}_0^\ell$ are the same. We use the symbol “ \equiv ” for “full equivalence” and the symbol “ $\stackrel{s}{\equiv}$ ” for “s-equivalence”. M_i denotes the i -th row of a matrix M and u denotes the largest entry of the matrix A . With this notation, we may rewrite the equation system $\mathbf{c} + A\mathbf{x} = B\mathbf{y}$ as follows:

$$\begin{aligned} \mathbf{c} + A\mathbf{x} = B\mathbf{y} &\equiv [-A \ B] \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c} \\ &\equiv [(uJ - A) \ B] \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c} + uJ \cdot \mathbf{x} \\ &\stackrel{s}{\equiv} [(uJ - A) \ B] \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c} + su \cdot \mathbf{1}^m \\ &\stackrel{\text{Cor. 3.3}}{\equiv} \left(\sum_{i=1}^m t_i (uJ - A)_i \right) \cdot \mathbf{x} + \left(\sum_{i=1}^m t_i B_i \right) \cdot \mathbf{y} \\ &= \sum_{i=1}^m t_i c_i + su \left(\sum_{i=1}^m t_i \right) \\ &\stackrel{s}{\equiv} \sum_{i=1}^m t_i c_i + \left(\sum_{i=1}^m t_i A_i \right) \cdot \mathbf{x} = \left(\sum_{i=1}^m t_i B_i \right) \cdot \mathbf{y} \end{aligned}$$

We move on to Stage 3 of the proof where we obtain the desired reduction to the 1-dimensional containment problem:

$$\begin{aligned}
L(\mathbf{c}, P) \subseteq L(\mathbf{0}, Q) &\Leftrightarrow \forall \mathbf{x} \in \mathbb{N}_0^k, \exists \mathbf{y} \in \mathbb{N}_0^\ell : \mathbf{c} + A\mathbf{x} = B\mathbf{y} \\
&\Leftrightarrow \forall \mathbf{x} \in \mathbb{N}_0^k, \exists \mathbf{y} \in \mathbb{N}_0^\ell : \\
&\quad \underbrace{\sum_{i=1}^m t_i c_i}_{=:c} + \underbrace{\left(\sum_{i=1}^m t_i A_i \right)}_{=: (p_1, \dots, p_k) =: \mathbf{p}^\top} \cdot \mathbf{x} = \underbrace{\left(\sum_{i=1}^m t_i B_i \right)}_{=: (q_1, \dots, q_\ell) =: \mathbf{q}^\top} \cdot \mathbf{y} \\
&\Leftrightarrow L(c, \mathbf{p}) \subseteq L(0, \mathbf{q})
\end{aligned}$$

The validity of the 2nd equivalence follows by case analysis:

Case 1: $\mathbf{x} \in K_s$.

We make use of the relation $\stackrel{s}{\equiv}$ that we had established before: within the restricted domain $K_s \times \mathbb{N}_0^\ell$, the solution set for the equation system $\mathbf{c} + A\mathbf{x} = B\mathbf{y}$ coincides with solution set for the single equation $c + \mathbf{p}^\top \mathbf{x} = \mathbf{q}^\top \mathbf{y}$.

Case 2: $\mathbf{x} \in \mathbb{N}^k \setminus K_s$.

We make use of the promise (7) and of the observation that the validity of $\mathbf{c} + A\mathbf{x} = B\mathbf{y}$ implies the validity of $c + \mathbf{p}^\top \mathbf{x} = \mathbf{q}^\top \mathbf{y}$ (because the aggregation of valid equations always yields another valid equation).

It is easy to see that the reduction mapping $(\mathbf{c}, P, Q) \mapsto (c, \mathbf{p}, \mathbf{q})$ is logspace-computable. This concludes the proof. \bullet

Combining the restrictions of dimensionality 1 and unary encoding of numbers, the containment problem for linear sets becomes solvable in polynomial time:

Theorem 3.5 *The containment problem for 1-dimensional linear sets with a unary encoding of numbers is in P.*

Proof Consider an input instance given by (the unary encoding of) c, P, c', P' with $c, c' \in \mathbb{N}_0$ and $P, P' \subset \mathbb{N}$. Let g (resp. g') be the greatest common divisor of the periods in P (resp. in P'). We make the following observation:

Claim: The containment $c + \langle P \rangle \subseteq c' + \langle P' \rangle$ is possible only if $c' \leq c$ and if g' is a divisor of g and of $c - c'$.

Given the assertion in the claim, we can accomplish the proof as follows. Setting $c_0 = c - c'$, our original question, “ $c + \langle P \rangle \subseteq c' + \langle P' \rangle$?”, is equivalent to “ $c_0 + \langle P \rangle \subseteq \langle P' \rangle$?”. We may now even assume that $g' = 1$ (because, if necessary, we can divide all numerical parameters by g'). If 1 is among the periods of P' , then the answer to “ $c_0 + \langle P \rangle \subseteq \langle P' \rangle$?” is clearly “yes”. Suppose now that $1 \notin P'$. It is well known that $\langle P' \rangle$ contains all but finitely many natural numbers [6]. Let $F(P')$ (called the *Frobenius number* of P') denote the largest number in \mathbb{N}

that is not contained in $\langle P' \rangle$. It is well known that $F(P') < (\max(P') - 1) \cdot (\min(P') - 1)$ [1]. The questions “ $x \in c_0 + \langle P \rangle$?” and “ $x \in \langle P' \rangle$?” can be answered for all $x < (\max(P') - 1) \cdot (\min(P') - 1)$ in the obvious way by dynamic programming. Given the answers to these questions, we can immediately decide whether $c_0 + \langle P \rangle \subseteq \langle P' \rangle$.

All that remains to be done is proving the above claim. Suppose that

$$c + \langle P \rangle \subseteq c' + \langle P' \rangle . \quad (8)$$

This obviously implies that $c' \leq c$. It is furthermore obvious that $\langle P \rangle \subseteq g \cdot \mathbb{N}_0$ and $\langle P' \rangle \subseteq g' \cdot \mathbb{N}_0$. Moreover, by the definition of the Frobenius number, $s := g \cdot F\left(\left(\frac{1}{g} \cdot \langle P \rangle\right)\right)$ is the largest multiple of g that does not belong to $\langle P \rangle$. Hence $c + s + g, c + s + 2g \in c + \langle P \rangle$ and, because of (8), there must exist $q_2 > q_1 \geq 1$ such that $c + s + g = c' + q_1 g'$ and $c + s + 2g = c' + q_2 g'$. Now we obtain $g = (q_2 - q_1)g'$ so that g' is a divisor of g . Since $(c - c') + \langle P \rangle \subseteq \langle P' \rangle \subseteq g' \cdot \mathbb{N}_0$ and $\langle P \rangle$ contains only multiples of g' (because it only contains multiples of g), it follows that g' must also be a divisor of $c - c'$, which concludes the proof of the claim and the proof of the theorem. \bullet

4 Proof of Theorem 2.9

It is easy to see that the containment problem for simple unary $(+, \cup)$ -expressions is a member of the complexity class Π_2^P . In somewhat more detail, let E and E' be two simple unary expressions of the form (2). Then $L(E) \subseteq L(E')$ iff

$$\forall a \in \{1, 2\}^s, \exists a' \in \{1, 2\}^{s'} : \sum_{i=1}^s \mathbf{B}_{\mathbf{ia}_i} = \sum_{i=1}^{s'} \mathbf{B}'_{\mathbf{ia}'_i} .$$

The membership in Π_2^P is now immediate from a well known characterization of Π_2^P due to Wrathall [8]: $L \in \Pi_2^P$ iff there exists a polynomial q and a language $L_0 \in \mathcal{P}$ such that

$$L = \{x \mid (\forall y_1 \text{ with } |y_1| \leq q(|x|)) (\exists y_2 \text{ with } |y_2| \leq q(|x|)) : \langle y_1, y_2, x \rangle \in L_0\} .$$

It remains to show that it is log-hard in Π_2^P . To this end, we will design a logspace reduction from $\overline{\mathcal{B}}_2^{DNF}$ to this problem. Let $f(X_1, X_2)$ be an instance of $\overline{\mathcal{B}}_2^{DNF}$ (as described in Example 2.4). Since f employs only finitely many variables, we may assume that $X_i = \{x_{i1}, \dots, x_{in}\}$ for $i = 1, 2$ and some $n \geq 1$. As a DNF-formula, f is the disjunction of Boolean monomials, say $f = M_1 \vee \dots \vee M_m$. We may clearly assume that none of the monomials contains the same variable twice. We will transform $f(X_1, X_2)$ into simple unary $(+, \cup)$ -expressions E_1 and E_2 such that

$$(\forall X_1, \exists X_2 : f(X_1, X_2) = 0) \Leftrightarrow (L(E_1) \subseteq L(E_2)) . \quad (9)$$

For all $i = 1, \dots, n$ and $j = 1, \dots, m$, let

$$\mathbf{b}_{\mathbf{1i}}[j] = \begin{cases} 1 & \text{if } x_{1i} \in M_j \\ 0 & \text{otherwise} \end{cases} ,$$

i.e., the binary vector $\mathbf{b}_{1i} \in \{0, 1\}^m$ indicates in which monomials the variable x_{1i} actually occurs. Let $\mathbf{b}'_{1i} \in \{0, 1\}^m$ denote the corresponding vector with indicator bits for the occurrences of $\overline{x_{1i}}$ within M_1, \dots, M_m . Let the vectors $\overline{\mathbf{b}_{1i}}$ and $\overline{\mathbf{b}'_{1i}}$ be obtained from \mathbf{b}_{1i} and \mathbf{b}'_{1i} , respectively, by bitwise negation. Clearly, the bits of these vectors indicate the non-occurrences of x_{1i} resp. $\overline{x_{1i}}$ within M_1, \dots, M_m . Let $\mathbf{b}_{2i}, \mathbf{b}'_{2i}, \overline{\mathbf{b}_{2i}}, \overline{\mathbf{b}'_{2i}}$ be the corresponding vectors with indicator bits for the occurrences resp. non-occurrences of the variable x_{2i} . We now define a couple of $(+, \cup)$ -expressions:

$$\begin{aligned} E'_1 &= \sum_{i=1}^n (\mathbf{1}^m \cup \mathbf{1}^m) \quad \text{and} \quad E_1 = E'_1 + \sum_{i=1}^n (\overline{\mathbf{b}_{1i}} \cup \overline{\mathbf{b}'_{1i}}) \\ E'_2 &= \sum_{j=1}^m \sum_{i=1}^{2n-1} (\mathbf{e}_j^m \cup \mathbf{0}^m) \quad \text{and} \quad E_2 = E'_2 + \sum_{i=1}^n (\mathbf{b}_{2i} \cup \mathbf{b}'_{2i}) . \end{aligned}$$

The following immediate observations will prove useful:

1. $L(E'_1) = \{n \cdot \mathbf{1}^m\}$ and $L(E'_2) = \{0, \dots, 2n-1\}^m$.
2. $L(E_1) \subseteq \{n, \dots, 2n\}^m$ and $L(E_2) \supseteq \{n, \dots, 2n-1\}^m$.

Note that the only vectors of $L(E_1)$ which might perhaps not belong to $L(E_2)$ are the ones with at least one component of size $2n$. The following definitions take care of these ‘‘critical vectors’’. We say that a partial assignment of the variables in $X_1 \cup X_2$ *annuls* M_j if one of the literals contained in M_j is set to 0. Let $\mathbf{y} \in \{n, \dots, 2n\}^m$. An assignment $A_1 : X_1 \rightarrow \{0, 1\}$ is said to be an X_1 -*assignment of type* \mathbf{y} if the following holds:

$$\forall j = 1, \dots, m : (\mathbf{y}[j] = 2n \Leftrightarrow A_1 \text{ does not annihilate } M_j) .$$

We say that $A_2 : X_2 \rightarrow \{0, 1\}$ is an X_2 -*assignment of type* \mathbf{y} if the following holds:

$$\forall j = 1, \dots, m : (\mathbf{y}[j] = 2n \Rightarrow A_2 \text{ annihilates } M_j) .$$

The desired equivalence (9) is easy to derive from the following claims:

Claim 1: For every $\mathbf{y} \in L(E_1)$, there exists an X_1 -assignment A_1 of type \mathbf{y} .

Claim 2: For every $A_1 : X_1 \rightarrow \{0, 1\}$, there exists $\mathbf{y} \in L(E_1)$ such that A_1 is an X_1 -assignment of type \mathbf{y} .

Claim 3: For every $\mathbf{y} \in \{n, \dots, 2n\}^m$:

$$\mathbf{y} \in L(E_2) \Leftrightarrow (\exists A_2 : X_2 \rightarrow \{0, 1\} : A_2 \text{ is an } X_2\text{-assignment of type } \mathbf{y}) .$$

Proof of Claim 1: Pick any $\mathbf{y} \in L(E_1)$. It follows that \mathbf{y} is of the form

$$\mathbf{y} = n \cdot \mathbf{1}^m + \sum_{i=1}^n \widetilde{\mathbf{b}_{1i}} \quad \text{with} \quad \widetilde{\mathbf{b}_{1i}} \in \{\overline{\mathbf{b}_{1i}}, \overline{\mathbf{b}'_{1i}}\} . \quad (10)$$

If $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}}_{1i}$, we set $A_1(x_{1i}) = 0$ else, if $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}'_{1i}}$, we set $A_1(x_{1i}) = 1$. We claim that A_1 is of type \mathbf{y} . This can be seen as follows. Pick any $j \in \{1, \dots, m\}$. An inspection of (10) reveals the following:

- Suppose that $\mathbf{y}[j] = 2n$. It follows that $\widetilde{\mathbf{b}}_{1i}[j] = 1$ for $i = 1, \dots, n$. Hence, if $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}}_{1i}$, then $A_1(x_{1i}) = 0$, $\overline{\mathbf{b}}_{1i}[j] = 1$ and, therefore, $x_{1i} \notin M_j$. Similarly, if $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}'_{1i}}$, then $A_1(x_{1i}) = 1$, $\overline{\mathbf{b}'_{1i}}[j] = 1$ and, therefore, $\overline{x_{1i}} \notin M_j$. Since these observations hold for all $i = 1, \dots, n$, we may conclude that A_1 does not annul M_j .
- Suppose that $\mathbf{y}[j] \leq 2n - 1$. Then there exists $i \in \{1, \dots, n\}$ such that $\widetilde{\mathbf{b}}_{1i}[j] = 0$. Hence, if $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}}_{1i}$, then $A_1(x_{1i}) = 0$, $\overline{\mathbf{b}}_{1i}[j] = 0$ and, therefore, $x_{1i} \in M_j$. Similarly, if $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}'_{1i}}$, then $A_1(x_{1i}) = 1$, $\overline{\mathbf{b}'_{1i}}[j] = 0$ and, therefore, $\overline{x_{1i}} \in M_j$. It follows that A_1 does annul M_j .

The above discussion shows that A_1 is of type \mathbf{y} , indeed.

Proof of Claim 2: Given any $A_1 : X_1 \rightarrow \{0, 1\}$, we set $\mathbf{y} = n \cdot \mathbf{1}^m + \sum_{i=1}^n \widetilde{\mathbf{b}}_{1i}$ where $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}}_{1i}$ if $A_1(x_{1i}) = 0$ and, similarly, $\widetilde{\mathbf{b}}_{1i} = \overline{\mathbf{b}'_{1i}}$ if $A_1(x_{1i}) = 1$. Note that, with this definition of \mathbf{y} , A_1 is precisely the X_1 -assignment that we had chosen in the proof of Claim 1. As argued in the proof of Claim 1 already, A_1 is of type \mathbf{y} .

Proof of Claim 3: Pick any $\mathbf{y} \in \{n, \dots, 2n\}^m$. Suppose first that $\mathbf{y} \in L(E_2)$. It follows that \mathbf{y} is of the form

$$\mathbf{y} = \mathbf{y}' + \sum_{i=1}^n \widetilde{\mathbf{b}}_{2i} \quad \text{with } \mathbf{y}' \in \{0, \dots, 2n - 1\}^m \quad \text{and } \widetilde{\mathbf{b}}_{2i} \in \{\mathbf{b}_{2i}, \mathbf{b}'_{2i}\} . \quad (11)$$

If $\widetilde{\mathbf{b}}_{2i} = \mathbf{b}_{2i}$, we set $A_2(x_{2i}) = 0$ else, if $\widetilde{\mathbf{b}}_{2i} = \mathbf{b}'_{2i}$, we set $A_2(x_{2i}) = 1$. We claim that A_2 is of type \mathbf{y} . Consider an index $j \in \{1, \dots, m\}$ such that $\mathbf{y}[j] = 2n$. An inspection of (11) reveals that there exists $i \in \{1, \dots, n\}$ such that $\widetilde{\mathbf{b}}_{2i}[j] = 1$. If $\widetilde{\mathbf{b}}_{2i} = \mathbf{b}_{2i}$, then $A_2(x_{2i}) = 0$, $\mathbf{b}_{2i}[j] = 1$ and, therefore, $x_{2i} \in M_j$. Similarly, if $\widetilde{\mathbf{b}}_{2i} = \mathbf{b}'_{2i}$, then $A_2(x_{2i}) = 1$, $\mathbf{b}'_{2i}[j] = 1$ and, therefore, $\overline{x_{2i}} \in M_j$. In any case, A_2 annuls M_j and we may conclude that A_2 is of type \mathbf{y} .

Suppose now that there exists an X_2 -assignment A_2 that is of type $\mathbf{y} \in \{n, \dots, 2n\}^m$. We define $\mathbf{y}'' = \sum_{i=1}^n \widetilde{\mathbf{b}}_{2i}$ where $\widetilde{\mathbf{b}}_{2i} = \mathbf{b}_{2i}$ if $A_2(x_{2i}) = 0$ and, similarly, $\widetilde{\mathbf{b}}_{2i} = \mathbf{b}'_{2i}$ if $A_2(x_{2i}) = 1$. Since A_2 is of type \mathbf{y} , it annuls every M_j with $\mathbf{y}[j] = 2n$. It follows that, for every $j \in \{1, \dots, m\}$ with $\mathbf{y}[j] = 2n$, there exists $i \in \{1, \dots, n\}$ such either $x_{2i} \in M_j$ and $A_2(x_{2i}) = 0$ or $\overline{x_{2i}} \in M_j$ and $A_2(x_{2i}) = 1$. In both cases, we have that $\widetilde{\mathbf{b}}_{2i}[j] = 1$. It follows from this discussion that $\mathbf{y}''[j] \geq 1$ for every j with $\mathbf{y}[j] = 2n$. Obviously $\mathbf{y}''[j] \leq n$ for all $j = 1, \dots, m$. Since $L(E_2) = \{0, \dots, 2n - 1\}^m$ and $\mathbf{y} \in \{n, \dots, 2n\}^m$, there exists $\mathbf{y}' \in L(E_2)$ such that $\mathbf{y} = \mathbf{y}' + \mathbf{y}''$. This decomposition of \mathbf{y} shows that $\mathbf{y} \in L(E_2)$.

We are ready now for proving (9). Assume first that the condition on the left hand-side of (9) is valid. Pick any $\mathbf{y} \in L(E_1)$. Pick an X_1 -assignment A_1 of type \mathbf{y} (application of Claim 1). It follows that the monomials M_j with $\mathbf{y}[j] = 2n$ are not yet annulled by A_1 . According to the left hand-side of (9), there must exist an assignment $A_2 : X_2 \rightarrow \{0, 1\}$ that annuls them. In other words: A_2 is an X_2 -assignment of type \mathbf{y} . We may now conclude from Claim 3 that $\mathbf{y} \in L(E_2)$, as desired.

Suppose now that $L(E_1) \subseteq L(E_2)$. Pick any assignment $A_1 : X_1 \rightarrow \{0, 1\}$. Pick $\mathbf{y} \in L(E_1)$ such A_1 is an X_1 -assignment of type \mathbf{y} (application of Claim 2). It follows that only the monomials M_j with $\mathbf{y}[j] = 2n$ are not yet annulled by A_1 . Since \mathbf{y} , as an element of $L(E_1)$, must satisfy $\mathbf{y} \in \{n, \dots, 2n\}^m$ and must furthermore belong to $L(E_2)$, we may conclude from Claim 3 that there exists an X_2 -assignment $A_2 : X_2 \rightarrow \{0, 1\}$ of type \mathbf{y} . In other words: A_2 annuls all monomials M_j with $\mathbf{y}[j] = 2n$. It follows from this discussion that the condition on the left hand-side of (9) is valid, which concludes the proof.

5 Open Problems

In the proof of our hardness results, we made essential use of the fact that $\langle P \rangle$ contains all linear combinations of the periods in P with coefficient vectors from $\mathbb{N}_0^{|P|}$. We would be interested to know whether the computational complexity of the containment problem is still the same when we deal with coefficient vectors from $\mathbb{N}^{|P|}$ (thereby ruling out 0-coefficients).

Acknowledgements. Many thanks go to Dmitry Chistikov and Christoph Haase who pointed my attention to [2], a paper that (without mentioning this explicitly) yields the log-hardness of the containment problem for linear sets of variable dimension.

References

- [1] Alfred Brauer. On a problem of partitions. *American Journal of Mathematics*, 64(1):299–312, 1942.
- [2] Dmitry Chistikov, Christoph Haase, and Simon Halfon. Context-free commutative grammars with integer counters and resets. *Theoretical Computer Science*, 2016. In press. Online version: <https://doi.org/10.1016/j.tcs.2016.06.017>.
- [3] F. Glover and R. E. D. Woolsey. Aggregating diophantine equations. *Zeitschrift für Operations Research*, 16:1–10, 1972.
- [4] Thiet-Dung Huynh. The complexity of semilinear sets. *Elektronische Informationsverarbeitung und Kybernetik*, 18(6):291–338, 1982.
- [5] Rohit J. Parikh. On context-free languages. *Journal of the Association on Computing Machinery*, 13(4):570–581, 1966.

- [6] José C. Rosales and Pedro A. García-Sánchez. *Numerical Semigroups*. Springer, 2009.
- [7] Larry J. Stockmeyer. The polynomial-time hierarchy. *Theoretical Computer Science*, 3(1):1–22, 1977.
- [8] Celia Wrathall. Complete sets and the polynomial-time hierarchy. *Theoretical Computer Science*, 3(1):23–33, 1976.