Bestimme (a) das glob. Min., (b) das glob. Max.

von $f(x,y) = x^2+y^2-4xy+1$ unter $x^2+y^2 \le 1$, $y \ge 0 \Leftrightarrow -y \le 0$

(a) $L(x,y) = -x^2-y^2+4xy-1-\lambda_1(x^2+y^2-1)-\lambda_2(-y)$

$L_x = -2x+4y-2\lambda_1 x = 0$ (i)

$L_y = -2y+4x-2\lambda_1 y+\lambda_2 = 0$ (ii)

$x^2+y^2 \le 1$, $\lambda_1 \ge 0$, $\lambda_1(x^2+y^2-1)=0$ (iii)

$y \ge 0$, $\lambda_2 \ge 0$, $\lambda_2 y = 0$ (iv)

(I): "=,=", $x^2+y^2=1$, $y=0 \Rightarrow x^2=1$, (i) $\Rightarrow -2x(1+\lambda_1)=0$

$\Rightarrow \lambda_1 = -1$ #

(II): "=,<", $x^2+y^2=1$, $y>0$, $\lambda_2=0$ $\overset{(i),(ii)}{\Rightarrow}$ $\lambda_1 = \frac{-2x+4y}{2x} = -1+2\frac{y}{x}$

$= \frac{-2y+4x}{2y} = -1+2\frac{x}{y}$

$\Rightarrow \frac{y}{x} = \frac{x}{y} \Leftrightarrow x^2=y^2 \Rightarrow 2y^2=1 \Leftrightarrow y = \frac{1}{\sqrt{2}}$ $(y>0)$

$\Rightarrow x = \pm y$, $x=-y$ #, da $\lambda_1 < 0$

$x=y \Rightarrow \lambda_1 = 1$, also $P_1 = \left(\frac{1}{\sqrt{2}} \middle/ \frac{1}{\sqrt{2}}\right)$, $\lambda_1=1$, $\lambda_2=0$

(III): "<,=", $x^2+y^2<1$, $y=0$, $\lambda_1=0$, (i) $\Rightarrow x=0$, (ii) $\Rightarrow \lambda_2=0$,

also $P_2 = (0/0)$ mit $\lambda_1=\lambda_2=0$

(IV): "<,<", $x^2+y^2<1$, $y>0$, $\lambda_1=\lambda_2=0$, (i) $\Rightarrow 2x=4y \Leftrightarrow x=2y$

(ii) $\Rightarrow 2y=4x \Leftrightarrow y=2x$

$\Rightarrow x = 4x \Leftrightarrow x=0 \Rightarrow y=0$ # in $y>0$

$f(P_1) = \frac{1}{2}+\frac{1}{2}-4\cdot\frac{1}{2}+1 = 0$

$f(P_2) = 1$

} ← glob. Min.

(Extremwertsatz: $x^2+y^2 \le 1$, $y \ge 0$ } Halbkreis)

(b) $L(x,y) = x^2+y^2-4xy+1-\lambda_1(x^2+y^2-1)-\lambda_2(-y)$

$L_x = 2x+4y-2\lambda_1 x = 0$ (i)

$L_y = 2y-4x-2\lambda_1 y+\lambda_2 = 0$ (ii)

, (iii), (iv) wie (a)

① "=,=": $x^2+y^2=1$, $y=0 \Rightarrow x^2=1$ $\overset{(i)}{\Rightarrow}$ $\lambda_1=1$, (ii) $\lambda_2=4x$

$x=-1$ #, da $\lambda_2<0$, $x=1$, $\lambda_2=4$, also

$P_1=(1|0)$, $\lambda_1=1$, $\lambda_2=4$

② "=,<": $x^2+y^2=1$, $y>0$, $\lambda_2=0$, $\overset{(i),(ii)}{\Rightarrow}$ $\lambda_1 = \frac{2x-4y}{2x} = 1-2\frac{y}{x}$

$= \frac{2y-4x}{2y} = 1-2\frac{x}{y}$

$\Rightarrow \frac{y}{x}=\frac{x}{y} \Leftrightarrow x^2=y^2 \Rightarrow 2y^2=1 \Leftrightarrow y=\frac{1}{\sqrt{2}}$ $(y>0)$

$x=y$ # zu $\lambda_1>0$, $x=-y \Rightarrow \lambda_1=3$, also

$P_2=\left(-\frac{1}{\sqrt{2}}\,|\,\frac{1}{\sqrt{2}}\right)$, $\lambda_1=3$, $\lambda_2=0$

③ "<,=": $x^2+y^2<1$, $y=0$, $\lambda_1=0$ $\overset{(i)}{\Rightarrow}$ $x=0$ $\overset{(ii)}{\Rightarrow}$ $\lambda_2=0$, also

$P_3=(0|0)$, $\lambda_1=\lambda_2=0$

④ "<,<": $x^2+y^2<1$, $y>0$, $\lambda_1=\lambda_2=0$ $\overset{(i),(ii)}{\Rightarrow}$ $\left.\begin{matrix}2x=4y\\2y=4x\end{matrix}\right\} \Leftrightarrow x=y=0$ # $y>0$

$f(P_1)=2$

$f(P_2)=\frac{1}{2}+\frac{1}{2}-4\cdot(-\frac{1}{2})+1=4$ $\Big\}$ ← glob. Max. (Extremwertsatz s.o.)

$f(P_3)=1$