1. $\dot{x} = x - 4y$, $x(0)=1$, $A = \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}$, $\mathrm{Spur}(A) = -2$, $|A| = 1$
$\dot{y} = x - 3y$, $y(0) = 0$

$\Rightarrow \ddot{x} + 2\dot{x} + x = 0$, $r = -1$, $w = 0$ $\Rightarrow x(t) = e^{-t}(A + Bt)$

$x(0) = 1 \Rightarrow A = 1 \Rightarrow x(t) = e^{-t}(1 + Bt)$

$\dot{x} = -e^{-t}(1+Bt) + e^{-t} \cdot B = e^{-t}(B - 1 - Bt)$

$\dot{x}(0) = 1 - 4 \cdot 0 = 1 \overset{!}{=} B - 1 \Rightarrow B = 2 \Rightarrow x(t) = e^{-t}(1+2t)$

$\Rightarrow y(t) = -\frac{1}{4}(\dot{x} - x) = -\frac{1}{4}\left(e^{-t}(1-2t) - e^{-t}(1+2t) \right)$

$= -\frac{1}{4} e^{-t} \cdot (-4t) = t \cdot e^{-t}$

2. $\dot{x} = 5x - 2y$, $x(0) = 1$, $\dot{x}(0) = 5 - 4 = 1$
$\dot{y} = 13x - 5y$, $y(0) = 2$

$A = \begin{pmatrix} 5 & -2 \\ 13 & -5 \end{pmatrix}$, $\mathrm{Spur}(A) = 0$, $|A| = 1$

$\Rightarrow \ddot{x} + x = 0$, $a = 0$, $b = 1 > 0$ $\Rightarrow x(t) = A\sin(t) + B\cos(t)$

$x(0) = 1 \Rightarrow B = 1 \Rightarrow x(t) = A\sin(t) + \cos(t)$

$\dot{x} = A\cos(t) - \sin(t)$, $\dot{x}(0) = 1 \Rightarrow A = 1$

$\Rightarrow x(t) = \sin(t) + \cos(t)$

$\Rightarrow y(t) = -\frac{1}{2}(\dot{x} - 5x) = -\frac{1}{2}\left(\cos(t) - \sin(t) - 5(\sin(t) + \cos(t))\right)$

$= -\frac{1}{2}(-4\cos(t) - 6\sin(t)) = 3\sin(t) + 2\cos(t)$

3. $\dot{x} = -3x + 4y - 4t^2 - 6$, $x(0) = 1$, $\dot{x}(0) = -3 + 4 - 6 = -5$
$\dot{y} = -y + t(t+2)$, $y(0) = 1$

Variante 1: Rekursiv

$\dot{y} + y = 0 \Rightarrow y = Ae^{-t}$, $u^* = Bt^2 + Ct + D$

$\dot{u}^* = 2Bt + C$, $\dot{u}^* + u^* = 2Bt + C + Bt^2 + Ct + D \overset{!}{=} t^2 + 2t$

$\Rightarrow B = 1$, $2 + C = 2 \Rightarrow C = 0$, $C + D = 0 \Rightarrow D = 0$

$\Rightarrow y(t) = Ae^{-t} + t^2$, $y(0) = 1 \Rightarrow A = 1$

$\Rightarrow y(t) = e^{-t} + t^2$

$\dot{x} + 3x = 0 \Rightarrow x(t) = Ae^{-3t}$, $f(t) = 4y - 4t^2 - 6$

$f(t) = 4e^{-t} + 4t^2 - 4t^2 - 6 = 4e^{-t} - 6$, $u^* = Be^{-t} + C$

$\dot{u}^* = -Be^{-t}$, $\dot{u}^* + 3u^* = -Be^{-t} + 3Be^{-t} + 3C \overset{!}{=} 4e^{-t} - 6$

$\Rightarrow 2B = 4 \Leftrightarrow B = 2$, $3C = -6 \Leftrightarrow C = -2$

$\Rightarrow x(t) = Ae^{-3t} + 2e^{-t} - 2$, $x(0) = 1 \Rightarrow A + 2 - 2 = 1$

$\Leftrightarrow A = 1 \Rightarrow x(t) = e^{-3t} + 2e^{-t} - 2$

Variante 2. Lineare DGL

$A = \begin{pmatrix} -3 & 4 \\ 0 & -1 \end{pmatrix}$, $b_1 = -4t^2 - 6$, $b_2 = t^2 + 2t$, $Spur(A) = -4$, $|A| = 3$ $(\Rightarrow$ stabil$)$

$\ddot{x} + 4\dot{x} + 3x = f(t) = \dot{b}_1 - |A_1| = -8t - \left|\begin{pmatrix} -4t^2-6 & 4 \\ t^2+2t & -1 \end{pmatrix}\right| = -8t - (4t^2 + 6 - 4t^2 - 8t) = -6$

homogen: $\Rightarrow$ $r = -2$, $w = 1 = 5$, $r_{1,2} = -2 \pm 1$, $r_1 = -1$, $r_2 = -3$

$\Rightarrow$ $x(t) = Ae^{-t} + Be^{-3t}$

partikulär: $u^*(t) = \frac{-6}{3} = -2$ $\Rightarrow$ $x(t) = Ae^{-t} + Be^{-3t} - 2$

$x(0) = 1$ $\Rightarrow$ $A + B - 2 \overset{!}{=} 1$ $\Rightarrow$ $A + B = 3$

$\dot{x} = -Ae^{-t} - 3Be^{-3t}$, $\dot{x}(0) = -A - 3B \underset{(s.o.)}{\overset{!}{=}} -5$ $\Bigg\}$ $\Rightarrow -2B = -2$ $\Rightarrow$ $B = 1$

$\Rightarrow$ $A = 2$

$\Rightarrow$ $x(t) = 2e^{-t} + e^{-3t} - 2$

$\Rightarrow$ $y(t) = \frac{1}{4}(\dot{x} + 3x + 4t^2 + 6) = \frac{1}{4}\left(-2e^{-t} - 3e^{-3t} + 3(2e^{-t} + e^{-3t} - 2) + 4t^2 + 6\right)$

$= \frac{1}{4}(4e^{-t} + 4t^2) = e^{-t} + t^2$

4. $\dot{x} = \frac{(y+1)^3}{4x}$, $\dot{y} = \frac{(y+1)^2}{2x}$, $x(1) = 1 = y(1)$

1. $\frac{\dot{y}}{\dot{x}} = \frac{(y+1)^2}{2x} \cdot \frac{4x}{(y+1)^3} = \frac{2}{y+1} \Rightarrow \dot{h} = \frac{2}{h+1}$ (Separabel)

$\int 2\,dt = 2X + A \overset{!}{=} \int h+1\,dh = \frac{1}{2}h^2 + h$

$\Leftrightarrow h^2 + 2h = 4x + 2A \Leftrightarrow (h+1)^2 = 4X + 2A + 1$

2. $\dot{x} = \frac{(h+1)^3}{4x} = \frac{(4X+2A+1)^{3/2}}{4x}$, $\dot{x}(1) = \frac{2^3}{4} = 2 \overset{!}{=} \frac{(4+2A+1)^{3/2}}{4}$

$\Leftrightarrow (5+2A)^{3/2} = 8 \Leftrightarrow (5+2A)^{1/2} = 2 \Leftrightarrow 5+2A = 4$

$\Leftrightarrow 2A = -1 \Leftrightarrow A = -\frac{1}{2} \Rightarrow \dot{x} = \frac{(4x)^{3/2}}{4x} = (4x)^{1/2} = 2\sqrt{x}$

(separabel) $\Rightarrow \int 2\,dt = 2t + B \overset{!}{=} \int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}$

$\Leftrightarrow \sqrt{x} = t + \frac{B}{2} \Rightarrow x(t) = \left(t + \frac{B}{2}\right)^2$

$x(1) = 1 \Rightarrow \left(1 + \frac{B}{2}\right)^2 = 1 \Leftrightarrow B = 0$

$\Rightarrow x(t) = t^2$

3. $(y+1)^2 = (h+1)^2 = 4X \Leftrightarrow y+1 = \pm 2\sqrt{X} = \pm 2t$

$y(1) = 1 \Rightarrow y+1 = 2t \Leftrightarrow y(t) = 2t - 1$