Lösungsansätze zur Probeklausur

Aufgabe 1 : (a) $L = -x(y^2-1) - \lambda_1(x^2+y^2-4) - \lambda_2\cdot(-y)$

$L_x = -(y^2-1) - 2\lambda_1 x = 0$ (i)
$L_y = -2xy - 2\lambda_1 y + \lambda_2 = 0$ (ii) .
$\lambda_1 \geq 0$, $x^2+y^2 \leq 4$, $\lambda_1(x^2+y^2-4) = 0$ (iii)
$\lambda_2 \geq 0$, $y \geq 0$, $\lambda_2 y = 0$ (iv)

(b) I "<,<" : $x^2+y^2 < 4$, $y > 0$, $\lambda_1 = \lambda_2 = 0$
$\Rightarrow \ldots \Rightarrow$ $P_1 = (0/1)$

II "<,=" : $x^2+y^2 < 4$, $y = 0$, $\lambda_1 = 0$ $\Rightarrow$ #

III "=,<" : $x^2+y^2 = 4$, $y > 0$, $\lambda_2 = 0$
$\Rightarrow \ldots \Rightarrow$ $P_2 = (-1/\sqrt{3})$, $\lambda_1 = 1$

IV "=,=" : $x^2+y^2 = 4$, $y = 0$
$\Rightarrow \ldots \Rightarrow$ $P_3 = (2/0)$, $\lambda_1 = \frac{1}{4}$, $\lambda_2 = 0$

(c) $f(P_1) = 0$, $f(P_2) = 2 = f(P_3)$
Zulässiger Bereich = Halbkreis (abgeschlossen + beschränkt)
Extremwertsatz $\Rightarrow$ P_2, P_3 globale Maxima von f
= globale Minima von $-f$

Aufgabe 2 : (a) $\dot{x} = \underbrace{x}_{g(x)}\cdot\underbrace{\left(\frac{1}{t}-1\right)}_{f(t)}$ separabel

$\int \frac{1}{x}\,dx = \ln(|x|) = \int \frac{1}{t} - 1\,dt = \ln(|t|) - t + A$
$\Rightarrow \ldots\ldots \Rightarrow$ $x(t) = t\cdot e^{1-t}$

(b) Homogen : $x(t) = Ae^{t} + Be^{-t}$
Partikulär : $u^*(t) = C\cdot\sin(2t) + D$ $\Rightarrow \cdots \Rightarrow$ $u^*(t) = -\sin(2t)+1$
$\Rightarrow$ $x(t) = Ae^{t} + Be^{-t} - \sin(2t) + 1$

(c) Homogen : $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, Spur$(A) = 4$, $|A| = 3$
$\Rightarrow$ $x(t) = Ae^{t} + Be^{3t}$

Partikulär: $u^*(t) = C\sin(t) + D\cos(t) \Rightarrow \cdots \Rightarrow$

$u^*(t) = 2\sin(t) - \cos(t) \Rightarrow X(t) = Ae^{t} + Be^{3t} + 2\sin(t) - \cos(t)$

$X(0) = 0$, $\dot{x}(0) = 5 \Rightarrow \cdots \Rightarrow X(t) = e^{3t} + 2\sin(t) - \cos(t)$

$\Rightarrow Y(t) = e^{3t} - \cos(t) - 3\sin(t)$

Aufgabe 3 (a) $\underset{u\geq 0}{\text{Max.}} \int_0^{10} -2x - 8u - 0{,}1u^2 \, dt$, $\dot{x} = u$, $x(0) = 20$, $x(10) = 200$

$H = -2x - 8u - 0{,}1u^2 + p\cdot u$ (konkav in (x,u))

u maximiert H, d.h. $\begin{cases} H_u = -8 - 0{,}2u + p = 0, & \text{falls } u > 0 \\ H_u = -8 + p \leq 0, & \text{falls } u = 0 \end{cases}$

$\dot{p} = -H_x = 2$; Bedingungen hinreichend, da H konkav

(b) $\dot{p} = 2 \Rightarrow p(t) = 2t + A$

$\Rightarrow u^*(t) = \begin{cases} 0 \\ 5(p(t) - 8) \end{cases}$ falls $\begin{matrix} p(t) \leq 8 & (\text{Fall } 1) \\ p(t) > 8 & (\text{Fall } 2) \end{matrix}$

$\Rightarrow X^*(t) = \begin{cases} x(0) = 20 \\ 5\cdot\int p(t) - 8 \, dt \end{cases}$ falls $\begin{matrix} p \leq 8 & (\text{Fall } 1) \\ p > 8 & (\text{Fall } 2) \end{matrix}$

Teste, ob $p(t) > 8$ möglich ist $\forall t \in [0,10]$:

Fall 2 $\Rightarrow x^*(t) = 5\cdot\int 2t + A - 8 \, dt = 5\cdot(t^2 + (A-8)t + B)$

$x^*(0) = 20 \Rightarrow B = 4$, $x^*(10) = 200 \Rightarrow \cdots \Rightarrow A = -6{,}4$

$\Rightarrow p(t) = 2t - 6{,}4 > 8 \Leftrightarrow t > 7{,}2$, also Fall 2 allein nicht möglich!

$p(t)$ stetig und monoton steigend $\Rightarrow \exists t^*$ mit $p(t^*) = 8$

und $\begin{cases} p(t) < 8 \\ p(t) > 8 \end{cases}$ für $\begin{matrix} t < t^* \\ t > t^* \end{matrix}$ $\Rightarrow p(t^*) = 2t^* + C = 8$
$\Leftrightarrow C = 8 - 2t^*$

$\Rightarrow p(t) = 2t + 8 - 2t^* = 2(t - t^*) + 8$, $p(t) - 8 = 2(t - t^*)$

$\Rightarrow u^*(t) = \begin{cases} 0 \\ 10(t - t^*) \end{cases}$ für $\begin{matrix} t \leq t^* \\ t > t^* \end{matrix}$

$\Rightarrow X^*(t) = \begin{cases} 20 \\ 5(t - t^*)^2 + D \end{cases}$ für $\begin{matrix} t \leq t^* \\ t > t^* \end{matrix}$

X^* stetig in $t^* \Rightarrow D = 20$, $x^*(10) = 200 \Rightarrow \cdots \Rightarrow$

$10 - t^* = \pm 6 \overset{t^* \leq 10}{\Rightarrow} t^* = 4 \Rightarrow C = 0$

$\Rightarrow p(t) = 2t$, $u^*(t) = \begin{cases} 0 \\ 10(t-4) \end{cases}$ für $\begin{matrix} t \leq 4 \\ t > 4 \end{matrix}$

$x^*(t) = \begin{cases} 20 \\ 5(t-4)^2 + 20 \end{cases}$ für $\begin{matrix} t \leq 4 \\ t > 4 \end{matrix}$

Aufgabe 4

(a) $H^c = -(u-a)^2 + \lambda \cdot u$ (konkav in u)

$H^c_u = -2(u-a) + \lambda = 0 \iff \hat{u} = \frac{1}{2}\lambda + a$

$\dot{\lambda} - r\lambda = -H^c_x = 0 \iff \lambda(t) = A \cdot e^{rt}$

$\Rightarrow u^*(t) = \frac{1}{2} A e^{rt} + a$

$\Rightarrow x^*(t) = \frac{1}{2} \cdot \frac{1}{r} \cdot A e^{rt} + at + B$

$x(0) = 0 \to \frac{A}{2r} + B = 0$, $x(T) = aT \Rightarrow \frac{A}{2r} e^{rT} + aT + B = aT$

$\Rightarrow \dots \Rightarrow A = 0 = B \Rightarrow \lambda(t) \equiv 0$, $u^*(t) = a$, $x^*(t) = at$

(b) $H = -2tx - u^2 + \mu \cdot u$ (konkav in x, u)

$H_u = -2u + \mu = 0 \iff \hat{u} = \frac{1}{2}\mu$, falls $\mu \geq 0$

$\dot{\mu} = -H_x = 2t \Rightarrow \mu(t) = t^2 + A$, $A \geq 0$

$\Rightarrow u^*(t) = \frac{1}{2}(t^2 + A) \Rightarrow x^*(t) = \frac{1}{2}\left(\frac{1}{3}t^3 + At + B\right)$

$x^*(0) = 1 \Rightarrow B = 2$, $x^*(1) = \frac{1}{2}\left(\frac{1}{3} + A + 2\right)$

$S'(x) = \frac{1}{x - \frac{1}{6}}$, $S'(x(1)) = \frac{1}{\frac{1}{6} + \frac{A}{2} + 1 - \frac{1}{6}} \overset{!}{=} \mu(1) = 1 + A$

$\overset{A \geq 0}{\Rightarrow} \dots \Rightarrow A = 0 \Rightarrow \mu(t) = t^2$, $u^*(t) = \frac{t^2}{2}$, $x^*(t) = \frac{t^3}{6} + 1$

Aufgabe 5

(a) $t = T$: $h_T(u) = \ln(x)$, u^*_T beliebig, $J_T(x) = \ln(x)$

$t = T-1$: $h_{T-1}(u) = -e \cdot u + \ln(x \cdot u) = -eu + \ln(x) + \ln(u)$ (konkav)

$h'_{T-1}(u) = -e + \frac{1}{u} = 0 \iff u^*_{T-1} = \frac{1}{e} = e^{-1} \in [0;1]$

$J_{T-1}(x) = \ln(x) - 2$

$t = T-2$: $h_{T-2}(u) = -eu + \ln(x) + \ln(u) - 2$

$\Rightarrow \dots \Rightarrow u^*_{T-2} = \frac{1}{e}$, $J_{T-2}(x) = \ln(x) - 4$

$\vdots$

$\Rightarrow u^*_t = \frac{1}{e}$; $J_{T-k}(x) = \ln(x) - 2k$, $t = 0, .., T$

$\Rightarrow x_1 = x_0 \cdot \frac{1}{e} = \frac{1}{e}$, $x_2 = x_1 \cdot u_1 = \frac{1}{e^2} = e^{-2}$, ..., $x_t = e^{-t}$

(b) $H = \begin{cases} -u^2 + 22u - 4x + \mu \cdot (x+u) & \text{für } t = 0, \dots, 3 \\ -u^2 + 22u - 4x & t = T = 4 \end{cases}$

$t = T = 4$: $H_u = -2u + 22 = 0 \iff u^*_4 = 11$, $\mu_4 = 0$, $\mu_3 = -4$

$t \leq 3$: $H_u = -2u + 22 + \mu = 0 \iff u^*_t = \frac{1}{2}\mu_t + 11$

$\mu_{t-1} = -4 + \mu_t \iff \mu_t = \mu_{t-1} + 4 \iff \mu_t = \mu_0 + 4t$

$\mu_2 = -4 + \mu_3 = -8 = \mu_0 + 8 \iff \mu_0 = -16$, $\mu_t = 4t - 16$

$\Rightarrow u^*_t = 2t + 3$, $u^*_0 = 3$, $u^*_1 = 5$, $u^*_2 = 7$, $u^*_3 = 9$

$\Rightarrow x^*_1 = x_0 + u_0 = 4$, $x^*_2 = 9$, $x^*_3 = 16$, $x^*_4 = 25$